Bulletin of the **S**ection of **L**ogic
Volume 9/1 (1980), pp. 39–43

reedition 2010 [original edition, pp. 39–45]

Viktor Finn
Revas Grigolia

BOCHVAR'S ALGEBRAS AND CORRESPONDING PROPOSITIONAL CALCULI

This is an abstract of the paper which is to appear in "Disallowance po neoclassicists logikam i teorii mnozhestv" ("Nauka").

In [1] D. A. Bochvar formulated a 3-valued logic. He analyzed the paradoxes of Russel and Weyl, and by means of the logic he proved that the paradox formulae were meaningless.

In this paper the class of algebras (B_n-algebras) corresponding to n-valued generalizations of the Bochovar's 3-valued logic is investigated. The class is defined axiomatically. The axiomatization for Bochovar's n-valued logic B_n is obtained on the basis of algebraic axiomatization.

1.

A B_n-algebra $(2 < n < \aleph_0)$ is a universal algebra $\mathcal{A} = \langle A, \cup, \cap, \sim, J_0, \ldots, J_{n-1}, 0, 1\rangle$, where A is a nonempty set of elements, 0 and 1 are constant elements of $A, \cup$ and $\cap$ are binary operations on elements of A, and $\sim$, $J_0, \ldots, J_{n-1}$ are unary operations on elements of A obeying the following axioms:

A1. $x \cup x = x$
A2. $x \cup y = y \cup x$
A3. $x \cup (y \cup z) = (x \cup y) \cup z$
A4. $x \cap (y \cup z) = (x \cap z) \cup (x \cap y)$
A5. $\sim\sim x = x$
A6. $\sim 1 = 0$

A7. $\sim(x\cup y)=\sim x\cap\sim y$
A8. $0\cup x=x$
A9. $J_{n-1}J_ix=J_ix,\ \ 0\le i\le n-1$
A10. $J_0J_ix=\sim J_ix,\ \ 0\le i\le n-1$
A11. $J_iJ_jx=0,\ \ 0<i<n-1,\ \ 0\le j\le n-1$
A12. $J_i(\sim x)=J_{n-1-i}x$
A13. $J_ix=\sim(J_0x\cup\ldots\cup J_{i-1}x\cup J_{i+1}x\cup\ldots\cup J_{n-1}x)$
A14. $J_ix\cup\sim J_ix=1,\ \ 0\le i\le n-1$
A15. $(J_ix\cup J_kx)\cap J_ix=J_ix,\ \ 0\le i,k\le n-1$
A16. $x\cup Jix=x,\ \ n-1\ge i\ge n-1-i$
A17. $J_k(x\cup y)=\bigcup_{j=0}^{k}(J_kx\cap J_jy)\cup\bigcup_{i=0}^{k}(J_ix\cap J_ky)\ \ 0\le k<[\frac{n}{2}]$
A18. $J_k(x\cup y)=\bigcup_{i=1}^{n-1}(J_ix\cap J_ky)\ \cup\ \bigcup_{i=k}^{n-1}(J_kx\cap J_iy)\ \cup$
$\bigcup_{i=k}^{n-1}(J_ix\cap J_k\sim y)\ \cup\ \bigcup_{i=k}^{n-1}(J_kx\cap J_i\sim y)\ \cup$
$\bigcup_{i=k}^{n-1}(J_i\sim x\cap J_ky)\ \cup\ \bigcup_{i=k}^{n-1}(J_k\sim x\cap J_iy),\ \ n-1\ge k\ge[\frac{n+1}{2}]$
A19. $(\forall i)(0\le i\le n-1)J_ix=J_iy\Rightarrow x=y$
$[m]$ is the largest integer k such that $k\le m$.

B_n-algebras are quasi-lattices in the sense of Płonka [2] with the operation of involution $\sim$ for which De Morgan axioms hold, and with unary J-operations $J_0,\ldots,J_{n-1}$.

The algebra $\mathbb{B}_n=\langle R_n,\cup,\cap,\sim,J_0,\ldots,J_{n-1},0,1\rangle$, where $R_n=\{0,\frac{1}{n-1},\ldots,\frac{n-2}{n-1},1\}$, $\sim x=1-x$, $x\cup y=min(max(x,y),\ max(\sim x,x),max(\sim y,y))$, $x\cap y=max(min(x,y),min(\sim,x),min(\sim y,y))$, $J_ix=\begin{cases}1,x=\frac{1}{n-1}\\ 0,x\neq\frac{i}{n-1}\end{cases}$, $0\le i\le n-1$, is an example of the B_n-algebra.

