Global Stability with Time-Delay of a Primal-Dual Congestion Control

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Abstract

This paper concerns network congestion control, focusing on a recently proposed "primal-dual" control law that has been shown to give local stability and desirable equilibrium properties for networks of arbitrary capacity and a fixed bound on delay. In this paper we study global stability for these non-linear, delay differential equations, focusing for simplicity on the scenario of a single flow and bottleneck link. We first provide bounds on the state variables that hold for any choice of parameters. Then we give results regarding global stability, based on Lyapunov-Razumikhin methods and singular perturbation theory.

1 Introduction

In recent years large strides have been taken in analytical approaches to network congestion control. These methods, originating in the academic community [8, 16, 13, 9, 2, 10, 11] are quickly evolving into viable alternatives to correct deficiencies of current Internet protocols [1, 18]. Using a framework proposed in [8, 16] based on fluid-flow models and utility functions, tools from convex optimization and control theory have come into play to propose new control laws that achieve equilibrium objectives of efficient network utilization, fair resource allocation and low delay, and that can be proven to converge to these equilibrium points. Control can be included in a decentralized way either at network sources or at links; following [8], it has become customary to denote by "primal" control laws those that contain dynamics at sources, but static functions at links, and "dual" laws those where the opposite holds. In this vein, the case where both control laws are dynamic is named "primal-dual".

If one neglects network delay, global stability results for various families of control laws can be established in very general networks; indeed, [8, 3] show Lyapunov proofs for both "primal" and "dual" type laws. "Primal-dual" laws are more difficult to analyze (see [14] for a time-scale analysis), but some cases can be treated by passivity theory methods, as shown recently in [17]. When feedback delay is included in the problem (an essential feature in large-scale networks) the designs become harder, especially if one wants to achieve stability by a decentralized control in a *scalable* way over arbitrary networks. The most successful approach has been to start from linearized designs in which such scalable results can be obtained [2, 10, 5], and build from these linearizations the global, nonlinear control laws.

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The question that remains after this is whether the resulting control laws are *globally* stabilizing. This question, involving both nonlinearity and delay is significantly more difficult, so one is forced to restrict the attention to simpler networks, for instance including a single bottleneck; some recent contributions are [12, 15, 4].

In this paper we follow a line of work initiated in [2], in which we developed a "dual" congestion control system that achieves high utilization, low delay and local scalable stability. We addressed the global stability aspects for the single bottleneck case in [4]. Further work in [5], strongly driven by our efforts to deploy these protocols [1], has convinced us of the need to include a "primal" component to these laws, that allows us to control in a fair way the resource allocation in equilibrium. To guarantee local stability in these "primal-dual" laws, in [5] we employ a separation of time-scales argument, running the "fairness" loop slower than a commonly agreed bound on the delay. This background is briefly reviewed in Section 2.

This paper tackles the global stability aspects of this new "primal-dual" system, extending the work in [4]. We focus on the case of a single link and source. First, in Section 3, we show that the trajectories of this system are bounded; moreover, we can state absolute bounds in which the variables must lie after an initial transient, and these hold independently of whether the system is locally stable or not. Armed with these bounds, in Section 4 we proceed to give global stability results, based on a singular perturbation method. Indeed, the fast "boundary-layer" dynamics in this system corresponds to the "dual" laws already studied in [4]; supplementing this with a study of the slow dynamics, a global result can be given. We also present some simulation results, which demonstrate this time-scale analysis. Conclusions are given in Section 5.

2 "Dual" and "Primal-Dual" Controls

In this section we briefly review the congestion control problem, presenting the "dual" control of [2] its local scalable stability and the global stability analysis in [4]; then we describe the "primal-dual" control of [5] and its local stability.

We are concerned with a system of L communication links shared by a set of S sources. The routing matrix R, of dimensions $L \times S$, is defined by

$$R_{li} = \begin{cases} 1 & \text{if source } i \text{ uses link } l \\ 0 & \text{otherwise} \end{cases}.$$

Each source i has an associated transmission rate $x_i(t)$; the set of transmission rates determines the aggregate flow $y_l(t)$ at each link, by the equation

$$y_l(t) = \sum_i R_{li} x_i(t - \tau_{li}^f), \tag{1}$$

in which the forward transmission delays τ_{li}^f between sources and links are accounted for. We assume each link has a capacity c_l in packets per second.

The next step is to model the feedback mechanism which communicates to sources the congestion information about the network. We associate with each link l a congestion measure or *price* $p_l(t)$ [8, 16] and assume sources have access to the *aggregate* price of all links in their route,

$$q_i(t) = \sum_l R_{li} p_l(t - \tau_{li}^b). \tag{2}$$

Here again we allow for backward delays τ_{li}^b in the feedback path from links to sources. The total round-trip-time (RTT) for the source thus satisfies

$$\tau_i = \tau_{li}^f + \tau_{li}^b$$

for every link in the source's path. This quantity can be estimated by sources in real time.

