THE NUMBER OF REAL ZEROS OF A CLASS OF RANDOM ALGEBRAIC POLYNOMIALS (II)

By J. E. A. DUNNAGE

[Received 26 September 1969]

1. Introduction

THE Rademacher functions $r_0(t)$, $r_1(t)$,... are defined on the interval $[0, 1]$ and take just the three values 0, -1, $+1$; and if the Lebesgue measure on $[0, 1]$ is used as a probability measure, they form a sequence of independent random variables taking the values -1, $+1$ each with probability $\frac{1}{2}$. Littlewood and Offord, in their paper (4), found estimates, above and below, for the most likely number of real zeros of the polynomial

$$f_0(x, t) = \sum_{\nu=0}^{n} r_\nu(t) x^\nu. \tag{1.1}$$

Since then, much more precise results have been found and these original estimates are now obsolete, although (4) contains a wealth of other information which remains as interesting as ever. But the methods used by Littlewood and Offord have the merit that they can be adapted to give results when the coefficients of the polynomial do not all have the same probability distribution, and a study of what happens then has been made in (1). However, when Theorem 2 of that paper, which sets a lower bound to the probable number of real zeros, is applied to $f_0(x, t)$, it does not give such a good result as (4) Theorem 2: a degree of refinement has been lost in the generalization. The reason for this is that the central limit theorem was used to calculate various probabilities, and the purpose of the present paper is to avoid this by taking a narrower (though still wide) class of distributions than in (1) and thus to produce a lower bound which does give a result as good as the original one of Littlewood and Offord for the polynomial $f_0(x, t)$.

Let $\eta_0(\omega)$, $\eta_1(\omega)$,... be identically distributed independent random variables which are symmetrically distributed about the origin and which possess moments of all orders. We shall see shortly that the requirement that $\eta_0(\omega)$, $\eta_1(\omega)$,... be identically distributed can be dropped, but its retention does allow us to simplify a little the presentation, though not the substance, of the proof. Let

$$\mathrm{E}(\eta_\nu^k) = \begin{cases} 0 & k \text{ odd} \\ \mu_k & k \text{ even.} \end{cases}$$

Quart. J. Math. Oxford (2), 21 (1970), 309–19.

We shall suppose throughout the paper that the η_ν are normalized so that

$$\mu_2 = 1, \tag{1.2}$$

and it will be convenient to write

$$\mu_{2k} = \lambda_{2k}^{2k}.$$

It is well known that $\lambda_2 \leqslant \lambda_4 \leqslant \ldots$ and therefore if $x > 0$ there is a largest integer p for which

$$\sqrt{(2p)}\lambda_{2p} \leqslant x; \tag{1.3}$$

we shall denote this integer by $I(x)$.

We shall suppose that the η_ν are defined on a probability space $(\Omega, \mathscr{F}, P)$. All random events that we have to consider will then correspond to P-measurable subsets of this space, and the probability that an event occurs or the measure of a subset of Ω will be denoted by $P(.)$.

We are going to study the polynomial

$$f(x, \omega) = \sum_{\nu=0}^{n} a_\nu \eta_\nu(\omega) x^\nu,$$

where the a_ν are any given real non-zero numbers. We write

$$\tau_n = \frac{\min|a_\nu|}{\max|a_\nu|} \quad (0 \leqslant \nu \leqslant n) \tag{1.4}$$

and

$$J_n = I\{2\tau_n \sqrt{(\log\log n)}\}. \tag{1.5}$$

The main result that we shall prove is

THEOREM 1. *There is a fixed integer n_0 and an absolute constant C such that if $n > n_0$, the probability that $f(x, \omega)$ has fewer than*

$$\{(\log\log\log n)^{-1} \exp(\tfrac{1}{2}J_n) - 14\}/(1200\mu_4^2)$$

real zeros does not exceed

$$C\mu_4^4(\log\log\log n)^{-1}\{1 + (\log\log\log n)^3 \exp(-\tfrac{1}{2}J_n)\}\exp(-\tfrac{1}{2}J_n).$$

There are two remarks that can be made conveniently at this point.

(i) Several inequalities which follow are valid when n is large enough, i.e. when n exceeds some appropriate integer. On each occasion, this appropriate integer will be absolute, independent of any parameters or probability distribution, and we shall let the largest of these integers be n_0. Hence each time we meet one of these inequalities we may say simply that it holds 'when $n > n_0$'.

