Galois connections among fuzzy groups, similarities and distances.

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Abstract

Given a set S, we define Galois connections between the lattice of fuzzy subgroups of transformations in S, the lattice of the similarities in S and the lattice of the distances in S.

1 Introduction

Many authors, such as Chakraborty and Das ([2]), Valverde and Jacas ([5],[11]), Ovchinikov ([8]) have widely studied an interesting class of fuzzy relations: the similarities. Another basic notion is the one of fuzzy subgroup proposed by Rosenfeld in [9]. In [3] it was showed that, given a set S, any fuzzy subgroup of transformations in S is associated with a similarity on S and any similarity on S is associated with a fuzzy subgroup of transformations in S.

In this paper we show that such a link is a Galois connection between the lattice of the fuzzy subgroups of transformations in S and the lattice of similarities on S. Also, we investigate the notion of extensionality in terms of degree of invariance. More precisely, in Section 2, we give some preliminary notions. In Section 3, we observe that the natural correspondence between equivalences on a set S and subgroups of transformations in S is a Galois connection. In Section 4, using an extension principle for operators, we extend the previous Galois connection to a Galois connection between the lattice of the fuzzy subgroups of transformations and the lattice of similarities on S. In Sections 5 and 6 we recall the well-known relationship between the notion of similarity and that of distance (see for examples Ruspini in [10] and Valverde in [11]) to state a Galois connection between the lattice of the fuzzy subgroups and the one of the distances. Finally, in Section 7 we relate our results to the notion of extensionality given by Klawonn and Castro in [7]. For sake of simplicity, we refer to the lattice $[0,1]$, but the most of our results can be extended to any complete and completely distributive lattice.

2 Preliminaries

A *fuzzy subset* of a set S is any map s from S to $[0,1]$. We denote by $\mathcal{F}(S)$ the class of the fuzzy subsets of S. The fuzzy subsets of $S \times S$ are also called *fuzzy relations*. The *inclusion relation* is defined by setting, for any s and s' in $\mathcal{F}(S)$,

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$s \subseteq s'$ iff $s(x) \leq s'(x)$ for every $x \in S$. Denote by $\vee$ and $\wedge$ the maximum and the minimum operator in $[0,1]$ and, by $\neg$ the map defined by setting $\neg(x) = 1 - x$. Then, the *union* $s \cup s'$ and the *intersection* $s \cap s'$ of s and s' are defined by setting, for every $x \in S$, $(s \cup s')(x) = s(x) \vee s'(x)$ and $(s \cap s')(x) = s(x) \wedge s'(x)$. Likewise, if $(s_i)_{i \in I}$ is a family of fuzzy subsets then we define $\bigcap_{i \in I} s_i$ and $\bigcap_{i \in I} s_i$ by setting $\bigcup_{i \in I} s_i(x) = \bigvee_{i \in I} s_i(x)$ and $\bigcap_{i \in I} s_i(x) = \bigwedge_{i \in I} s_i(x)$. Finally, we define the *complement* $\sim s$ of s by setting $(\sim s)(x) = \neg s(x)$. In this way, $\mathcal{F}(S)$ becomes an algebraic structure $(\mathcal{F}(S), \cup, \cap, \sim)$ which is a complete lattice with an involution. As a matter of fact, such a structure is the direct power of the structure $([0,1], \vee, \wedge, \neg)$ with index set S. We say that a fuzzy subset s is "crisp" provided that $s(x) \in \{0,1\}$ for every $x \in S$. We can identify the class of subsets of S with the class of crisp subsets of S by identifying any subset X of S with the related characteristic function c_X. Given a fuzzy subset s of S, for every $\lambda \in [0,1]$ the subsets

$$C(s,\lambda) = \{x \in S \mid s(x) \geq \lambda\} \text{ and } O(s,\lambda) = \{x \in S \mid s(x) > \lambda\} \qquad (1)$$

are called the *closed* λ-*cut* and the *open* λ-*cut* of s, respectively.

We can interpret the connective "and" by suitable binary operations on $[0,1]$. Usually, one refers to the class of t-norms defined as follows.

Definition 1. A binary operation $*$ in $[0,1]$ is called a *triangular norm*, in brief a *t-norm*, if it is associative, commutative, non decreasing in both variables and has as unit 1.

A t-norm $*$ is called *Archimedean* if $x * x < x$ for any $x \in [0,1]$. Given a continuous triangular norm $*$, we can define the *implication* and the *equivalence* respectively as follows

$$x \to_* y = max\{z \mid x * z \leq y\} \quad , \quad x \leftrightarrow_* y = (x \to_T y) * (y \to_T x). \qquad (2)$$

As an example, if $x * y = \max\{x + y - 1, 0\}$ is the t-norm of Lukasiewicz, then the related implication is given by $\min\{1 - x + y, 1\}$ and the equivalence by $1 - |x - y|$. If $*$ is the t-norm of the minimum, then the related implication is 1 if $x \leq y$, y otherwise. The equivalence is 1 if $x = y$, $x \wedge y$ otherwise. If $*$ is the t-norm of the product, then the related implication is equal to 1 if $x \leq y$, $\frac{y}{x}$ otherwise. The equivalence is 1 if $x = y$, $\frac{x \wedge y}{x \vee y}$ otherwise. In the following $*$ always denotes a continuous t-norm.

The notion of fuzzy subgroup is defined as a many valued extension of the classical notion of subgroup (as an example see [1] and [9]).

Definition 2. Let $(G, \cdot, ^{-1}, e)$ be a group. Then a $*$-*fuzzy subgroup*, in brief a *fuzzy subgroup*, of G is a fuzzy subset $\tilde{G}$ of G such that

i) $\tilde{G}(e) = 1$,

ii) $\tilde{G}(x^{-1}) \geq \tilde{G}(x)$ for every $x \in G$,

iii) $\tilde{G}(x \cdot y) \geq \tilde{G}(x) * \tilde{G}(y)$ for every $x, y \in G$.