A "dual" control is developed in [2] with dynamic in the link and static function in the source. In the link side, the "price" is controlled as

$$\dot{p}_l = \left(\frac{y_l - c_{0l}}{c_{0l}} \right)_{p_l}^+, \tag{3}$$

working with a *virtual* capacity $c_{0l} < c_l$, by which the network could have almost full utilization and empty queue at equilibrium. Equation (3) uses the "positive-projection" ' notation [17], namely

$$(f(x))_x^+ = \begin{cases} f(x), \text{if } x > 0 \text{ or } x = 0 \text{ and } f(x) > 0; \\ 0 \qquad \text{if } x = 0 \text{ and } f(x) < 0, \end{cases}$$

A static control law is applied in the sources

$$x_i = x_{\max,i}\, e^{-\frac{\alpha_i q_i}{M_i \tau_i}}. \tag{4}$$

Here M_i is the number of the bottleneck links the flow from this source goes through, τ_i is the RTT, and $x_{\max,i}$ is a maximum rate parameter, which can vary for each source, and can also depend on M_i, τ_i (but not on q_i). The following local stability result holds, where $\bar{R}$ denotes the reduced routing matrix obtained by eliminating non-bottleneck links.

Theorem 1 ([2]). *Suppose the matrix $\bar{R}$ is of full row rank, and that $\alpha_i < \frac{\pi}{2}$. Then the "dual" link and source control laws (1-4) give a locally stable system for arbitrary delays and link capacities.*

In [4], the global stability of the "dual" is analyzed through small-gain theory and Razuminhin's theorm. One result shows that a system with single bottleneck link is globally stable, under somewhat tighter restrictions on α.

Theorem 2 ([4]). *For a given "dual" system with single bottleneck link and heterogenous flows, there exists a positive constant Θ such that for $\alpha < \Theta$, the origin of the system is globally asymptotically stable.*

In the single-flow case, a condition of the global stability on α is given as

$$\alpha < \frac{c_0^2}{x_{max}^2}. \tag{5}$$

In order to obtain the very general scalable stability in the "dual" control, some restrictions apply to the sources' demand curves (or their utility functions). To leave the utility functions completely up to the sources, new *dynamics* are added at the sources. Such a

"primal-dual" solution is developed in [5], by taking the same link control (3) as that of the "dual", which maintains high utilization and small equlibrium queues, and a new source control as

$$\tau_i \dot{\xi}_i = \beta_i (U_i'(x_i) - q_i), \tag{6}$$

$$x_i = x_{m,i} e^{(\xi_i - \frac{\alpha_i q_i}{M_i \tau_i})}. \tag{7}$$

Here $U(x)$ is a strictly concave increasing utility function which leads to "elastic" flows [8]. At equilibrium, $q_i^* = U_i^{'}(x_i^*)$, maximizing the local benefit $U_i(x_i) - q_i x_i$. When the parameter β_i is small, the system works in two different time scales. In the fast time scale when ξ varies slowly, the system works like the dual to track the utilization. And the slow time scale is applied to track the resource allocation. A local stability theorem is given in [5].

Theorem 3 ([5]). *Assume that for every source i, $\tau_i \leq \bar{\tau}$. With the rank assumptions of Theorem 1, the "primal-dual" control, with $\alpha_i \leq \alpha < \frac{\pi}{2}$ and $\beta = \eta \frac{\alpha_i}{M_i \bar{\tau}}$, the closed loop is linearly stable for a small enough $\eta \in (0, 1)$ depending only on α.*

Simulations show that such control laws seem also globally stabilizing under the same conditions as the local stability, but this is not easy to establish analytically. In what follows, we present partial results for the case of a single link and flow. In this case, the system equations can be simplified to the following:

$$\tau \dot{\xi} = \beta (U'(x) - q), \tag{8}$$

$$x = x_m e^{(\xi - \frac{\alpha q}{\tau})}, \tag{9}$$

$$\dot{q} = \left(\frac{x(t-\tau) - c_0}{c_0} \right)_q^+. \tag{10}$$

Some properties of the variables are

$$x \geq 0, \quad U^{'}(x) > 0, \quad U''(x) < 0 \tag{11}$$

$$q \geq 0, \quad \dot{q} \geq -1. \tag{12}$$

For many (not all) of the results below, we must further restrict the utility function by the following assumption.

Assumption 1. $U^{'}(x) \leq U_{max}^{'}$.

3 Eventually Bounded States

In this section, we will show that under very weak conditions (milder than those needed for local stability), the system variables ξ, q, x will remain bounded. Furthermore, we will be able to give a priori bounds that must hold "eventually", after a sufficiently long transient. For convenience, we introduce the following terminology:

Definition 1. *A function $f(t)$ is "eventually bounded" by $\bar{f}$ if there exists T such that $f(t) \leq \bar{f}$ for any $t \geq T$.*

The following three propositions contain bounds on each of the variables x, ξ and q.

Proposition 4. *The rate $x(t)$ of the system (8-10) is eventually bounded by $\bar{x}$, depending only on the parameters c_0, α, τ, β and the utility function.*

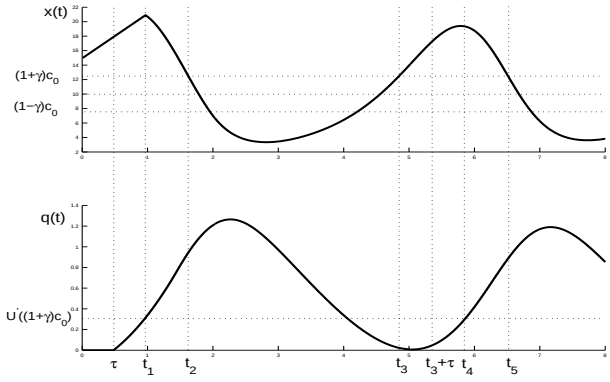

Figure 1: Upper bound of rate

Proof: A sketch of the argument is given first, details follow. The key observations we build on are:

- Consider a small $\gamma > 0$, and assume the rate $x(t)$ is remaining above $(1+\gamma)c_0$ for some time. Then we can show that after a fixed bounded time, the rate is bounded by a known decreasing exponential.
- If the rate is above $(1+\gamma)c_0$ and increasing, its rate of increase is bounded by a known exponential.