We shall always use C, C_1, C_2 to denote positive absolute constants, not necessarily the same from one occurrence to another.

(ii) As already stated, it is not necessary to suppose that the η_ν all have the same probability distribution. Theorem 1 remains true if we

take μ_{2k} to be the largest moment of order $2k$ of the random variables $\eta_0, \eta_1, \ldots, \eta_n$; for it will be apparent that Lemma 2 below remains true in these circumstances, and I have already remarked in (2) that Lemma 3 does also. It is upon these two lemmas that the proof of Theorem 1 essentially depends.

Theorem 1 becomes more vivid in the special case when the η_ν are bounded. We have

THEOREM 2. *Suppose that the η_ν are independent random variables symmetrically distributed about the origin and satisfy* (i) $\mathbf{E}(\eta_\nu^2) = 1$, (ii) $P(|\eta_\nu| \leqslant \alpha) = 1$ *for a fixed* α. *If* $n > n_0$, *the probability that* $f(x, \omega)$ *has fewer than*

$$\{(\log\log\log n)^{-1}(\log n)^{\tau_n/\alpha^2} - 14\}/(1200\alpha^8)$$

real zeros does not exceed

$$C\alpha^{16}(\log\log\log n)^{-1}(\log n)^{-\tau_n^2/\alpha^2}\{1+(\log\log\log n)^3(\log n)^{-\tau_n^2/\alpha^2}\}.$$

If we want to apply Theorem 2 to the polynomial f_0 defined in (1.1) we must take $\alpha = \tau_n = 1$, when we find that the probability that $f_0(x)$ has fewer than $C_1 \log n/\log\log\log n$ real zeros is at most $C_2(\log n \log\log\log n)^{-1}$. This is what Littlewood and Offord found in (4) Theorem 2.

2. Preliminary lemmas

The lemmas grouped in this section are not specifically concerned with $f(x, \omega)$ and so have an interest apart from the use that we make of them.

LEMMA 1. *Let $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_n$ be independent events such that*

$$P(\mathscr{E}_\nu) \geqslant p \quad (\nu = 1, 2, \ldots, n),$$

where $0 < p < 1$. *The probability that at most $\frac{1}{2}pn$ of the n events $\mathscr{E}_\nu$ occur does not exceed*

$$(2/e)^{\frac{1}{2}np} < 2/(n^2p^2).$$

I am grateful to the referee for showing me the following proof. My own was much longer and gave slightly less.

Proof. Let N be the number of events that occur. Then for $0 < z < 1$ and $0 \leqslant r \leqslant n$,

$$P(N \leqslant r) \leqslant z^{-r}\mathbf{E}(z^N). \tag{2.1}$$

We next introduce random variables N_ν defined by

$$N_\nu = \begin{cases} 1 & \text{if } \mathscr{E}_\nu \text{ occurs,} \\ 0 & \text{if } \mathscr{E}_\nu \text{ does not occur.} \end{cases}$$

Since the $\mathscr{E}_\nu$ are independent, so are the N_ν, and since $N = \sum_1^n N_\nu$ it

follows that

$$\mathbf{E}(z^N) = \prod_{\nu=1}^{n} \mathbf{E}(z^{N_\nu}). \tag{2.2}$$

But $$\mathbf{E}(z^{N_\nu}) = 1-(1-z)P(\mathscr{E}_\nu) \leqslant 1-(1-z)p$$

and therefore from (2.2)

$$\mathbf{E}(z^N) \leqslant \{1-(1-z)p\}^n \leqslant \exp\{-np(1-z)\}.$$

Hence from (2.1)

$$P(N \leqslant r) \leqslant z^{-r}\exp\{-np(1-z)\},$$

and the lemma follows if we take $z = \frac{1}{2}$ and $r = \frac{1}{2}pn$.

The inequality to be proved next is similar to one found by Paley and Zygmund for infinite series of Rademacher functions. They showed [(6) Lemma 2 and (7) Lemma 19] that if $a_0, a_1, \ldots$ are real numbers such that $\sum_0^\infty a_\nu^2$ is convergent, the measure of the set of t where

$$\Big|\sum_0^\infty a_\nu r_\nu(t)\Big| \geqslant \tfrac{1}{2}\Big(\sum_0^\infty a_\nu^2\Big)^{\frac{1}{2}}$$

is not less than $\frac{1}{16}$.