In the following, we call *closure system* a class $\mathcal{C}$ of fuzzy subsets of S such that the intersection of any family of elements of $\mathcal{C}$ is an element of $\mathcal{C}$. Given a fuzzy subset s, we say that $\bar{s} = \cap\{s' \in \mathcal{C} \mid s' \supseteq s\}$ is the fuzzy subset generated

by s. Any closure system is a complete lattice in which the join of a family $(s_i)_{i\in I}$ of elements is the fuzzy subset generated by the union $\bigcup_{i\in I} s_i$ and the meet is the usual intersection. The following proposition holds.

Proposition 3. *Let $F(G)$ be the class of fuzzy subgroups of G. Then $F(G)$ is a closure system and therefore a complete lattice.*

The notion of similarity or fuzzy equivalence is on the basis of fuzzy set theory (see for example [11]).

Definition 4. A *$*$-similarity,* in brief *similarity,* on a domain S is a fuzzy relation $\mathcal{R} : S \times S \to [0,1]$ such that the following properties hold

i) $\mathcal{R}(x,x) = 1$ for every $x \in S$ (reflexivity),

ii) $\mathcal{R}(x,y) \geq \mathcal{R}(y,x)$ for every $x, y \in S$ (symmetry),

iii) $\mathcal{R}(x,z) \geq \mathcal{R}(x,y) * \mathcal{R}(y,z)$ for every $x, y, z \in S$ ($*$-transitivity).

The set of similarities is ordered by the inclusion relation. Moreover we have the following proposition.

Proposition 5. *Let $E(S)$ be the class of all similarities on S. Then $E(S)$ is a closure system and therefore a complete lattice.*

If $*$ corresponds to the minimum we can characterize the just given notions in terms of cuts. Indeed, one proves that $\widetilde{G}$ is a fuzzy subgroup of a group G if and only if all its closed cuts are classical subgroups of G. Also, $\mathcal{R}$ is a similarity on a set S if and only if every closed cut $C(\mathcal{R}, \lambda)$ is an equivalence relation on S.

3 A Galois connection

If S is a set, we denote by Σ_S the group of transformations of S.

Definition 6. Let $G \subseteq \Sigma_S$ be a set of transformations. Then we define a relation $\varphi(G)$ on S by setting

$$\varphi(G) = \{(x,y) \in S \times S \mid \exists g \in G : g(x) = y\}. \tag{3}$$

Definition 7. Let $R \subseteq \mathcal{P}(S \times S)$ be a relation. Then we define a subset $\psi(R)$ of Σ_S by setting

$$\psi(R) = \{f \in \Sigma_S \mid xRf(x) \text{ for any } x \in S\}. \tag{4}$$

In the following we write also R_G to denote $\varphi(G)$ and G_R to denote $\psi(R)$.

Proposition 8. *Let $G \subseteq \Sigma_S$ be a subgroup of Σ_S. Then R_G is an equivalence relation on S. Let R be an equivalence relation on S. Then G_R is a subgroup of Σ_S.*

The following proposition shows the connection between G and $\psi(\varphi(G))$ and between R and $\varphi(\psi(R))$.

Proposition 9. *Let G be a subgroup of Σ_S and R an equivalence relation on S. Then*

$$\psi(\varphi(G)) \supseteq G \quad \textit{and} \quad \varphi(\psi(R)) = R. \tag{5}$$

Proof. We have

$$\begin{aligned} \psi(\varphi(G)) &= \{f \in \Sigma_S \mid \forall x \ (x, f(x)) \in \varphi(G)\} \\ &= \{f \in \Sigma_S \mid \forall x \ \exists g \in G : g(x) = f(x)\} \supseteq G. \end{aligned}$$

Also, given an equivalence relation R, we have

$$\begin{aligned} (x, y) &\in \varphi(\psi(R)) \Leftrightarrow \exists g \in \psi(R) : g(x) = y \\ &\Leftrightarrow \exists g : \forall z \ zRg(z) \ \textit{and} \ g(x) = y \Rightarrow xRy. \end{aligned}$$

Conversely, let xRy and let's define the following trasposition

$$< x, y > (z) = \begin{cases} x \text{ if } z = y \\ y \text{ if } z = x \\ z \text{ otherwise.} \end{cases} \tag{6}$$

Then we have that $< x, y > \in \psi(R)$ and $< x, y > (x) = y$. Thus $(x, y) \in \varphi(\psi(R))$. □

We denote by $R(S)$ the set of all equivalences on S and by $G(\Sigma_S)$ the set of all subgroups of Σ_S. It is well known that both $(R(S), \subseteq)$ and $(G(\Sigma_S), \subseteq)$ are closure systems and therefore complete lattices.

We recall the definition of Galois connection between two ordered sets.

Definition 10. Let S and S' be ordered sets. A *Galois connection* between S and S' is any pair (φ, ψ) of maps $\varphi : S \to S'$ and $\psi : S' \to S$ such that:

i) φ and ψ are both order preserving,

ii) $\psi(\varphi(a)) \geq a \quad \forall a \in S$ and $\varphi(\psi(a')) \leq a' \quad \forall a' \in S'$.

We say that (φ, ψ) is *surjective* if φ is surjective. Observe also that a Galois connection is surjective iff $\varphi\psi$ is the identity mapping. The following theorem shows that the correspondence between subgroups and equivalences is a Galois connection.

Theorem 11. *The mappings $\varphi : G(\Sigma_S) \to R(S)$ and $\psi : R(S) \to G(\Sigma_S)$ define a surjective Galois connection between the complete lattices $(G(\Sigma_S), \subseteq)$ and $(R(S), \subseteq)$.*

Proof. It is immediate to prove that ψ and φ are both order preserving. Then the proof follows from Proposition 9. □

4 Extension principle and Galois connection

We recall the definition and some properties of an extension principle for operators given in [4].

Definition 12. Let $S_1, ..., S_n, S$ be sets and $J : \mathcal{P}(S_1) \times ... \times \mathcal{P}(S_n) \to \mathcal{P}(S)$ an operator. Then we define the fuzzy operator $J^* : \mathcal{F}(S_1) \times ... \times \mathcal{F}(S_n) \to \mathcal{F}(S)$, called the *canonical extension* of J, by setting for any $s_1 \in \mathcal{F}(S_1)$, ..., $s_n \in \mathcal{F}(S_n)$ and $x \in S$

$$J^*(s_1, ..., s_n)(x) = \bigvee \{\lambda \in [0,1] \mid x \in J(C(s_1, \lambda), ..., C(s_n, \lambda))\}. \tag{7}$$

The following proposition is immediate.