These observations imply first, that from any initial condition the rate must eventually reach $(1+\gamma)c_0$; furthermore, if it were to exceed this value again, it can only do so for a fixed time, in which we have a bound on the increase rate. From here we can derive an absolute bound that cannot be exceeded after the first time that the rate reaches $(1+\gamma)c_0$. Now we give the argument in detail. Figure 1 shows the arguments on the bound of $x(t)$.

First, for any initial value p_{t_0} with $t_0 \in [-\tau_b, 0]$, x_{t_0} with $t_0 \in [-\tau_f, 0]$, $q_0 \geq 0$ and $x(0) > c_0$, we claim that there exists $t_2 \geq 0$ such that $x(t_2) \leq (1+\gamma)c_0$. Otherwise, assume that $x(t) > (1+\gamma)c_0$ for any $t \geq 0$, then for any $t > \tau$, $\dot{q}(t) > \gamma$ from (10) , i.e., $q(t) > q(\tau) + \gamma(t-\tau)$. By some time t_1, the price reaches $U^{'}((1+\gamma)c_0)$ if $q(\tau) < U^{'}((1+\gamma)c_0)$ as in the figure, i.e.,

$$t_1 = \gamma^{-1}[U^{'}((1+\gamma)c_0 - q(\tau))]^{+} + \tau. \tag{13}$$

For any $t > t_1$, $q(t) > q(\tau) + \gamma(t-\tau) > U^{'}((\gamma+1)c_0) > U'(x(t))$, which leads to $\dot{\xi}(t) < 0$ from (8). Then, for $t > t_1$,

$$\dot{x}(t) = x(t)(\dot{\xi} - \frac{\alpha}{\tau}\dot{q}(t)) < -x(t)\frac{\alpha}{\tau}\dot{q}(t) < -\frac{\alpha}{\tau}\gamma x(t),$$

which means that $x(t)$ decreases exponentially as

$$x(t) < x(t_1)e^{-\frac{\alpha}{\tau}\gamma(t-t_1)},$$

such that $x(t) < (\gamma+1)c_0$ when $t > t_2$ by taking

$$t_2 = t_1 - \frac{\tau}{\alpha\gamma}\log(\frac{(\gamma+1)c_0}{x(t_1)}), \tag{14}$$

a contradiction. Therefore, for any value $x_0 > (1+\gamma)c_0$, after a time no greater than t_2, the rate goes under $(\gamma+1)c_0$. Furthermore, the maximum of $x(t)$ in $[0, t_2]$ lies in the time $[0, t_1]$.

When $t > t_2$, there are two possibilities. One is that $x(t)$ is kept equal or less than $[(1+\gamma)c_0]$, converging to equilibrium or oscillating. In the other case, $x(t)$ may arrive and will go above $[(1+\gamma)c_0]$, for instance, at a time t_3. Similarly as above, the rate can not stay above $(1+\gamma)c_0$ for ever. Two periods exist before the rate goes down $(1+\gamma)c_0$ again. After, at most, time $t_3+\tau$, the price $q(t)$ begins to increase with $\dot{q} > \gamma$. If $q(t_3+\tau) < U^{'}((1+\gamma)c_0$, then the price reaches the point $U^{'}((1+\gamma)c_0$ by some time t_4, such that $\dot{\xi} \leq 0$, and the rate begins to decrease exponentially similarly as in the first step. By some time t_5, the rate goes down $(1+\gamma)c_0$ again. The only difference from the first step is that $x(t_3) = (1+\gamma)c_0$ instead of any other value, by which we can find the upper bound of x.

We first show that the time period $[t_3, t_4]$, when the rate arrives its maximum point, is upper bounded. If $q(t_3+\tau) > U^{'}((\gamma+1)c_0$, $t_4 = t_3+\tau$. Therefore, from (13),

$$t_4 \leq \gamma^{-1}[U^{'}((\gamma+1)c_0) - q((t_3+\tau))]^{+} + \tau. \tag{15}$$

Since

$$q(t_3+\tau) \geq 0$$

we have a upper bound of $t_4 - t_3$ as

$$\Delta t_I = \gamma^{-1}U^{'}((\gamma+1)c_0) + \tau. \tag{16}$$

Next, noticing that the rate $x(t) > (\gamma+1)c_0$ during the time $[t_3, t_4]$, we have, from (8-10)

$$\dot{x} = x(\beta(U^{'}(x)) - q - \frac{\alpha}{\tau}\dot{q}) \tag{17}$$

$$\leq x(\beta U^{'}((\gamma+1)c_0) + \frac{\alpha}{\tau}), \tag{18}$$

since

$$q > 0 \quad \text{and} \quad \dot{q} \geq -1,$$

and $U^{'}(x)$ is decreasing with x. So the maximum $x(t)$ is bounded.