LEMMA 2. *Let*

$$S_n(\omega) = \sum_{\nu=0}^{n} a_\nu \eta_\nu(\omega),$$

where the a_ν are any real numbers. If Ω_0 is the set of ω for which

$$|S_n(\omega)| \geqslant \tfrac{1}{2}\Big(\sum_{\nu=0}^{n} a_\nu^2\Big)^{\frac{1}{2}},$$

then $$P(\Omega_0) \geqslant 3/(16\mu_4).$$

Proof. If Ω_1 is the complement of Ω_0 we have

$$\int_{\Omega_1} S_n^2(\omega)\, dP \leqslant \tfrac{1}{4}\sum_{\nu=0}^{n} a_\nu^2;$$

and since $$\int_{\Omega} S_n^2(\omega)\, dP = \sum_0^n a_\nu^2$$

on account of (1.2), $$\tfrac{3}{4}\sum_0^n a_\nu^2 \leqslant \int_{\Omega_0} S_n^2(\omega)\, dP,$$

whence $$\tfrac{9}{16}\Big(\sum_0^n a_\nu^2\Big)^2 \leqslant \Big(\int_{\Omega_0} S_n^2(\omega)\, dP\Big)^2$$

$$\leqslant P(\Omega_0)\int_{\Omega} S_n^4(\omega)\, dP. \tag{2.3}$$

Now
$$\int_\Omega S_n^4(\omega)\,dP = \mathbf{E}\{(a_0\eta_0+a_1\eta_1+\ldots+a_n\eta_n)^4\}$$
$$= \sum\nolimits_1 \frac{4!}{(2\nu_0)!\,(2\nu_1)!\ldots(2\nu_n)!}\,(\lambda_{2\nu_0}a_0)^{2\nu_0}\ldots(\lambda_{2\nu_n}a_n)^{2\nu_n}, \tag{2.4}$$

where $\sum_1$ is taken over all non-negative integers ν_i such that

$$\nu_0+\nu_1+\ldots+\nu_n = 2.$$

But if $\nu_i \geqslant 0$ and $\nu_0+\nu_1+\ldots+\nu_n = m$ where m is a positive integer, it is known [(3) Hilfssatz 1] that

$$\frac{(\nu_0)!\,(\nu_1)!\ldots(\nu_n)!}{(2\nu_0)!\,(2\nu_1)!\ldots(2\nu_n)!} \leqslant \frac{1}{2^m};$$

and since also $\lambda_{2\nu} \leqslant \lambda_4$ in (2.4), we have

$$\int_\Omega S_n^4(\omega)\,dP \leqslant \frac{4!\,\lambda_4^4}{2^2(2!)}\sum\nolimits_1 \frac{2!}{(\nu_0)!\,(\nu_1)!\ldots(\nu_n)!}\,a_0^{2\nu_1}\ldots a_n^{2\nu_n}$$
$$= 3\mu_4\Big(\sum_0^n a_\nu^2\Big)^2. \tag{2.5}$$

From this and (2.3) we have

$$\tfrac{9}{16} \leqslant 3\mu_4 P(\Omega_0),$$

which completes the proof.

LEMMA 3. *If $S_n(\omega)$ is defined as in Lemma 2 and if $\Lambda > 0$,*

$$P\Big\{|S(\omega)| > \Lambda\Big(\sum_0^n a_\nu^2\Big)^{\frac{1}{2}}\Big\} < \sqrt{(2e)}\exp\{-I(\Lambda)\}.$$

Proof. This lemma is little more than a restatement of (2) Theorem 1. According to that theorem, if m is a positive integer,

$$P\Big\{|S_n(\omega)| > \sqrt{(2m)}\lambda_{2m}\Big(\sum_0^n a_\nu^2\Big)^{\frac{1}{2}}\Big\} < \sqrt{(2e)}e^{-m}.$$

If we take $m = I(\Lambda)$, then from the very definition of I given immediately after (1.3),

$$\Lambda \geqslant \sqrt{(2m)}\lambda_{2m}.$$

Hence
$$P\Big\{|S_n(\omega)| > \Lambda\Big(\sum_0^n a_\nu^2\Big)^{\frac{1}{2}}\Big\}$$
$$\leqslant P\Big\{|S_n(\omega)| > \sqrt{(2m)}\lambda_{2m}\Big(\sum_0^n a_\nu^2\Big)^{\frac{1}{2}}\Big\}$$
$$< \sqrt{(2e)}\exp\{-I(\Lambda)\}.$$

This proves the lemma.