Proposition 13. *If $J : \mathcal{P}(S_1) \times ... \times \mathcal{P}(S_n) \to \mathcal{P}(S)$ is an operator, then*

$$J \text{ is order preserving} \Leftrightarrow J^* \text{ is order preserving.}$$

Now the following theorem can be given.

Theorem 14. *Let $\psi : \mathcal{P}(S \times S) \to \mathcal{P}(\Sigma_S)$ be defined as in (4) and let $\psi^* : \mathcal{F}(S \times S) \to \mathcal{F}(\Sigma_S)$ be the canonical extension of ψ. Then*

$$\psi^*(\mathcal{R})(f) = \bigwedge_{x \in S} \mathcal{R}(x, f(x)). \tag{8}$$

Proof. Indeed, by Definition 12, we have

$$\begin{aligned} \psi^*(\mathcal{R})(f) &= \bigvee \{\lambda \in [0,1] \mid f \in \psi(C(\mathcal{R}, \lambda))\} \\ &= \bigvee \{\lambda \in [0,1] \mid \mathcal{R}(x, f(x)) \geq \lambda \quad \forall x \in S\} \\ &= \bigwedge_{x \in S} \mathcal{R}(x, f(x)). \end{aligned}$$

□

In a geometrical interpretation, we can interpret $\mathcal{R}(x, y)$ as a valuation of the claim that x is "near" to y. Then $\psi^*(\mathcal{R})$ is the fuzzy set of transformations f such that, for any point x, the image $f(x)$ is "near" to x. In the following sometimes we write $\widetilde{G}_{\mathcal{R}}$ instead of $\psi^*(\mathcal{R})$. The following theorem was proved in [3].

Theorem 15. *Let $\mathcal{R}$ be a $*$-similarity in S. Then $\psi^*(\mathcal{R})$ is a $*$-fuzzy subgroup of Σ_S.*

Proof. $\mathcal{R}$ is reflexive so we have that $\psi^*(\mathcal{R})(i_s) = \bigwedge_{x \in S} \mathcal{R}(x, x) = 1$. Also, since $\mathcal{R}$ is symmetric,

$$\begin{aligned} \psi^*(\mathcal{R})(f) &= \bigwedge_{x \in S} \mathcal{R}(x, f(x)) = \bigwedge_{y \in S} \mathcal{R}(f^{-1}(y), f(f^{-1}(y))) \\ &= \bigwedge_{y \in S} \mathcal{R}(f^{-1}(y), y) \leq \bigwedge_{y \in S} \mathcal{R}(y, f^{-1}(y)) = \psi^*(\mathcal{R})(f^{-1}). \end{aligned}$$

Finally, since $\mathcal{R}$ is $*$-transitive,

$$\begin{aligned}
\psi^*(\mathcal{R})(f) * \psi^*(\mathcal{R})(g) &= (\bigwedge_{x\in S} \mathcal{R}(x, f(x))) * (\bigwedge_{y\in S} \mathcal{R}(y, g(y))) \\
&= \bigwedge_{x\in S} \mathcal{R}(x, f(x)) * \bigwedge_{x'\in S} \mathcal{R}(f(x'), g(f(x'))) \\
&= \bigwedge_{x,x'\in S} (\mathcal{R}(x, f(x)) * \mathcal{R}(f(x'), g(f(x')))) \\
&\leq \bigwedge_{x\in S} (\mathcal{R}(x, f(x)) * \mathcal{R}(f(x), g(f(x)))) \\
&\leq \bigwedge_{x\in S} \mathcal{R}(x, g(f(x))) = \psi^*(\mathcal{R})(g \circ f).
\end{aligned}$$

□

Theorem 16. *Let $\phi : \mathcal{P}(\Sigma_S) \to \mathcal{P}(S \times S)$ be defined as in (3) and let $\varphi^* : \mathcal{F}(\Sigma_S) \to \mathcal{F}(S \times S)$ be the canonical extension of φ. Then*

$$\varphi^*(\widetilde{G})(x, y) = \bigvee_{g\in\Sigma_S} \left\{ \widetilde{G}\ (g) \mid g(x) = y \right\} \tag{9}$$

Proof. By Definition 12, we have

$$\begin{aligned}
\varphi^*(\widetilde{G})(x, y) &= \bigvee \left\{ \lambda \in [0, 1] \mid (x, y) \in \varphi(C(\widetilde{G}, \lambda)) \right\} \\
&= \bigvee \left\{ \lambda \in [0, 1] \mid \exists g \in C(\widetilde{G}, \lambda) \ : \ g(x) = y \right\} \\
&= \bigvee_{g\in\Sigma_S} \left\{ \widetilde{G}\ (g) \mid g(x) = y \right\}.
\end{aligned}$$

□

In accordance with such a theorem, we can consider $\varphi^*(\widetilde{G})(x, y)$ as a multivalued valuation of the claim that a transformation g in $\widetilde{G}$ exists such that $g(x) = y$. In the following we write $\mathcal{R}_{\widetilde{G}}$ to denote the fuzzy relation $\varphi^*(\widetilde{G})$. The following theorem was proved in [3].

Theorem 17. *Let $\widetilde{G}$ be a $*$-fuzzy subgroup of Σ_S, then $\varphi^*(\widetilde{G})$ is a $*$-similarity in S.*

Proof. Since $\widetilde{G}\ (i_s) = 1$, it is immediate that $\varphi^*(\widetilde{G})(x, x) = 1$. Since $\widetilde{G}\ (g^{-1}) \geq \widetilde{G}\ (g)$ for every $g \in \Sigma_S$,

$$\varphi^*(\widetilde{G})(x, y) \leq \bigvee_{g\in\Sigma_S} \left\{ \widetilde{G}\ (g^{-1}) \mid g^{-1}(y) = x \right\} = \varphi^*(\widetilde{G})(y, x)$$

for every $x, y \in S$. Finally, since $\widetilde{G}\ (h \circ g) \geq \widetilde{G}\ (g) * \widetilde{G}\ (h)$, then

