GENERIC BERNSTEIN-SATO POLYNOMIAL ON AN IRREDUCIBLE AFFINE SCHEME

ROUCHDI BAHLOUL

Abstract. Given p polynomials with coefficients in a commutative unitary integral ring $\mathcal{C}$ containing $\mathbb{Q}$, we define the notion of a generic Bernstein-Sato polynomial on an irreducible affine scheme $V \subset \mathrm{Spec}(\mathcal{C})$. We prove the existence of such a non zero rational polynomial which covers and generalizes previous existing results by H. Biosca. When $\mathcal{C}$ is the ring of an algebraic or analytic space, we deduce a stratification of the space of the parameters such that on each stratum, there is a non zero rational polynomial which is a Bernstein-Sato polynomial for any point of the stratum. This generalizes a result of A. Leykin obtained in the case $p = 1$.

Introduction and Main Results

Fix $n \geq 1$ and $p \geq 1$ two integers and $v \in \mathbb{N}^p$. Let $x = (x_1, \ldots, x_n)$ and $s = (s_1, \ldots, s_p)$ be two systems of variables. Let $\mathbf{k}$ be a field of characteristic [1] 0. Let $\mathbf{A}_n(\mathbf{k})$ be the ring of differential operators with coefficients in $\mathbf{k}[x] = \mathbf{k}[x_1, \ldots, x_n]$ and $\mathcal{D}$ (resp. $\mathcal{O}$) be the sheaf of rings of differential operators (resp. analytic functions) on $\mathbb{C}^n$ for which we denote by $\mathcal{D}_{x_0}$ (resp. $\mathcal{O}_{x_0}$) the fiber in x_0.

Let $f = (f_1, \ldots, f_p)$ be in $\mathbf{k}[x]^p$ (resp. $\mathcal{O}_{x_0}^p$) and consider the following functional identity:

$$b(s) f^s \in \mathbf{A}_n(\mathbf{k})[s] \cdot f^{s+v},$$

(resp. $\mathcal{D}_{x_0}[s]$ instead of $\mathbf{A}_n(\mathbf{k})[s]$) where $f^{s+v} = f_1^{s_1+v_1} \cdots f_p^{s_p+v_p}$. This identity takes place in the free module generated by f^s over $\mathbf{k}[x, \frac{1}{f_1 \cdots f_p}, s]$ (resp. $\mathcal{O}_{x_0}[\frac{1}{f_1 \cdots f_p}, s]$).

The set of such $b(s)$ is an ideal of $\mathbf{k}[s]$ (resp. $\mathbb{C}[s]$). This ideal is called the (global) Bernstein-Sato ideal of f (resp. local Bernstein-Sato ideal in x_0) and we denote it by $\mathcal{B}^v(f)$ (resp. $\mathcal{B}^v_{x_0}(f)$). When $p = 1$, this ideal is principal and its monic generator is called the Bernstein polynomial associated with f. Historically, I.N. Bernstein [Be] introduced the (global) Bernstein polynomial and proved its existence (i.e. the fact that it is not zero). J.E. Björk [Bj] has given the proof in the analytic case. Let us cite also M. Kashiwara [K] who proved, moreover, the rationality of the roots of the local Bernstein polynomial. For $p \geq 2$, the algebraic case can be easily treated in the same way as for $p = 1$. For the analytic case, the proof of the non nullity of $\mathcal{B}^v_{x_0}(f)$ is due to C. Sabbah ([S1] and [S2]). Let us also cite A. Gyoja [G] who proved that $\mathcal{B}^v_{x_0}(f)$ contains a non zero *rational* polynomial. The absolute Bernstein-Sato polynomial naturally leads to the notion of a generic Bernstein-Sato polynomial which we shall explain in what follows.

Let $\mathcal{C}$ be a unitary commutative integral ring with the following condition:
For any prime ideal $\mathcal{P} \subset \mathcal{C}$ and for any $n \in \mathbb{N} \smallsetminus \{0\}$, we have:

$$n \in \mathcal{P} \Rightarrow 1 \in \mathcal{P}.$$

[1] all the fields considered in this paper are of characteristic 0

This condition is equivalent to the fact that for any $\mathcal{P} \subset \mathcal{C}$, the fraction field of $\mathcal{C}/\mathcal{P}$ is of characteristic 0. Note that this condition is satisfied if and only if there exists an injective ring morphism $\mathbb{Q} \hookrightarrow \mathcal{C}$.