3. Proof of the theorems

We start by proving Theorem 1. Define

$$M = [324e^2\log\log n]+2 \tag{3.1}$$

and let j be the integer uniquely determined by the inequalities

$$(2j+1)\log M \leqslant \exp(\tfrac{1}{2}J_n) < (2j+3)\log M, \tag{3.2}$$

where J_n has been defined in (1.5). It is clear from (3.1) that

$$\log M > \log\log\log n \tag{3.3}$$

and that when $n > n_0$,

$$\log M < 2\log\log\log n. \tag{3.4}$$

Now if $\exp(\frac{1}{2}J_n) \leqslant 14\log\log\log n$, Theorem 1 is trivially true and so we may suppose in what follows that

$$\exp(\tfrac{1}{2}J_n) > 14\log\log\log n,$$

i.e. that

$$\exp(\tfrac{1}{2}J_n) > 7\log M$$

if we assume that $n > n_0$ so that (3.4) can be used. In this case it follows from the left-hand inequality of (3.2) that

$$j > 3. \tag{3.5}$$

We shall study the behaviour of $f(x,\omega)$ at the j points

$$x_m = 1 - \frac{1}{M^{2m}} \quad (m = 1, 2, \ldots, j),$$

and to do this we write

$$f(x_m,\omega) = A_m(\omega) + B_m(\omega),$$

where

$$A_m(\omega) = \sum_{M^{2m-1}+1}^{M^{2m+1}} a_\nu \eta_\nu(\omega) x_m^\nu$$

and

$$B_m(\omega) = \Big(\sum_{0}^{M^{2m-1}} + \sum_{M^{2m+1}+1}^{n}\Big) a_\nu \eta_\nu(\omega) x_m^\nu.$$

We also write

$$V_m = \tfrac{1}{2}\sqrt{\mathbf{E}(A_m^2)} = \tfrac{1}{2}\Big(\sum_{M^{2m-1}+1}^{M^{2m+1}} a_\nu^2 x_m^{2\nu}\Big)^{\frac{1}{2}}.$$

Let $\mathscr{E}_m$ be the event:

$$\textit{either } A_{2m}(\omega) \geqslant V_{2m} \textit{ and } A_{2m+1}(\omega) \leqslant -V_{2m+1}$$
$$\textit{or } A_{2m}(\omega) \leqslant -V_{2m} \textit{ and } A_{2m+1}(\omega) \geqslant V_{2m+1}.$$

There are $[\frac{1}{2}j]$ events $\mathscr{E}_m$.

LEMMA 4. *There is a set Ω_a of measure at most $C\mu_4^6 j^{-2}$ such that if $n > n_0$ and $\omega \notin \Omega_a$, at least $(j-2)/(300\mu_4^2)$ of the events $\mathscr{E}_m$ occur.*

Proof. Let Ω_m^+, Ω_m^- be the sets of ω for which $A_m(\omega) \geqslant V_m$ and $A_m(\omega) \leqslant -V_m$ respectively, and let Ω_m^* be the set of ω for which $\mathscr{E}_m$ occurs. Since the random variables $\eta_\nu(\omega)$ are symmetrically distributed about the origin, so is $A_m(\omega)$ and therefore

$$P(\Omega_m^+) = P(\Omega_m^-) = \tfrac{1}{2}P(|A_m(\omega)| \geqslant V_m).$$

Hence by Lemma 2,

$$P(\Omega_m^+) \geqslant 3/(32\mu_4), \qquad P(\Omega_m^-) \geqslant 3/(32\mu_4).$$

But
$$\Omega_m^* = (\Omega_{2m}^+ \cap \Omega_{2m+1}^-) \cup (\Omega_{2m}^- \cup \Omega_{2m+1}^+); \tag{3.6}$$

and so, since the two sets whose union forms the right-hand side of (3.6) are disjoint, and since $A_m(\omega)$ with different subscripts are independent random variables,

$$P(\Omega_m^*) = P(\Omega_{2m}^+)P(\Omega_{2m+1}^-)+P(\Omega_{2m}^-)P(\Omega_{2m+1}^+)$$
$$\geqslant \frac{9}{512\mu_4^2}.$$

Since $P(\Omega_m^*)$ is the probability that $\mathscr{E}_m$ occurs, Lemma 1 shows that the probability that at most $9[\frac{1}{2}j]/(1024\mu_4^2)$ of the events $\mathscr{E}_m$ occur does not exceed $C\mu_4^6/[\frac{1}{2}j]^2$. The lemma now follows because $[\frac{1}{2}j] > \frac{1}{2}j-1$ and because $[\frac{1}{2}j]^{-2} < Cj^{-2}$ in virtue of (3.5).