$\varphi^*(\widetilde{G})(x, y) * \varphi^*(\widetilde{G})(y, z) =$
$\bigvee_{g\in\Sigma_S} \left\{ \widetilde{G}\ (g) \mid g(x) = y \right\} * \bigvee_{h\in\Sigma_S} \left\{ \widetilde{G}\ (h) \mid h(y) = z \right\} =$

$$\bigvee_{g,h\in\Sigma_S}\left\{\widetilde{G}\ (g)*\widetilde{G}\ (h)\ \mid g(x)=y\ \text{ and }\ h(y)=z\right\}\leq$$
$$\bigvee_{g,h\in\Sigma_S}\left\{\widetilde{G}\ (h\circ g)\ \mid\ h(g(x))=z\right\}=\varphi^*(\widetilde{G})(x,z).$$ □

Notice that if $J:\mathcal{P}(S_1)\times...\times\mathcal{P}(S_n)\to\mathcal{P}(S)$ is any order-preserving operator, the following inclusions hold, for every $\lambda\in[0,1]$ and every $s_i\in\mathcal{F}(S_i)$,

$$\begin{aligned}O(J^*(s_1,...,s_n),\lambda)\ &\subseteq\ J(O(s_1,\lambda),...,O(s_n,\lambda))\\ &\subseteq\ J(C(s_1,\lambda),...,C(s_n,\lambda))\subseteq C(J^*(s_1,...,s_n),\lambda).\end{aligned}$$

The following proposition shows that a stronger property holds for the canonical extensions φ^* and ψ^*.

Proposition 18. *Let $\mathcal{R}$ be a fuzzy binary relation in a set S, $\widetilde{G}$ a fuzzy subset of Σ_S and $\lambda\in[0,1]$. Then*

$$O(\varphi^*(\widetilde{G}),\lambda)=\varphi(O(\widetilde{G},\lambda))\qquad;\qquad C(\psi^*(\mathcal{R}),\lambda)=\psi(C(\mathcal{R},\lambda)).\tag{10}$$

Proof. We have that

$$\begin{aligned}O(\varphi^*(\widetilde{G}),\lambda)\ &=\ \left\{(x,y)\mid\bigvee_{g\in\Sigma_S}\left\{\widetilde{G}\ (g)\mid g(x)=y\right\}>\lambda\right\}\\ &=\ \left\{(x,y)\mid\exists g:\widetilde{G}\ (g)>\lambda\ \text{ and }\ g(x)=y\right\}=\varphi(O(\widetilde{G},\lambda)).\end{aligned}$$

Also we have that

$$\begin{aligned}C(\psi^*(\mathcal{R}),\lambda)\ &=\ \{f\ \mid\ \psi^*(\mathcal{R})(f)\geq\lambda\}=\left\{f\ \mid\ \bigwedge_{x\in S}\mathcal{R}(x,f(x))\geq\lambda\right\}\\ &=\ \{f\ \mid\ \mathcal{R}(x,f(x))\geq\lambda\ \text{ for every } x\in S\}=\psi(C(\mathcal{R},\lambda)).\end{aligned}$$

□

Observe that, if $*$ is different from the minimum and $\mathcal{R}$ is a similarity, then $C(\mathcal{R},\lambda)$ is not an equivalence relation and, also, if $\widetilde{G}$ is a fuzzy subgroup, $O(\widetilde{G},\lambda)$ is not a subgroup, in general.

Proposition 19. *For every fuzzy subgroup $\widetilde{G}$ of Σ_S,*

$$\psi^*(\varphi^*(\widetilde{G}))\supseteq\widetilde{G}\ .\tag{11}$$

For every similarity $\mathcal{R}$ in S,

$$\varphi^*(\psi^*(\mathcal{R}))=\mathcal{R}.\tag{12}$$

Moreover, both ψ^* and φ^* are order preserving.

Proof. It is immediate that both ψ^* and $\varphi^*$are order preserving. Also,

$\psi^*(\varphi^*(\widetilde{G}))(f) = \bigwedge_{x\in S} \varphi^*(\widetilde{G})(x, f(x)) = \bigwedge_{x\in S} \bigvee_{g\in\Sigma_S} \left\{\widetilde{G}\ (g) \mid g(x) = f(x)\right\}$

$= \bigvee_{g\in\Sigma_S} \bigwedge_{x\in S} \left\{\widetilde{G}\ (g) \mid g(x) = f(x)\right\} = \bigvee_{g\in\Sigma_s} \widetilde{G}\ (g) \geq \widetilde{G}\ (f).$

Furthermore,

$$\begin{aligned}
\varphi^*(\psi^*(\mathcal{R}))(x,y) &= \bigvee_{g\in\Sigma_S} \{\psi^*(\mathcal{R})(g) \mid g(x) = y\} \\
&= \bigvee_{g\in\Sigma_S} \left\{ \bigwedge_{z\in S} \mathcal{R}(z, g(z)) \mid g(x) = y \right\} \\
&\leq \bigvee_{g\in\Sigma_S} \{\mathcal{R}(x, g(x)) \mid g(x) = y\} = \mathcal{R}(x,y).
\end{aligned}$$

Conversely, let $< x, y >$ be the trasposition defined in (6). Then we have that

$$\begin{aligned}
\varphi^*(\psi^*(\mathcal{R}))(x,y) &= \bigvee_{g\in\Sigma_S} \left\{ \bigwedge_{z\in S} \mathcal{R}(z, g(z)) \mid g(x) = y \right\} \\
&\geq \bigwedge_{z\in S} \mathcal{R}(z, \bar{g}\ (z)) = \mathcal{R}(x,y).
\end{aligned}$$

□

We can express the properties of φ^* and ψ^* as follows.

Theorem 20. *The pair of maps (φ^*, ψ^*) defines a surjective Galois connection between the complete lattice of the fuzzy subgroups $(F(\Sigma_S), \subseteq)$ and the complete lattice of the similarities $(E(S), \subseteq)$.*

5 Pseudometrics, similarities and fuzzy subgroups.

We call *extended pseudometric* on S any map $d : S \times S \to [0, +\infty]$ such that

$$d(x,x) = 0,\ d(x,y) = d(y,x),\ \ d(x,y) \leq d(x,z) + d(z,y)$$

for any $x, y, z \in S$,where we assume that $x + (+\infty) = (+\infty) + x = +\infty$ for any $x \in [0, +\infty]$.