The same arguments can be applied once the rate goes above $(1+\gamma)c_0$ again. Therefore, the rate $x(t)$ is eventually bounded, i.e., after the first time it goes down $(1+\gamma)c_0$ at the time

$$t_x = t_1, \tag{19}$$

the rate is bounded by an absolute value

$$\bar{x} = (1+\gamma)c_0 e^{\left((\beta U^{'}((1+\gamma)c_0)+\frac{\alpha}{\tau})(\gamma^{-1}U^{'}((1+\gamma)c_0)+\tau)\right)}, \tag{20}$$

which depends on the parameters but without any restriction on them such as α and β.

■

We remark that, in the proof of the last proposition, it's not required that ξ and q are bounded, or acts just as shown in the figure. Furthermore, the bound $\bar{x}$ is an increasing function with respect to both α and β. Next we argue that the state $\xi(t)$ is actually bounded based on the bound $\bar{\xi}$ and Assumption 1.

Proposition 5. *The state $\xi(t)$ of the system (8-10) is eventually bounded by $\bar{\xi}$ after applying Assumption 1 and bound of rate $\bar{x}$ in (20) without restriction on the parameters.*

Proof: We consider the dynamics beginning from the time t_x in (19), after which we can bound x by $\bar{x}$ in (20). Then, by applying the dynamics (8-9) and Assumption 1, we have

$$\dot{\xi} + \beta q = \beta U^{'}(x) \leq \beta U^{'}_{max}; \tag{21}$$

$$\xi - \frac{\alpha}{\tau} q = \log(\frac{x}{x_m}) \leq \log \frac{\bar{x}}{x_m}. \tag{22}$$

Define constants

$$C_1 = \beta(U^{'}_{max} + \frac{\tau}{\alpha} \log(\frac{\bar{x}}{x_m}) \tag{23}$$

$$C_2 = -\beta \frac{\tau}{\alpha} < 0, \tag{24}$$

then we have, with the initial state ξ_{t_x} at time t_x,

$$\dot{\xi} \leq C_1 + C_2 \xi \tag{25}$$

$$= C_1 + C_2 \xi_{t_x} + C_2 \int_{t_x}^{t} \dot{\xi}(\sigma) d\sigma. \tag{26}$$

By applying the Gronwall-Bellman Inequality [7], we have,

$$\dot{\xi} \leq (C_1 + C_2 \xi_{t_x}) e^{C_2 (t - t_x)},$$

such that

$$\xi(t) \leq \frac{C_1 + C_2 \xi_{t_x}}{C_2} (e^{C_2 (t - t_x)} - 1) + \xi_{t_x} \tag{27}$$

$$= (\frac{C_1}{C_2} + \xi_{t_x}) e^{C_2 (t - t_x)} - \frac{C_1}{C_2} \tag{28}$$

$$= (\xi_{t_x} - (\log(\frac{\bar{x}}{x_m}) + \frac{\alpha}{\tau} U^{'}_{max})) e^{-\beta \frac{\tau}{\alpha} (t - t_x)} + (\log(\frac{\bar{x}}{x_m}) + \frac{\alpha}{\tau} U^{'}_{max}). \tag{29}$$

Since $\dot{\xi}$ is upper bounded, ξ cannot escape in the time $[0, t_x]$ from ξ_0. Therefore, we have one bound of ξ after time t_x, which may depend on the initial state,

$$\hat{\xi} = \max\{\xi_{t_x}, (\log(\frac{\bar{x}}{x_m}) + \frac{\alpha}{\tau} U^{'}_{max})\}. \tag{30}$$

However, if we define another bound with some positive number Δ_ξ

$$\bar{\xi} = \log(\frac{\bar{x}}{x_m}) + \frac{\alpha}{\tau} U^{'}_{max} + \Delta_\xi, \tag{31}$$

then we are sure there exists a time t_ξ

$$t_\xi = \frac{\alpha}{\beta\tau}\left(\log(|\xi_{t_x} - (\log(\frac{\bar{x}}{x_m}) + \frac{\alpha}{\tau}U'_{max})|) - \log(\Delta_{xi})\right)^+ + t_x, \tag{32}$$

after which the state ξ is eventually bounded by $\bar{\xi}$ for any initial value. Again, this bound is independent of the local stability condition.

■

With the bound of ξ, we can further argue that the state q is eventually bounded, too. We give one bound in the following proposition.

Proposition 6. *The state $q(t)$ of the system (8-10) is eventually bounded by $\bar{q}$ after applying Assumption 1 and bound $\bar{\xi}$ in (31), independent of the initial state and local stability condition.*

Proof: Since x is upper bounded, $\dot{q}$ is also upper bounded from (10) such that it cannot escape during the time $[0, t_\xi]$. So we consider the bound of q after time t_ξ. We follow similar arguments as that on the bound of x in Proposition 4.