LEMMA 5. *There is a set Ω_b of measure at most*

$$\sqrt{(2e)}j\exp(-J_n)$$

such that if $\omega \notin \Omega_b$ and if $n > n_0$, then

$$|B_m(\omega)| < V_m$$

simultaneously for $m = 1, 2, \ldots, j$.

Proof. Let
$$\Lambda = \tau_n(M-1)^{\frac{1}{2}}/9e;$$

then by (3.1)
$$\Lambda > 2\tau_n\sqrt{(\log\log n)},$$

so that
$$I(\Lambda) \geqslant J_n. \tag{3.7}$$

Next, let
$$\sum_1^2 = \sum_0^{M^{2m-1}} a_\nu^2 x_m^{2\nu}$$

and
$$\sum_2^2 = \sum_{M^{2m+1}+1}^{n} a_\nu^2 x_m^{2\nu}.$$

If $K = \max_{0 \leqslant \nu \leqslant n} |a_\nu|$, we can show exactly as in (**1**) 454 that if $n > n_0$,

$$\sum_1 < 2KM^{m-\frac{1}{2}}, \qquad \sum_2 < KM^{m-\frac{1}{2}} \tag{3.8}$$

and
$$V_m > 3KM^{m-\frac{1}{2}}\Lambda. \tag{3.9}$$

Now

$$P\{|B_m(\omega)| > \Lambda(\textstyle\sum_1+\sum_2)\} \leqslant P\{|B_m(\omega)| > \Lambda(\sum_1^2+\sum_2^2)^{\frac{1}{2}}\}$$
$$< \sqrt{(2e)}\exp\{-I(\Lambda)\} \quad \text{(by Lemma 3)}$$
$$\leqslant \sqrt{(2e)}\exp(-J_n)$$

by (3.7). Thus outside a set Ω' (say) of measure at most $\sqrt{(2e)}\exp(-J_n)$,

$$|B_m(\omega)| \leqslant \Lambda(\textstyle\sum_1+\sum_2)$$
$$< V_m$$

from (3.8) and (3.9). The lemma follows when we consider all the j values of m because the measure of the union of all the sets Ω' does not exceed $\sqrt{(2e)}j\exp(-J_n)$.

We complete the proof of Theorem 1 as follows. Since

$$f(x_m, \omega) = A_m(\omega)+B_m(\omega),$$

if $|A_m| > V_m$ and $|B_m| < V_m$, $f(x_m)$ has the same sign as A_m. If, therefore, the event $\mathscr{E}_m$ occurs and if also $|B_{2m}| < V_{2m}$ and $|B_{2m+1}| < V_{2m+1}$, $f(x_{2m})$ and $f(x_{2m+1})$ have opposite signs, and so $f(x)$ has at least one real zero between x_{2m} and x_{2m+1}. We can now see from Lemmas 4 and 5 that outside the exceptional set $\Omega_a \cup \Omega_b$, the number of real zeros of $f(x, \omega)$ cannot be less than $(j-2)/(300\mu_4^2)$; and from (3.4) and the right-hand inequality of (3.2), this quantity is not less than

$$\{(\log\log\log n)^{-1}\exp(\tfrac{1}{2}J_n)-14\}/(1200\mu_4^2). \tag{3.10}$$

But Lemmas 4 and 5 show that

$$\begin{aligned} P(\Omega_a \cup \Omega_b) &\leqslant P(\Omega_a)+P(\Omega_b) \\ &\leqslant C\mu_4^6\{j^{-2}+j\exp(-J_n)\}, \end{aligned} \tag{3.11}$$

remembering that (1.2) implies that $\mu_4 \geqslant 1$. From the left-hand inequality of (3.2) and from (3.3),

$$j \leqslant (\log\log\log n)^{-1}\exp(\tfrac{1}{2}J_n);$$

while from the right-hand inequality of (3.2) and from (3.5),

$$\exp(\tfrac{1}{2}J_n) < 3j\log M,$$

whence from (3.4)