Proposition 21. *The class $\mathcal{C}$ of all extended pseudometrics defined on S is closed with respect to the join operator. Consequently, $\mathcal{C}$ is a complete lattice.*

Proof. Let $(d_i)_{i\in I}$ be a family of extended pseudometrics. We claim that $d(x,y) = \bigvee_{i\in I} d_i(x,y)$ is an extended pseudometric. Obviously, $d(x,x) = 0$ and $d(x,y) = d(y,x)$. Also, for every $i \in I$

$$d_i(x,y) \leq d_i(x,z) + d_i(z,y) \leq \bigvee_{i\in I} d_i(x,z) + \bigvee_{i\in I} d_i(z,y).$$

Consequently, $\bigvee_{i\in I} d_i(x,y) \leq \bigvee_{i\in I} d_i(x,z) + \bigvee_{i\in I} d_i(z,y)$. □

In the following, given an injective map $f : [0,1] \to [0,+\infty]$ we denote by $f^{[-1]}$ the *pseudoinverse* of f, i.e.

$$f^{[-1]}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f([0,1]) \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

We call *additive generator*, in brief *generator*, any continuous strictly decreasing function $f : [0,1] \to [0,+\infty]$ such that $f(1) = 0$. These notions enable us to recall the basic representation theorem for continous Archimedean t-norms (see for example [6]). Observe that if f is a generator, $f([0,1])$ is the interval $[0, f(0)]$. So, $f^{[-1]}(x) = f^{-1}(x)$ if $x \leq f(0)$ and $f^{[-1]}(x) = 0$ otherwise. In particular, $f^{[-1]}(f(x)) = x$ for any $x \in [0,1]$ and $f(\ f^{[-1]}(x)) = x$ if $x \leq f(0)$ and $f(f^{[-1]}(x)) = f(0)$ otherwise.

Theorem 22. *Let $*$ be a function from $[0,1]^2$ into $[0,1]$. Then $*$ is a continuous Archimedean t-norm if and only if there exists a generator such that for all x, y in $[0,1]$*

$$x * y = f^{[-1]}(f(x) + f(y)). \tag{14}$$

The following duality between pseudometrics and similarities has been established by Valverde in [11] in the case of continuous Archimedean t-norms.

Theorem 23. *Let f be a generator of a continuous, Archimedean t-norm $*$. Then, for every extended pseudometric d, the fuzzy relation $g(d) : S \times S \to [0,1]$ defined by setting*

$$g(d)(x,y) = f^{[-1]}(d(x,y)) \tag{15}$$

is a $*$-similarity.

The converse also holds.

Theorem 24. *Let f be a generator of a continuous Archimedean t-norm $*$ and $\mathcal{R}$ a $*$-similarity. Then the function $h(\mathcal{R}) : S \times S \to [0,1]$ defined by setting*

$$h(\mathcal{R})(x,y) = f(\mathcal{R}(x,y)) \tag{16}$$

is an extended pseudometric.

We will write also $\mathcal{R}_d$ to denote $g(d)$ and $d_{\mathcal{R}}$ to denote $h(\mathcal{R})$. For example, if d is an extended pseudometric and $f(x) = -\ln(x)$, then the t-norm $*$ generated by f is the usual product and a similarity $\mathcal{R}_d$ is obtained by setting $\mathcal{R}_d(x,y) = e^{-d(x,y)}$ for any $x, y \in S$. Conversely, if $\mathcal{R}$ is a similarity with respect to the product and we set $d_{\mathcal{R}}(x,y) = -\ln(\mathcal{R}(x,y))$, then $d_{\mathcal{R}}$ is an extended pseudometric. Furthermore, if $d : S \times S \to [0,1]$ is a pseudometric and $f(x) = 1 - x$, then the t-norm $*$ generated by f is the t-norm of Lukasiewicz and a similarity $\mathcal{R}_d$ is obtained by setting $\mathcal{R}_d(x,y) = 1 - d(x,y)$ for any $x, y \in S$. Conversely, if $\mathcal{R}$ is a similarity with respect to the t-norm of Lukasiewicz and we set $d_{\mathcal{R}}(x,y) = 1 - \mathcal{R}(x,y)$, then $d_{\mathcal{R}}$ is a pseudometric.

Theorem 25. *Let f be a generator of a strict continuous and Archimedean t-norm $*$. Then, for every extended pseudometric d on S, the function $\beta(d) : \Sigma_s \to [0,1]$ defined by setting, for any $g \in \Sigma_S$*

$$\beta(d)(g) = f^{[-1]}(\bigvee_{x \in S} d(x, g(x))) \tag{17}$$

is a $*$-fuzzy subgroup of Σ_s.

Proof. Let $\mathcal{R}_d(x,y) = f^{[-1]}(d(x,y))$. By Theorem 23, $\mathcal{R}_d$ is a $*$-similarity. By Theorem 17 we have that the fuzzy subset $\beta(d)$ such that for any $g \in \Sigma_s$

$$\beta(d)(g) = \bigwedge_{x \in S} \mathcal{R}_d(x, g(x)) = \bigwedge_{x \in S} f^{[-1]}(d(x, g(x))) = f^{[-1]}(\bigvee_{x \in S} d(x, g(x)))$$

is a $*$-fuzzy subgroup. □

Notice that if d is an extended pseudometric in S, then by setting, for any $f, g \in \Sigma_S$

$$d(f,g) = \bigvee_{x \in S} d(f(x), g(x)), \tag{18}$$

we obtain an extended pseudometric in the space Σ_S. Then, by Theorem 23, $g(d) : \Sigma_S \times \Sigma_S \to [0,1]$ is a similarity in Σ_S. This means that we can interpret the fuzzy subgroup $\beta(d)$ as the fuzzy subset of transformations similar to the identity map $i_s : S \to S$.

Theorem 26. *Let f be a generator of a strict continuous and Archimedean t-norm $*$ and $\tilde{G}$ a $*$-fuzzy subgroup of Σ_S. Then the function $\alpha(\tilde{G}) : S \times S \to [0, +\infty]$ defined by setting*

$$\alpha(\tilde{G})(x,y) = f\left(\bigvee_{g \in \Sigma_S} \{\tilde{G}(g) \mid g(x) = y\}\right) \tag{19}$$

is an extended pseudometric on S.