PROPOSITION. *The class of all B_n-algebras is a quasi-variety but it is not a variety.*

$$x\le y\quad\text{iff}\quad\begin{array}{ll}J_i(x\cap y)=J_ix & [\frac{n+1}{2}]\le i\le n-1,\\ J_j(x\cup y)=J_jy & 0\le j<[\frac{n}{2}].\end{array}$$

THEOREM 1.1. *The relation $\le$ is a partially ordered relation on A.*

A subset F of the set A is a filter of the B_n-algebra $\mathcal{A}$ iff (i) $1 \in F$, (ii) if $x, y \in F$ then $x \cap y \in F$, (iii) if $x \in F$ and $x \leq y$ then $y \in F$, (iv) if $x \in F$ then $J_{n-1}x \in F$.

THEOREM 1.2. *If F is a filter, then the relation R on A defined by xRy iff $\sim J_i x \cup J_i y$, $J_i x \cup \sim J_i y \in F$ $[\frac{n+1}{2}] \leq 1 \leq n-1$, $\sim J_j x \cup J_j y$, $J_j x \cup \sim J_j y \in F$, $0 \leq j < [\frac{n}{2}]$, is a congruence relation.*

Let us consider the algebra $\mathbb{B}_m = \langle R_m, \cup, \cap, \sim, J_0, \ldots, J_{n-1}, 0, 1\rangle$. Let f be a mapping of the set $\{0, 1, \ldots, m-1\}$ into the set $\{0, 1, \ldots, n-1\}$ $(m < n)$ such that (1) $f(0) = 0$, (2) $f(m-1) = n-1$, (3) $\forall x, y \in \{0, \ldots, m-1\}$ $x \leq y = f(x) \leq f(y)$, (4) $f(m-1-i) = n-1-f(i)$ where $0 \leq i \leq m-1$. From the definition of f it follows that such f does not exist if m is odd and n is odd and n is even. The algebra

$$\mathbb{B}_m^f = \langle R_m, \cup, \cap, \sim, J_0, \ldots, J_{f(1)}, \ldots, J_{f(2)}, \ldots, J_{f(m-2)}, \ldots, J_{n-1}, 0, 1\rangle$$

where $J_{f(i)} = J_i x$ for $i \in \{0, \ldots, m-1\}$ and $J_k x = 0$ for $k \in \{0, \ldots, n-1\} - \{0, \ldots, m-1\}$ is a B_n-algebra.

LEMMA 1.3. *If the filter F is maximal, then $\mathcal{A}/F$ is isomorphic to $\mathbb{B}_m^f$ for suitable f and m, where $2 \leq m \leq n$ if n is odd, and $m = 2k$, $2 \leq m \leq n$ if n is even.*

Representation theorem. Every B_n-algebra $\mathcal{A}$ is isomorphic to the subdirect product of algebras $\mathbb{B}_m^f$, where $2 \leq m \leq n$ if n is odd, and $m = 2k$, $2 \leq m \leq n$ is n is even.

The formulae of the logic B_n are constructed by means of propositional variables and the connectives $\cup, \cap, \sim, J_0, \ldots, J_{n-1}$ (where $\cup, \cap$ are binary and $\sim, J_0, \ldots, J_{n-1}$ are unary) in the usual way. We shall denote them by $\alpha, \beta, \gamma \ldots$. Formulae of the form $J_i\alpha$, $\sim J_i\alpha$ will be denoted by $\xi, \eta, \zeta, \ldots$. We introduce the following abbreviations:

$$\alpha \supset \beta \rightleftharpoons \sim \alpha \cup \beta, 0 \rightleftharpoons \sim J_{n-1}\alpha \cap J_{n-1}\alpha,$$
$$1 \rightleftharpoons \sim J_{n-1}\alpha \cup J_{n-1}\alpha,$$
$$\alpha \equiv \beta \rightleftharpoons \bigcap_{i=0}^{n-1} ((J_i\alpha \supset J_i\beta) \cap (J_i\beta \supset J_i\alpha)).$$

Now we shall construct the calculi B_n by giving a finite number of axiom schemes and inference rules of modus ponens:

$B_n1.$ $(\alpha \cup \alpha) \equiv \alpha$
$B_n2.$ $(\alpha \cup \beta) \equiv (\beta \cup \alpha)$
$B_n3.$ $(\alpha \cup (\beta \cup \gamma) \equiv ((\alpha \cup \beta) \cup \gamma)$
$B_n4.$ $(\alpha \cap (\beta \cup \alpha)) \equiv ((\alpha \cap \beta) \cup (\alpha \cap \gamma))$
$B_n5.$ $\sim (\sim (\alpha)) \equiv \alpha$
$B_n6.$ $\sim (\sim J_{n-1}\alpha \cup J_{n-1}\alpha) \equiv (\sim J_{n-1}\alpha \cap J_{n-1}\alpha)$
$B_n7.$ $\sim (\alpha \cup \beta) \equiv (\sim \alpha \cap \sim \beta)$
$B_n8.$ $((\sim J_{n-1}\alpha \cap J_{n-1}\alpha) \cup \beta) \equiv \beta$
$B_n9.$ $J_{n-1}\xi \equiv \xi$
$B_n10.$ $J_0\xi \equiv \sim \xi$
$B_n11.$ $J_i\xi \equiv (\sim J_{n-1}\alpha \cap J_{n-1}\alpha),\ 0 < i < n-1$
$B_n12.$ $J_i(\sim \alpha) \equiv J_{n-1-i}\alpha,\ 0 \leq i \leq n-1$
$B_n13.$ $J_i\alpha \equiv (J_0\alpha \cup \ldots \cup J_{i-1}\alpha \cup J_{i+1}\alpha \cup \ldots \cup J_{n-1}\alpha)\ \ 0 \leq i \leq n-1$
$B_n14.$ $(J_i\alpha \cup \sim J_i\alpha) \equiv (J_{n-1}\alpha \cup \sim J_{n-1}\alpha),\ 0 \leq i < n-1$
$B_n15.$ $((J_i\alpha \cup J_k\beta) \cap J_i\alpha) \equiv J_i\alpha,\ 0 \leq i, k \leq n-1$
$B_n16.$ $(\alpha \cup J_i\alpha) \equiv \alpha,\ n-1 \geq i \geq n-1-i.$
$B_n17.$ $J_k(\alpha \cup \beta) = \bigcup_{j=0}^{k}(J_k\alpha \cap J_j\beta) \cup \bigcup_{i=0}^{k}(J_i\alpha \cap J_k\beta),\ 0 \leq k < [\frac{n}{2}]$
$B_n18.$ $J_k(\alpha \cup \beta) = \bigcup_{i=k}^{n-1}(J_i\alpha \cap J_k\beta) \cup \bigcup_{i=k}^{n-1}(J_k\alpha \cap J_i\beta) \cup$
$\bigcup_{i=k}^{n-1}(J_i\alpha \cap J_k \sim \beta) \cup \bigcup_{i=k}^{n-1}(J_k\alpha \cap J_i \sim \beta) \cup$
$\bigcup_{i=k}^{n-1}(J_i \sim \alpha \cap J_k\beta) \cup \bigcup_{i=k}^{n-1}(J_k \sim \alpha \cap J_i\beta),$
$n-1 \geq k \geq [\frac{n+1}{2}].$
$B_n19.$ $\xi \supset (\eta \supset \zeta)$
$B_n20.$ $(\xi \supset (\eta \supset \zeta)) \supset ((\xi \supset \eta) \supset (\xi \supset \zeta))$
$B_n21.$ $(\xi \cap \eta) \supset \xi$
$B_n22.$ $(\xi \cap \eta) \supset \eta$
$B_n23.$ $(\xi \supset \eta) \supset ((\xi \supset \zeta) \supset (\xi \supset (\eta \cap \zeta)))$
$B_n24.$ $\xi \supset (\xi \cup \eta)$
$B_n25.$ $\eta \supset (\xi \cup \eta)$
$B_n26.$ $(\xi \supset \zeta) \supset ((\eta \supset \zeta) \supset ((\xi \cup \eta) \supset \zeta)))$
$B_n27.$ $(\xi \supset \eta) \supset (\sim \eta \supset \sim \xi)$
$B_n28.$ $\xi \supset \sim\sim \xi$
$B_n29.$ $\sim\sim \xi \supset \xi$

Inference rule: $\dfrac{\alpha, \alpha \supset \beta}{\beta}$

A formula α is said to be a tautology if α considered as an algebraic polinom has the value 1 for each assignment of variables by elements of the algebra $\mathbb{B}_n$.

COMPLETENESS THEOREM. *For each formula α, α is a theorem $B_n(\vdash B)$ iff α is a tautology.*

Note that the Bochvar's 3-valued logic has already been formalized in several different ways [2,3,5]. Some authors treat this logic as the nonsense-logic or the logic of significance.

References

[1] D. A. Bochvar, *Ob odnom trojokhznachnom ischislenii i ego primenenii k analizu paradoksov klassicheskogo rasshirennogo phunktisionalnego ischislenia*, **Matematicheski sbornik** 4 (1938), no 2, pp. 287-308.

[2] J. Płonka, *On distributive quasi-lattices*, **Fundamenta Mathematicae** LX (1967), pp. 191–200.

[3] V. K. Finn, *On the functional properties and axiomatization of D. A. Bochvar's three-valued logic*, **IV International Congress for Logic, Methodology and Philosophy of Science**, Bucharest (1971), pp. 22–23.

[4] Z. Goddard and R. Routley, **The logic of significance and context**, N. Y. (1973), v. 1.

[5] K. Piróg-Rzepecka, **Systems of nonsense-logics**, Warszawa-Wrocław, 1977.

Department of Semiotic Problems of Informatics of All-Union Institute of Scientific and Technical Information, the Academy of Science of USSR, Moscov

Department of Logic
The Institute of Cybernetics of the Academy of Sciences of the Georgian SSR, Tbilissi