$$j^{-1} < 6\log\log\log n\exp(-\tfrac{1}{2}J_n)$$

if $n > n_0$. Thus if $n > n_0$,

$$\begin{aligned} &P(\Omega_a \cup \Omega_b) \\ &< C\mu_4^6(\log\log\log n)^{-1}\{1+(\log\log\log n)^3\exp(-\tfrac{1}{2}J_n)\}\exp(-\tfrac{1}{2}J_n) \end{aligned} \tag{3.12}$$

from (3.11). This completes the proof of Theorem 1.

To deduce Theorem 2, we note that if $P(|\eta_\nu| \leqslant \alpha) = 1$, then $\lambda_{2p} \leqslant \alpha$ so that

$$I(x) \geqslant \tfrac{1}{2}(x/\alpha)^2$$

from (1.3), giving $\quad J_n \geqslant 2(\tau_n/\alpha)^2\log\log n$

from (1.5). Thus $\quad \exp(\tfrac{1}{2}J_n) \geqslant (\log n)^{\tau_n^2/\alpha^2}$;

and since $\mu_4 \leqslant \alpha^4$, Theorem 2 is now an easy consequence of Theorem 1.

4. Extensions of the Paley–Zygmund lemma

Lemma 2 can easily be made to apply to infinite series of random variables using basically the same argument as Paley and Zygmund.

If $\sum_{i=0}^{\infty} a_i^2 < \infty$, the a_i being real numbers, $S_n(\omega)$ converges in quadratic mean as $n \to \infty$ and therefore, since the η_i are independent, it converges almost surely to a random variable $S(\omega)$ which is equivalent to the quadratic mean limit [(5) 249]. Thus for almost all ω,

$$S(\omega) = \sum_{i=0}^{\infty} a_i \eta_i(\omega).$$

The proof of Lemma 2 can now be taken as far as (2.3) with $S_n(\omega)$ and $\sum_0^n a_i^2$ replaced everywhere by $S(\omega)$ and $\sum_0^\infty a_i^2$ respectively; and we can then show that

$$\mathbf{E}(S^4) \leqslant 3\mu_4\Big(\sum_0^\infty a_i^2\Big)^2$$

by letting $n \to \infty$ in (2.5), a process that is justified by Fatou's theorem. Hence

$$P\Big(|S(\omega)| \geqslant \tfrac{1}{2}\Big(\sum_0^\infty a_i^2\Big)^{\frac{1}{2}}\Big) \geqslant 3/(16\mu_4).$$

The independence of the η_i plays an essential role in the above argument, but it is interesting to see that it is possible to obtain a comparable result if the η_i merely obey certain orthogonality conditions. Of course, in this new situation $S_n(\omega)$ need not converge almost surely, but under the conditions we impose, it will still converge in quadratic mean, and we shall interpret the sum of an infinite series of random variables in this way. Thus we shall write

$$S(\omega) = \text{l.i.m.}\, S_n(\omega) = \sum_0^\infty a_i \eta_i(\omega).$$

Notice that it is still true that

$$S(\omega) = \sum_0^n a_i \eta_i(\omega) + \sum_{n+1}^\infty a_i \eta_i(\omega).$$

The theorem that we shall prove is as follows:

THEOREM 3. *Let $\eta_0(\omega)$, $\eta_1(\omega)$,... be random variables with finite fourth order moments satisfying*

(i) $\qquad \mathbf{E}(\eta_i^2) = 1, \qquad \mathbf{E}(\eta_i^4) \leqslant \mu \quad (i = 0, 1, 2,...)$

and the orthogonality conditions

(ii) $\qquad \mathbf{E}(\eta_i \eta_j) = \mathbf{E}(\eta_i \eta_j^3) = 0 \quad (i, j = 0, 1, 2,..., i \neq j).$

If $a_0, a_1,...$ are real numbers such that $\sum_0^\infty a_i^2 < \infty$, the measure of the ω-set where

$$|S(\omega)| \geqslant \tfrac{1}{2}\Big(\sum_0^\infty a_i^2\Big)^{\frac{1}{2}}$$

is not less than $3/(16\mu)$.