Proof. By Theorem 17,

$$\mathcal{R}_{\tilde{G}}(x,y) = \bigvee_{g \in \Sigma_S} \{\tilde{G}(g) \mid g(x) = y\} \tag{20}$$

is a $*$-similarity. Since $*$ is a continuous and Archimedean t-norm, by Valverde's theorem, $f(\mathcal{R}_{\tilde{G}}(x,y))$ is a pseudometric on S. □

In the following we write also $d_{\tilde{G}}$ to denote $\alpha(\tilde{G})$ and $\tilde{G}_d$ to denote $\beta(d)$. As an example, if d is the usual Euclidean distance and $f(x) = -\ln x$, then

$$\tilde{G}_d(g) = e^{-\bigvee_{x \in S} d(x, g(x))} = \bigwedge_{x \in S} e^{-d(x, g(x))}$$

defines a fuzzy subgroup with respect to the t-norm of product. For such a fuzzy subgroup, if g is a translation of length h, then $\tilde{G}_d(g) = e^{-h}$, if g is a rotation

$\tilde{G}_d(g) = e^{-\infty} = 0$. Conversely, if $\tilde{G}$ is a $*$-fuzzy subgroup with respect to the t-norm of the product, then the function d defined by setting

$$d_{\tilde{G}}(x,y) = -\ln(\bigvee_{g\in\Sigma_S}\{\tilde{G}(g) \mid g(x) = y\}) \tag{21}$$

for any $x, y \in S$, is an extended pseudometric.

Likewise, if d is a bounded pseudometric and $f(x) = 1 - x$, then

$$\tilde{G}_d(g) = 1 - \bigvee_{x\in S} d(x, g(x)) \tag{22}$$

is a fuzzy subgroup with respect to the t-norm of Lukasiewicz. Conversely, if $*$ is the t-norm of Lukasiewicz and $\tilde{G}$ is a $*$-fuzzy subgroup, then by setting

$$d_{\tilde{G}}(x,y) = 1 - (\bigvee_{g\in\Sigma_S}\{\tilde{G}(g) \mid g(x) = y\}) \tag{23}$$

we obtain a pseudometric.

In the following we denote by $D(S)$ the dual of $\mathcal{C}$, i.e. the complete lattice defined by considering the relation $\leq^d$ defined by setting, for every $d_1, d_2 \in \mathcal{C}$:

$$d_1 \leq^d d_2 \Leftrightarrow d_1 \geq d_2. \tag{24}$$

Also, given $\lambda \in [0, +\infty]$, we denote by $D(S)_\lambda$ the sublattice of $D(S)$ defined by the class of extended pseudometrics bounded by λ.

Theorem 27. *Let f be a generator of a continuous, Archimedean t-norm $*$, denote by $F(\Sigma_S)$ the lattice of $*$-fuzzy subgroups of Σ_S. Then the mappings $\alpha : F(\Sigma_S) \to D_{f(0)}(S)$ and $\beta : D_{f(0)}(S) \to F(\Sigma_S)$ define a surjective Galois connection between the complete lattices $F(\Sigma_S)$ and $D_{f(0)}(S)$.*

Proof. For every $\tilde{G}_i \in F(\Sigma_S)$ and $d_i \in D_{f(0)}(S)$ we have:

$$\begin{aligned}\tilde{G}_1 \subseteq \tilde{G}_2 \Rightarrow \alpha(\tilde{G}_1) \geq \alpha(\tilde{G}_2) \Rightarrow \alpha(\tilde{G}_1) \leq^d \alpha(\tilde{G}_2),\\ d_1 \leq^d d_2 \Rightarrow d_1 \geq d_2 \Rightarrow \beta(d_1) \leq \beta(d_2).\end{aligned}$$

In this way we have proved that α and β are both order preserving.

Furthermore, if i_s denotes the identity mapping,

$$\begin{aligned}\beta(\alpha(\tilde{G}))(\bar{g}) &= f^{[-1]}(\bigvee_{x\in S}\alpha(\tilde{G})(x, \bar{g}\ (x)))\\ &= f^{[-1]}(\bigvee_{x\in S}(f(\bigvee_{g\in\Sigma_S}\{\tilde{G}(g) \mid g(x) = \bar{g}\ (x)\})))\\ &= f^{[-1]}(\bigvee_{x\in S}(f(\bigvee_{g'\in\Sigma_S}\{\tilde{G}(\bar{g}\circ g') \mid g'(x) = x\})))\\ &\geq f^{[-1]}(\bigvee_{x\in S}(f(\tilde{G}(\bar{g}\circ i_s)))) = \tilde{G}(\bar{g}).\end{aligned}$$

To prove that $\alpha \circ \beta$ is the identity map, observe that

$$\begin{aligned}
\alpha(\beta(d))(\bar{x},\bar{y}) &= f(\bigvee_{g\in\Sigma_S}\left\{\beta(d)(g) \mid g(\bar{x}) = \bar{y}\right\}) \\
&= f(\bigvee_{g\in\Sigma_S}(f^{[-1]}(\bigvee_{x\in S}\left\{d(x,g(x)) \mid g(\bar{x}) = \bar{y}\right\}))) \\
&\geq f(\bigvee_{g\in\Sigma_S}(f^{[-1]}(d(\bar{x},g(\bar{x})) \mid g(\bar{x}) = \bar{y}))) \\
&= f(f^{[-1]}(d(\bar{x},\bar{y}))).
\end{aligned}$$

But also

$$\begin{aligned}
\alpha(\beta(d))(\bar{x},\bar{y}) &= f(\bigvee_{g\in\Sigma_S}(f^{[-1]}(\bigvee_{x\in S}(d(x,g(x))\mid g(\bar{x}) = \bar{y})))) \\
&\leq f(f^{[-1]}(\bigvee_{x\in S} d(x,<\bar{x},\bar{y}>(x)) \mid <\bar{x},\bar{y}>(\bar{x}) = \bar{y})) \\
&= f(f^{[-1]}(d(\bar{x},\bar{y}))).
\end{aligned}$$

In conclusion, since $d(\bar{x},\bar{y}) \leq f(0)$ we have that

$$\alpha(\beta(d))(\bar{x},\bar{y}) = f(f^{[-1]}(d(\bar{x},\bar{y}))) = d(\bar{x},\bar{y}).$$

□

6 Ultrametrics and fuzzy subgroups.

Since the minimum is not Archimedean, we cannot apply the theorems given in the previous section. Nevertheless, we can find similar results by referring to the notion of ultrapseudometric. Recall that an *ultrapseudometric* is a map $d : S \times S \to [0,1]$ such that, for any $x,y,z \in S$, $d(x,x) = 0$, $d(x,y) = d(y,x)$, $d(x,y) \leq d(x,z) \vee d(z,y)$. Any ultrapseudometric is a pseudometric, obviously.