We shall see $\mathcal{C}$ as the ring of coefficients or parameters. Indeed, let $f = (f_1, \ldots, f_p)$ in $\mathcal{C}[x]^p = \mathcal{C}[x_1, \ldots, x_n]^p$.

Let us denote by $\mathbf{A}_n(\mathcal{C})$ the ring of differential operators with coefficients in $\mathcal{C}[x]$, that is the $\mathcal{C}$-algebra generated by x_i and ∂_{x_i} $(i = 1, \ldots, n)$ where the only non trivial commutation relations are $[\partial_{x_i}, x_i] = 1$ for $i = 1, \ldots, n$ (hence $\mathcal{C}$ is in the center of $\mathbf{A}_n(\mathcal{C})$).

We denote by $\mathrm{Spec}(\mathcal{C})$ (resp. $\mathrm{Specm}(\mathcal{C})$) the set of prime (resp. maximal) ideals of $\mathcal{C}$ which is the spectrum of $\mathcal{C}$ (resp. the maximal spectrum). For an ideal $\mathcal{I} \subset \mathcal{C}$, we denote by $V(\mathcal{I}) = \{\mathcal{P} \in \mathrm{Spec}(\mathcal{C})\,;\, \mathcal{P} \supset \mathcal{I}\}$ the affine scheme defined by $\mathcal{I}$ and $V_m(\mathcal{I}) = V(\mathcal{I}) \cap \mathrm{Specm}(\mathcal{C})$. Remark that we shall only work with the closed subsets of $\mathrm{Spec}(\mathcal{C})$ and forget the sheaf structure of a scheme.

We are going to introduce the notion of a generic Bernstein-Sato polynomial of f on an irreducible affine scheme $V = V(\mathcal{Q}) \subset \mathrm{Spec}(\mathcal{C})$ (that is when $\mathcal{Q}$ is prime).
So let $\mathcal{Q}$ be a prime ideal of $\mathcal{C}$ and suppose that none of the f_j's is in $\mathcal{Q}[x]$.
The main result of this article is the following.

Theorem 1. *There exists $h \in \mathcal{C} \smallsetminus \mathcal{Q}$ and $b(s) \in \mathbb{Q}[s_1, \ldots, s_p] \smallsetminus 0$ such that*

$$h\, b(s) f^s \in \mathbf{A}_n(\mathcal{C})[s] f^{s+v} + \big(\mathcal{Q}[x, \frac{1}{f_1 \cdots f_p}, s]\big) f^s.$$

Such a $b(s)$ is called a (rational) generic Bernstein-Sato polynomial of f on $V = V(\mathcal{Q})$ (see the notation and the remark below).

In the case where $p = 1$, the generic and relative (not introduced here) Bernstein polynomial has been studied by F. Geandier in [Ge] and by J. Briançon, F. Geandier and P. Maisonobe in [Br-Ge-M] in an analytic context (where f is an analytic function of x). In [Bi] (see also [Bi2]), H. Biosca studied these notions with $p \geq 1$ in the analytic and the algebraic context (that which we are concerned with) and proved that when

- $\mathcal{C} = \mathbb{C}[a_1, \ldots, a_m]$ or
- $\mathcal{C} = \mathbb{C}\{a_1, \ldots, a_m\}$ and

$Q = (0)$ so that V is smooth and equal to $\mathbb{C}^m$ or $(\mathbb{C}^m, 0)$, we have a generic Bernstein-Sato polynomial. It does not seem straigthforward to adapt her proof to the case where $\mathcal{Q} \neq (0)$ (i.e. when V is singular). Let us also say that she did not mention the fact that the polynomial she constructed is rational even though a detailed study of her proof shows that it is. As it appears, our main result covers and generalizes the previous existing results in this affine situation.