Define a reference price $\check{q}$ such that, with some small number $0 < \gamma < 1$,

$$x_m e^{(\bar{\xi} - \frac{\alpha}{\tau}\check{q})} = (1-\gamma)c_0, \tag{33}$$

and assume that $q > \check{q}$ for some time. First we argue that it's impossible that $q > \check{q}$ for any $t > t_\xi$ with initial value $q_{t_\xi} > \check{q}$. Otherwise, q will decrease after time τ because $x(t) < (1-\gamma)c_0$ such that $\dot{q} < -\gamma$. By some time t_q, $q = \check{q}$, a contradiction. In the next time when $q = \check{q}$ and goes above it again, after *at most* τ, q will decrease until it reaches $\check{q}$. Therefore, after time t_q, since $\dot{q}$ is upper bounded because of $\bar{x}$, the price q should be bounded by

$$\bar{q} = \check{q}(1 + (\frac{\bar{x}}{c_0} - 1)\tau). \tag{34}$$

■

Finally we have the theorem on the bounds.

Theorem 7. *The rate $x(t)$, states $\xi(t)$ and $q(t)$ of the system (8-10) are all eventually bounded by some absolute values $\bar{x}$, $\bar{\xi}$ and $\bar{q}$ as those in (20), (31) and (34) respectively for any $t > T_u$ with $T_u \geq t_q$. These bounds are independent of initial states and local stability conditions, and T_u is limited for any limited initial values.*

4 Global stability with delays

In this section, we consider the global stability of the "primal-dual" control (8-10). Noticing that the principle of the design is to make the system working in different time scales to track the utilization (fast) and fairness (slow), one direct approach on its stability analysis is to explore its two-time-scale property.

At equilibria, $x^* = c_0$, $q^* = U^{'}(c_0)$, $\xi^* = \log(\frac{c_0}{x_m}) + \frac{\alpha}{\tau}U^{'}(c_0)$. Define $\delta\xi = \xi - \xi^*$, $\delta x = x - x^*$, $\delta q = q - q^*$, then we have,

$$\frac{d\delta\xi}{\beta dt} = \frac{1}{\tau}\left(U'(c_0+\delta x) - U'(c_0) - \delta q\right), \tag{35}$$

$$\delta x(t) = c_0\left(e^{\delta\xi(t)-\frac{\alpha}{\tau}\delta q(t)} - 1\right), \tag{36}$$

$$\frac{d\delta q}{dt} = \left(\frac{\delta x(t-\tau)}{c_0}\right)^+_{q_0+\delta q}. \tag{37}$$

By taking $\tilde{t} = \beta t$, we have

$$\frac{d\delta\xi}{d\tilde{t}} = \frac{1}{\tau}\left(U'(x_0+\delta x) - U'(x_0) - \delta q\right), \tag{38}$$

$$\delta x = c_0\left(e^{\delta\xi-\frac{\alpha}{\tau}\delta q} - 1\right), \tag{39}$$

$$\beta\frac{d\delta q}{d\tilde{t}} = \left(\frac{\delta x(\frac{\tilde{t}}{\beta}-\tau)}{c_0}\right)^+_{q_0+\delta q}, \tag{40}$$

which is in the form of a singular perturbation model with the origin an isolated equilibrium, fast dynamic δq and slow dynamic $\delta\xi$ when β is small.

Taking $\beta = 0$ in (39-40), we have an isolated root $\delta x = 0$, which means that

$$\delta q = \frac{\tau}{\alpha}\delta\xi. \tag{41}$$

Therefore, we have the reduced system

$$\frac{d\delta\xi}{d\tilde{t}} = -\frac{1}{\alpha}\delta\xi. \tag{42}$$

The boundary-layer system when ξ is considered as a constant,

$$\delta x = c_0\left(e^{\delta\xi-\frac{\alpha}{\tau}\delta q} - 1\right), \tag{43}$$

$$\frac{d\delta q}{dt} = \left(\frac{\delta x(t-\tau)}{c_0}\right)^+_{q_0+\delta q}, \tag{44}$$

is exactly a "dual" control with $x_{max} = x_m e^{\xi}$ in the source side as

$$x = (x_m e^{\xi})e^{-\frac{\alpha q}{\tau}}. \tag{45}$$

We first present two simulation results, in which the system shows behavior with different time scales.

4.1 Simulations by Matlab

The simulations are done by Matlab. From the system decomposition, q should be the fast dynamic, which converges much faster than the dynamic ξ to a steady state of the

boundary-layer system, i.e., $x = c_0$, for given ξ. In these so-called "quasi-steady" states, we should have

$$q = \frac{\tau}{\alpha}(\xi - \log(\frac{c_0}{x_m})). \tag{46}$$

In the simulations we take the utility function $U(x) = K\log(x)$, which leads to a proportionally fair allocation.

Consider a system (8-10) with the following parameters: $c_0 = 40, \alpha = 0.37, \tau = 0.2, x_m = 10, K = 20, \beta_0 = \alpha/(1.5625 * \tau)$. ($\beta_0$ is taken to make the loop close to the boundary of stability according to the local stability theorem.) We set the same initial states $x(0) = 0, p(0) = 0$, and differentiate those of the state ξ. The phase portraits of the state q and ξ with different values of β are shown in (Figure 2-Figure 4).

Figure 2 shows that the system is asymptotically stable for the various initial values of ξ. When the parameter β decreases, the system acts just as expected, presenting two different time scales. In Figure 4, the state p converges much faster than the state ξ at the beginning, while ξ is kept almost unchanged until they reach some equilibrium of the boundary system. Then the states go along the "line" as (46), where once ξ varies, the rates converge at once to the "quasi-steady" state until they arrive the equilibrium.