Proof. We first observe that Lemma 2 continues to hold under the conditions (i) and (ii); indeed, it is only the calculation of $\mathbf{E}(S_n^4)$ that requires any modification, and there, instead of

$$\mathbf{E}(\eta_i^2\eta_j^2) = \mathbf{E}(\eta_i^2)\mathbf{E}(\eta_j^2) = 1 \leqslant \mu$$

we have $$\mathbf{E}(\eta_i^2\eta_j^2) \leqslant \{\mathbf{E}(\eta_i^4)\mathbf{E}(\eta_j^4)\}^{\frac{1}{2}} \leqslant \mu.$$

Now let $$S(\omega) = S_n(\omega)+T_n(\omega), \tag{4.1}$$

where $$S_n(\omega) = \sum_0^n a_i\,\eta_i(\omega), \qquad T_n(\omega) = \sum_{n+1}^{\infty} a_i\,\eta_i(\omega);$$

and write $$\Lambda_n^2 = \mathbf{E}(S_n^2) = \sum_0^n a_i^2$$

and $$\lambda_n^2 = \mathbf{E}(T_n^2) = \sum_{n+1}^{\infty} a_i^2.$$

(This is not the meaning that was given to λ earlier in the paper, but no confusion results.) If

$$\Omega_S = \{\omega\colon |S_n(\omega)| \geqslant \tfrac{1}{2}\Lambda_n\}$$

and $$\Omega_T = \{\omega\colon |T_n(\omega)| < \lambda_n^{\frac{1}{2}}\},$$

then $$P(\Omega_S) \geqslant 3/(16\mu), \qquad P(c\Omega_T) \leqslant \lambda_n$$

by Lemma 2 and Markov's inequality respectively. Therefore

$$\begin{aligned} P\{c(\Omega_S \cap \Omega_T)\} &\leqslant P(c\Omega_S)+P(c\Omega_T) \\ &\leqslant P(c\Omega_S)+\lambda_n, \end{aligned}$$

from which $$\begin{aligned} P(\Omega_S \cap \Omega_T) &> P(\Omega_S)-\lambda_n \\ &\geqslant 3/(16\mu)-\lambda_n. \end{aligned} \tag{4.2}$$

If $\omega \in \Omega_S \cap \Omega_T$, (4.1) gives

$$\begin{aligned} |S(\omega)| &\geqslant |S_n(\omega)|-|T_n(\omega)| \\ &> \tfrac{1}{2}\Lambda_n-\lambda_n^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

Let $$\Omega_0 = \left\{\omega\colon |S(\omega)| \geqslant \tfrac{1}{2}\left(\sum_0^{\infty} a_i^2\right)^{\frac{1}{2}}\right\}$$

and, for $n = 1, 2,\ldots,$

$$\Omega_n = \{\omega\colon |S(\omega)| > \tfrac{1}{2}\Lambda_n-\lambda_n^{\frac{1}{2}}\}.$$

From (4.2) and (4.3)

$$P(\Omega_n) \geqslant 3/(16\mu)-\lambda_n. \tag{4.4}$$

Furthermore, $\Omega_0 = \bigcap_1^{\infty} \Omega_n$ and $\Omega_{n+1} \subset \Omega_n$ for $n = 1, 2,\ldots$ so that $\Omega_0 = \lim \Omega_n$, which means that

$$P(\Omega_0) = \lim_{n\to\infty} P(\Omega_n) \geqslant 3/(16\mu)$$

from (4.4). This completes the proof of the theorem.

REFERENCES

1. J. E. A. Dunnage, 'The number of real zeros of a class of random algebraic polynomials', *Proc. London Math. Soc.* (3) 18 (1968) 439–60.
2. —— 'Khintchine's lemma on sums of random variables', *J. London Math. Soc.* (to appear).
3. A. Khintchine, 'Über dyadische Brüche', *Math. Z.* 18 (1923) 109–16.
4. J. E. Littlewood and A. C. Offord, 'On the number of real roots of a random algebraic equation. II', *Proc. Cambridge Phil. Soc.* 35 (1939) 133–48.
5. Michel Loève, *Probability theory*, 3rd ed. (Van Nostrand, 1963).
6. R. E. A. C. Paley and A. Zygmund, 'On some series of functions, (1)', *Proc. Cambridge Phil. Soc.* 26 (1930) 337–57.
7. —— —— 'On some series of functions, (3)', ibid. 28 (1932) 190–205.

Chelsea College of Science and Technology
London, S.W. 3