Theorem 28. *Let $f : [0,1] \to [0,1]$ be a decreasing bijective function with $f(1) = 0$. Then if $\mathcal{R}$ is a $\wedge$-similarity on S, the equation*

$$d(x,y) = f(\mathcal{R}(x,y)) \tag{25}$$

defines an ultrapseudometric. Conversely, if $d : S \times S \to [0,1]$ is an ultrapseudometric, the equation

$$\mathcal{R}(x,y) = f^{-1}(d(x,y)) \tag{26}$$

defines a $\wedge$-similarity.

Proof. Obviously $d(x,x) = 0$ and $d(x,y) = d(y,x)$. Moreover, since f is decreasing, for every $x,y,z \in S$, from $\mathcal{R}(x,z) \geq \mathcal{R}(x,y) \wedge \mathcal{R}(y,z)$ it follows

$$f(\mathcal{R}(x,z)) \leq f(\mathcal{R}(x,y) \wedge \mathcal{R}(y,z)) = f(\mathcal{R}(x,y)) \vee f(\mathcal{R}(y,z)).$$

In a similar way the second part of the theorem can be proved. □

Observe that $\wedge$-fuzzy subgroups give rise to ultra-pseudometrics, and viceversa.

Theorem 29. *Let $\tilde{G}$ be a $\wedge$-fuzzy subgroup of Σ_S and $f : [0,1] \to [0,1]$ a decreasing bijective function with $f(1) = 0$. Then the function $\alpha'(\tilde{G}) : S \times S \to [0,1]$ defined by setting*

$$\alpha'(\tilde{G})(x,y) = f(\bigvee_{g \in \Sigma_S} \{\tilde{G}(g) | \, g(x) = y\}) \tag{27}$$

is an ultrapseudometric on S.

Proof. By Theorem 17, $\mathcal{R}_{\tilde{G}}(x,y)$ is a $*$-similarity. Since $*$ is the t-norm of the minimum, the thesis follows by Theorem 28 . □

Theorem 30. *Let $*$ be the t-norm of the minimum and $f : [0,1] \to [0,1]$ a decreasing bijective function with $f(1) = 0$. Then, for every ultrapseudometric d, the function $\beta'(d) : \Sigma_s \to [0,1]$ defined by setting, for any $g \in \Sigma_S$*

$$\beta'(d)(g) = f^{-1}(\bigvee_{x \in S} d(x, g(x))) \tag{28}$$

is a $\wedge$-fuzzy subgroup.

Proof. By Theorem 28 the relation $\mathcal{R}(x,y) = f^{-1}(d(x,y))$ is a $\wedge$-similarity and therefore, by Theorem 15, $\widetilde{G}_{\mathcal{R}}$ is a fuzzy subgroup. Moreover,

$$\widetilde{G}_{\mathcal{R}}(g) = \bigwedge_{x \in S} (\mathcal{R}(x, g(x))) = \bigwedge_{x \in S} f^{-1}(d(x, g(x))) = f^{-1}\left(\bigvee_{x \in S} d(x, g(x))\right).$$

□

If we denote by $(U(S), \leq^d)$ the complete lattice of the ultrapseudometrics defined on S, then the mappings $\alpha' : F(\Sigma_S) \to U(S)$ and $\beta' : U(S) \to F(\Sigma_S)$ define a Galois connection in a trivial way.

7 Extensionality and Similarity.

We relate our results to the interesting notion of extensionality given in [7].

Definition 31. A fuzzy set $\mu \in [0,1]^S$ is called *extensional* with respect to the $*$-similarity $\mathcal{R}$ on S if

$$\mu(x) * \mathcal{R}(x,y) \leq \mu(y) \tag{29}$$

holds for all $x, y \in S$.

Given a fuzzy subgroup $\widetilde{G}$ of Σ_S and $\mu \in [0,1]^S$, we say that μ is *extensional with respect to* $\widetilde{G}$ if is extensional with respect to $\mathcal{R}_{\widetilde{G}}$.

Theorem 32. *Let $\widetilde{G}$ be a fuzzy subgroup. Then μ is extensional with respect to $\widetilde{G}$ if and only if, for every $g \in \Sigma_S$ and $x \in S$*

$$\mu(x) * \widetilde{G}(g) \leq \mu(g(x)). \tag{30}$$

Proof. Using Definition 31 we have that, for any $x, y \in S$

$$\begin{aligned}\mu(x) * \mathcal{R}(x,y) \quad &\leq \quad \mu(y) \Leftrightarrow \mu(x) * \bigvee_{g \in \Sigma_S} \left\{ \widetilde{G}\ (g) \ \mid \ g(x) = y \right\} \leq \mu(y) \\ &\Leftrightarrow \quad \bigvee_{g \in \Sigma_S} (\mu(x) * \widetilde{G}\ (g)\ ) \leq \mu(g(x)) \Leftrightarrow \mu(x) * \widetilde{G}\ (g)\ \leq \mu(g(x))\end{aligned}$$

for every $g \in \Sigma_S$. □

In other words, the extensionality of μ with respect to $\widetilde{G}$ means that μ contains the images of its elements. One can prove that the class of extensional fuzzy subsets is a fuzzy closure system. As a consequence, given a fuzzy set μ, we may consider the smallest extensional fuzzy set containing μ.

Definition 33. Let $\mathcal{R}$ be a similarity on S and $\mu \in [0,1]^S$. Then the fuzzy set

$$\widetilde{\mu} = \bigcap \{\nu \mid \mu \subseteq \nu \text{ and } \nu \text{ is extensional w.r.t. } \mathcal{R}\} \tag{31}$$

is called the *extensional hull* of μ with respect to $\mathcal{R}$.