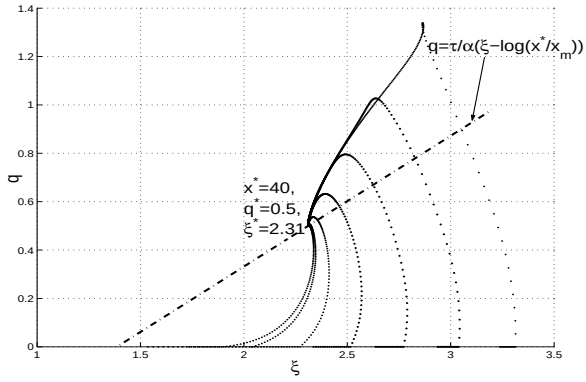

Figure 2: Two time-scales: one flow with $\beta = \beta_0$

4.2 Time-scale analysis

To get a theorem on the time-scale analysis, we consider only the dynamics after time T_u, after which the state ξ is upper bounded by $\bar{\xi}$ in (31), and the price is bounded by $\bar{q}$ in (34). We take T_u as the beginning time.

We first claim the stability of the reduced system (42).

Proposition 8. *The reduced system is exponentially stable.*

Since the boundary-layer system is exactly a "dual" control except that $x_{max} = x_m e^{\xi}$ in the source side, we have a claim as an extension of Theorem 2.

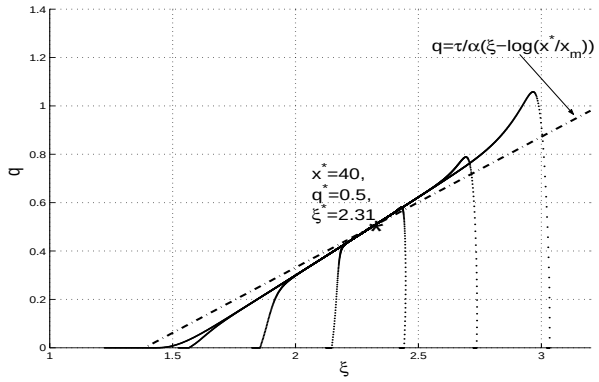

Figure 3: Two time-scales: one flow with $\beta = 0.1\beta_0$

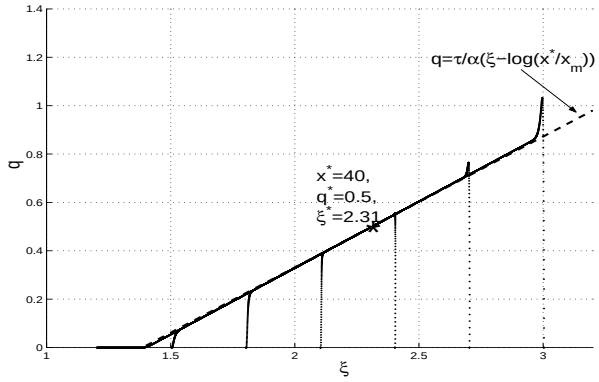

Figure 4: Two time-scales: one flow with $\beta = 0.01\beta_0$

Proposition 9. *The boundary-layer system is exponentially stable uniformly with t and ξ, by taking*

$$\alpha < \frac{c_0^2}{(x_m e^{\bar{\xi}})^2}.$$

Proof: The sketch of the proof is as follow. First, following the argument in the proof of the Razuminhin-type theorem in [4], we show that the system is globally asymptotically stable uniformly with ξ. Then, by invoking a lemma in [15], we argue that the system is also exponentially stable.

Take similar notations as those in [4], define $z(t) = \delta q(t)$, and

$$f(u) = e^{\frac{\alpha u}{\tau}} - 1, \tag{47}$$

then the system can be expressed as

$$\dot{z} = f(-z(t-\tau)), \tag{48}$$

with the constraint

$$z \geq -q^*,$$

where q^* is the equilibrium price as

$$q^* = \frac{\tau}{\alpha}\log(\frac{x_m e^{\xi}}{c_0}).$$

We consider the case when $x_m e^{\xi} > c_0$. First, based on the Proposition 3 in [4], the price deviation $z(t)$ will be eventually bounded after certain time, for instance t_b, i.e.,

$$z(t) \leq \Delta q \quad \text{for} \quad \forall t > t_b,$$

with

$$\Delta q = \tau(\frac{x_m e^{\xi}}{c_0} - 1).$$

Since the state q always bounded by $\bar{q}$, and ξ bounded by $\bar{\xi}$, t_b is also bounded from the argument in the Proposition 3 in [4], assumed a value T_b independent of ξ and q. Then, the nonlinearity $f(u)$ can be bounded into a sector conic since $-q^* \leq \delta q \leq \Delta q$, i.e.,

$$\gamma \leq \frac{f(u)}{u} \leq \rho, \quad \text{and} \quad |\frac{\partial f(u)}{\partial u}| \leq \zeta, \tag{49}$$

where

$$\rho = \frac{e^{\frac{\alpha q^*}{\tau}} - 1}{q^*}, \quad \gamma = \frac{1 - e^{-\frac{\alpha \Delta q}{\tau}}}{\Delta q}, \quad \zeta = \frac{\alpha}{\tau} e^{\frac{\alpha q^*}{\tau}}. \tag{50}$$