Given a fuzzy subgroup $\widetilde{G}$ and $\mu \in [0,1]^S$, we say that $\widetilde{\mu}$ is the *extensional hull* of μ with respect to $\widetilde{G}$ provided that $\widetilde{\mu}$ is the extensional hull of $\mathcal{R}_{\widetilde{G}}$. The following definition characterizes the extensional hull of a fuzzy subset.

Definition 34. Let μ be a fuzzy subset of S and $\widetilde{G}$ a fuzzy subgroup of Σ_S. The *extension of μ through $\widetilde{G}$* is the fuzzy subset $H_{\widetilde{G}}(\mu)$ defined by setting

$$H_{\widetilde{G}}(\mu)(y) = \bigvee_{y = g(x)} \mu(x) * \widetilde{G}\ (g). \tag{32}$$

The following proposition shows an alternative way to define $H_{\widetilde{G}}$.

Proposition 35. *Given the fuzzy subset μ and a fuzzy subgroup $\widetilde{G}$ of Σ_S, we have that*

$$H_{\widetilde{G}}(\mu)(y) = \bigvee_{f \in \Sigma_S} \mu(f(y)) * \widetilde{G}\ (f). \tag{33}$$

Proof. Indeed

$$\begin{aligned}H_{\widetilde{G}}(\mu)(y) \quad &= \quad \bigvee_{x = g^{-1}(y)} \mu(x) * \widetilde{G}\ (g) = \bigvee_{g \in \Sigma_S} \mu(g^{-1}(y)) * \widetilde{G}\ (g) \\ &= \quad \bigvee_{g \in \Sigma_S} \mu(g^{-1}(y)) * \widetilde{G}\ (g^{-1}) = \bigvee_{f \in \Sigma_S} \mu(f(y)) * \widetilde{G}\ (f).\end{aligned}$$

□

Proposition 36. *Let $\widetilde{G}$ be a fuzzy subgroup and μ a fuzzy subset. Then the extensional hull $\widetilde{\mu}$ of μ with respect to $\widetilde{G}$ coincides with the extension of μ through $\widetilde{G}$, i.e.*

$$\widetilde{\mu} = H_{\widetilde{G}}(\mu). \tag{34}$$

Proof. $H_{\widetilde{G}}(\mu)$ is extensional by Theorem 32. Since $\widetilde{G}\,(e) = 1$ and $x * 1 = x$, we have that, for every x,

$$H_{\widetilde{G}}(\mu)(x) = \bigvee_{x=g(x)} \mu(x) * \widetilde{G}\,(g) \geq \mu(x) * \widetilde{G}\,(e) = \mu(x)$$

and therefore $H_{\widetilde{G}}(\mu)(x) \geq \widetilde{\mu}\,(x)$.

Conversely, let ν be extensional w.r.t. $\mathcal{R}$ and $\mu \leq \nu$. Then we have

$\nu(y) \geq \nu(x) * \widetilde{G}\,(g) \geq \mu(x) * \widetilde{G}\,(g) \qquad \forall x \in S.$

So $\nu(y) \geq \bigvee_{y=g(x)} \mu(x) * \widetilde{G}\,(g) = H_{\widetilde{G}}(\mu)(y)$ that is $H_{\widetilde{G}}(\mu) \leq \widetilde{\mu}$. □

The following theorem given in [7] shows how a $*$-similarity can be derived from a given collection of fuzzy sets.

Theorem 37. *Let $*$ be a t-norm and $\leftrightarrow$ the related equivalence. Let $\mathcal{F}$ be a class of fuzzy subsets of S and define $\mathcal{R}_{\mathcal{F}}$ by setting*

$$\mathcal{R}_{\mathcal{F}}(x, y) = \bigwedge_{\mu \in \mathcal{F}} (\mu(x) \leftrightarrow \mu(y))\,. \tag{35}$$

Then $\mathcal{R}_{\mathcal{F}}$ is the greatest $*$-similarity on S such that all fuzzy sets in $\mathcal{F}$ are extensional with respect to $\mathcal{R}_{\mathcal{F}}$.

As we have showed before, we can associate the relation $\mathcal{R}_{\mathcal{F}}$ with a fuzzy subgroup which we denote with $\widetilde{G}_{\mathcal{F}}$. In order to obtain such a fuzzy subgroup directly from $\mathcal{F}$, we give the following definition.

Definition 38. Given a fuzzy subset $\mu \in [0,1]^S$, we call *degree of invariance* of μ with respect to the transformation f the value

$$In(f, \mu) = \bigwedge_{x \in S} (\mu(x) \leftrightarrow \mu(f(x)))\,. \tag{36}$$

If $\mathcal{F}$ denotes a class of fuzzy sets, we call degree of invariance of $\mathcal{F}$ with respect to f the value

$$In(f, \mathcal{F}) = \bigwedge_{\mu \in \mathcal{F}} In(f, \mu). \tag{37}$$

We can consider $In(f, \mathcal{F})$ as a multivalued valuation of the claim that every μ in $\mathcal{F}$ is an invariant w. r. t. f.

Theorem 39. *Let $\mathcal{F}$ be a class of fuzzy subsets and $\tilde{G}_{\mathcal{F}}$ the fuzzy subgroup associated with $\mathcal{R}_{\mathcal{F}}$. Then, for every $f \in \Sigma_S$,*

$$\tilde{G}_{\mathcal{F}}(f) = In(f, \mathcal{F}). \tag{38}$$

Proof. We have the following identities:

$$\begin{aligned} \tilde{G}_{\mathcal{F}}(f) &= \bigwedge_{x \in S} \mathcal{R}_{\mathcal{F}}(x, f(x)) = \bigwedge_{x \in S} \bigwedge_{\mu \in \mathcal{F}} (\mu(x) \leftrightarrow \mu(f(x))) \\ &= \bigwedge_{\mu \in \mathcal{F}} In(f, \mu) = In(f, \mathcal{F}). \end{aligned}$$

□

We can interpret the fuzzy subgroup $\tilde{G}_{\mathcal{F}}$ as the fuzzy group of transformations preserving the vague properties in $\mathcal{F}$.

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