Next we apply Razuminhin's theorem on the stability of a retarded functional differential equation [6] to show the exponential stability. Consider the positive definite function

$$V(z(t)) = \frac{1}{2} z^2(t), \tag{51}$$

and the set of all $z(t)$ such that, for some $\eta > 1$,

$$z^2(t - \sigma) \leq \eta^2 z^2(t), \quad \text{for} \quad \sigma \in [0, 2\tau]. \tag{52}$$

Then, we only need to show the derivative of V is decreasing along the solution of the functional differential equations for only part of the states (52). Specifically,

$$\begin{aligned}
&\dot{V}(z(t)) = z(t) f(-z(t-\tau)) \\
&= z(t)[f(-z(t)) - (f(-z(t)) - f(-z(t-\tau)))] \\
&= z(t)\left[f(-z(t)) + \int_{t-\tau}^{t} (\frac{\partial f}{\partial z}(-z(\sigma))\dot{z}(\sigma))d\sigma\right] \\
&\leq \left[z(t) f(-z(t)) + |z(t)| \int_{t-\tau}^{t} \zeta\rho |z(\sigma - \tau)| d\sigma\right] \\
&\leq \left[\frac{f(-z(t))}{z(t)} + \eta\rho\zeta\tau\right] z^2(t) \\
&\leq \left[-\gamma + \eta\rho\zeta\tau\right] z^2(t) \\
&\leq -\mu z^2(t),
\end{aligned}$$

for some $\mu > 0$, provided that,

$$\mu \leq \gamma - \eta\rho\zeta\tau. \tag{53}$$

If such conditions can be imposed, Theorem 4.2 [P152, [6]] implies that the origin of z, is globally asymptotically stable.

Taking the value of α as in the proposition, with the assumption that $x_m e^{\xi} > c_0$, we have,

$$\alpha < \frac{c_0^2}{(x_m e^{\xi})^2} < \frac{\log \frac{x_m e^{\xi}}{c_0}}{\frac{x_m e^{\xi}}{c_0} - 1},$$

such that

$$\Delta q < \frac{\tau}{\alpha} \log \frac{x_m e^{\xi}}{c_0} = q^*.$$

So we can take an *relaxed* upper bound of the state as

$$\Delta q = q^* = \frac{\tau}{\alpha} \log \left(\frac{x_m e^{\xi}}{c_0} \right).$$

Therefore we only require that

$$\begin{aligned}
\mu &\leq \alpha \frac{1 - \frac{c_0}{x_m e^{\xi}}}{\tau \log \left(\frac{x_m e^{\xi}}{c_0} \right)} - \eta \alpha^2 \frac{\frac{x_m e^{\xi}}{c_0} - 1}{\tau \log \left(\frac{x_m e^{\xi}}{c_0} \right)} \frac{x_m e^{\xi}}{c_0} \\
&= \frac{\alpha(1 - \frac{c_0}{x_m e^{\xi}})}{\tau \log \left(\frac{x_m e^{\xi}}{c_0} \right)} \left(1 - \eta \alpha \left(\frac{x_m e^{\xi}}{c_0} \right)^2 \right) \\
&= \frac{1}{\varsigma} \frac{c_0^2}{(x_m e^{\bar{\xi}})^2} \frac{(1 - \frac{c_0}{x_m e^{\xi}})}{\tau \log \left(\frac{x_m e^{\xi}}{c_0} \right)} \left(1 - \eta \frac{1}{\varsigma} \frac{c_0^2}{(x_m e^{\bar{\xi}})^2} \left(\frac{x_m e^{\xi}}{c_0} \right)^2 \right) \\
&= \phi(\xi).
\end{aligned}$$

We can take η such that $\eta < \varsigma$ and $\mu > 0$. Noticing $\phi(\xi)$ is a decreasing function of ξ, we can take

$$\mu = \phi(\bar{\xi}), \tag{54}$$

i.e., the value of ϕ when ξ is taken its eventual bound. Therefore we have a common η and μ such that the system is globally asymptotically for all $\xi \leq \bar{\xi}$.

To finish the proof, we apply the Lemma 2.1 in [15], which claims that such a system is also exponentially stable, i.e., taking

$$y(t) = \sup_{t - 2\tau \leq \sigma \leq t} \frac{1}{2} z^2(\sigma),$$

then it's argued in [15] that

$$y(t) \leq \eta y(t_0) e^{-\nu(t - t_0)},$$

where

$$\nu = \frac{\mu \log(\eta)}{\log(\eta) + 2\tau\mu}.$$

Here we consider the dynamics beginning from the time $t_0 = T_b$ to apply the Razuminhin's theorem. Recalling that $z(t)$ is upper bounded by q^* after this time, we could take $y(t_0) =$

$\frac{1}{2}q^{*2}$. However, we need to count the period from the beginning to T_b. Noticing that the price is bounded by $\bar{q}$, and this period of time is bounded by T_b, one relaxed exponential bound is

$$y(t) \leq \eta \frac{1}{2}\bar{q}^2 e^{-\nu(t-T_b)}, \tag{55}$$

which means that the boundary-layer system is exponentially stable, uniformly with ξ.

■

Invoking Theorem 9.3 in [7], from Proposition 8 and Proposition 9, we have the theorem

Theorem 10. *The system (8-10) is exponentially stable after time T_u by taking some value $\alpha < \frac{c_0^2}{(x_m e^{\bar{\xi}})^2}$, and small enough β.*

We remark here that one case is ignored during the analysis of the boundary-layer system. We only consider the case $x_m e^{\xi} > c_0$ there. When $x_m e^{\xi} < c_0$, the fast dynamic q will converge very fast to $q = 0$ and $x = x_m e^{\xi}$. After that, the slow dynamic ξ will increase monotonically until x goes over c_0.

Another problem is that $\bar{\xi}$ is dependent on α and β. Noticing that the bounds are increasing functions of α and β, we can get the bounds $\bar{\xi}$ and $\bar{q}$ with certain values of the parameters, for instance, $\alpha = 1$ and the value of β based on Theorem 3, then applying the $\bar{\xi}$ to the time-scale analysis to get a condition on α and β which should be more conservative than those from Theorem 3.

Until now, we prove the system (8-10) is eventually bounded, and after T_u, it's exponentially stable. Therefore, with Theorem 7 and Theorem 10, we claim the global stability.

Theorem 11. *The states of the "primal-dual" control system (8-10) are eventually bounded. Furthermore, the system is globally asymptotically stable if taking some value $\alpha < \frac{c_0^2}{(x_m e^{\bar{\xi}})^2}$, and small enough β.*

5 Conclusion

In this paper, we analyze the global stability of a scalable "primal-dual" control with a simple scenario. We find some general bounds of the states without restriction on the parameters of the system. Then we prove the control with delay is globally asymptotically stable by time-scale analysis and Lyapunov-Razuminhin approach. Simulation results are described to show that the analysis is reasonable. Some future problems exist. For instance, it's reasonable that the Assumption 1 on the utility could be generally true. Also, the bounds are very conservative. More accurate bounds may be explored.

References

[1] FAST Kernel: Background Theory and Experimental Results C. Jin, D. Wei, S. H. Low, G. Buhrmaster, J. Bunn, D. H. Choe, R. L. A. Cottrell, J. C. Doyle, H. Newman, F. Paganini, S. Ravot, S. Singh, Presented at the First International Workshop on Protocols for Fast Long-Distance Networks, February 3-4, 2003, CERN, Geneva, Switzerland. http://netlab.caltech.edu/FAST.

[2] Fernando Paganini, John C. Doyle,and Steven H. Low, Scalable laws for stable network congestion control, in *Proceedings of 2001 CDC*, Decemeber 2001.

[3] F. Paganini, "A Global Stability Result in Network Flow Control", *Systems & Control Letters* 46 (3) (2002) pp. 153-163

[4] Z. Wang, F. Paganini, Global Stability with Time Delay in Network Congestion Control, to appear in *Proceedings of 2002 CDControl*, Las Vegas, Dec., 2002.

[5] F. Paganini, Z. Wang, S. Low, J. Doyle, A new TCP/AQM for Stable Operation in Fast Networks, to appear in *IEEE Infocom* 2003.

[6] Jack K. Hale, Sjoerd M. Verduyn Lunel, Introduction to functional differential equations, Springer-Verlag, 1993.

[7] Hassan K. Khalil, Nonlinear systems, second edition, Prentice-Hall, Inc., 1996.

[8] Frank P. Kelly, Aman Maulloo, and David Tan, Rate control for communication networks: Shadow prices, proportional fairness and stability, *Journal of Operations Research Soceity*, vol. 49, no.3, pp 237-252, March 1998.

[9] R. Johari and D. Tan, End-to-end congestion control for the Internet: delays and stability, *IEEE/ACM Transactions on Networking* 9(2001) 818-832.

[10] Glenn Vinnicombe, On the stability of networks operating TCP-like congestion control, to appear on IFAC'02. http://www-control.eng.cam.ac.uk/gv/internet/index.html.

[11] Chris Hollot, Vishal Misra, Don Towsley, and Wei-Bo Gong, A control theoretic analysis of RED, in *Proceedings of IEEE Infocom*, April 2001,

[12] C.V.Hollot, Y. Chait, "Nonlinear stability analysis for a class of TCP/AQM schemes", *Proc. IEEE CDC 2001*, Dec. 2001.

[13] S. Kunniyur, R. Srikant, "End–to–end congestion control schemes: utility functions, random losses and ECN marks", *Proc. IEEE Infocom*, 2000.

[14] Srisankar Kunniyur and R. Srikant, A time-scale decomposition approach to adaptive ECN marking, in *Proceedings of IEEE infocom*, April 2001,

[15] S. Deb, R. Srikant, "Global Stability of Congested Controllers for the Internet", in *Proceeding of IEEE CDC*, 2002.

[16] Steven H. Low and David E. Lapsley, Optimization flow control, I: basic algorithm and convergence, *IEEE/ACM Transactions on Networking*, vol.7, no.6,pp861-874, December 1999.

[17] John T. Wen, Murat Arcak, A Unifying Passivity Framework for Network Flow Control, Accepted by IEEE Infocom 2003.

[18] Tom Kelly, Scalable TCP: Improving Performance in Highspeed Wide Area Networks, Submitted for publication, December 2002. http://www-lce.eng.cam.ac.uk/ ctk21/papers.