Notation. *Let $\mathcal{P}$ be a prime ideal of $\mathcal{C}$. For c in $\mathcal{C}$, denote by $[c]_{\mathcal{P}}$ the class of c in the quotient $\mathcal{C}/\mathcal{P}$ and $(c)_{\mathcal{P}} = \frac{[c]_{\mathcal{P}}}{1}$ this class viewed in the fraction field of $\mathcal{C}/\mathcal{P}$. We naturally extend these notations to $\mathcal{C}[x]$, $\mathbf{A}_n(\mathcal{C})$ and $\mathcal{C}[x, \frac{1}{f_1 \cdots f_p}, s]$.*

Remark. *Using these notations, we can see that the polynomial $b(s)$ of theorem 1 is a Bernstein-Sato polynomial of $(f)_{\mathcal{P}}$ for any $\mathcal{P} \in V(\mathcal{Q}) \smallsetminus V(h)$. This justifies the name of a generic Bernstein-Sato polynomial on $V(\mathcal{Q})$.*

As an application of theorem 1, we obtain some consequences :

Corollary 2. *Fix a positive integer d and a field* **k**.
For each $j = 1, \ldots, p$, take $f_j = \sum_{|\alpha| \leq d} a_{\alpha,j} x^\alpha$ with $\alpha \in \mathbb{N}^n$ and $a_{\alpha,j}$ an indeterminate.

Take $a = (a_{\alpha,j})$ *for* $|\alpha| \le d$ *and* $j = 1, \ldots, p$ *such that we see* $f = (f_1, \ldots, f_p)$ *in* $\mathbf{k}[a][x]^p$. *Denote by* m *the number of the* $a_{\alpha,j}$*'s.*
Then there exists a finite partition of $\mathbf{k}^m = \cup W$ *where each* W *is a locally closed subset of* $\mathbf{k}^m$ *(i.e.* W *is a difference of two Zariski closed sets) such that for any* W*, there exists a polynomial* $b_W(s) \in \mathbb{Q}[s_1, \ldots, s_p] \smallsetminus 0$ *such that for each* a_0 *in* W*,* $b_W(s)$ *is in* $\mathcal{B}^v(f(a_0, x))$.

Remark.

- *This corollary generalizes to the case* $p \ge 2$ *the main result of A. Leykin* [L] *and J. Briançon and Ph. Maisonobe* [Br-Mai] *in the case* $p = 1$.
- *There is another way to generalize these results: Given a well ordering* $<$ *on* $\mathbb{N}^p$ *compatible with sums, it is possible to prove the existence of a partition* $\mathbf{k}^m = \cup W$ *into locally closed subsets with the following property: For any* W*, there exists a finite subset* $G_W \subset \mathbf{k}[a][x]$ *such that for any* $a_0 \in W$*, the set* $G_W(a_0)$ *is a* $<$*-Gröbner basis of the Bernstein-Sato ideal* $\mathcal{B}^v(f(a_0, x))$*, see* [Br-Mai] *and* [Ba].

Proof of Corollary 2. We remark that we can give the same statement as in corollary 2 for any algebraic subset $Y \subset \mathbf{k}^m$ as a space of parameters. The statement of corollary 2 will then follow from the proof of this more general statement, that we shall give by an induction on the dimension of Y. If $\dim Y = 0$, the result is trivial. Suppose $\dim Y \ge 1$. Write $Y = V_m(Q_1) \cup \cdots \cup V_m(Q_r)$ where the Q_i's are prime ideals in $\mathbf{k}^m$ (we identify the maximal ideals of $\mathbf{k}[a]$ and the points of $\mathbf{k}^m$). For each i, let $h_i \in \mathbf{k}[a] \smallsetminus Q_i$ and $b_i(s) \in \mathbb{Q}[s] \smallsetminus 0$ be the h and $b(s)$ of theorem 1 applied to Q_i. Now, write

$$Y = \Big(\bigcup_{i=1}^{r} V_m(Q_i) \smallsetminus V_m(h_i) \Big) \bigcup Y',$$

with $Y' = \bigcup \big(V_m(Q_i) \cap V_m(h_i) \big)$ for which $\dim Y' < \dim Y$. Apply the induction hypothesis to Y'. We obtain that Y is a union (not necessarily disjoint) of locally closed subsets V such that for each V there exists $b_V(s) \in \mathbb{Q}[s] \smallsetminus 0$ which is in $\mathcal{B}^v(f(a_0, x))$ for any $a_0 \in V$. Let us show now how to obtain the annouced partition. Let B be the set of the obtained polynomials b_V's. Set $B = \{b_1, \ldots, b_e\}$. For any $i = 1, \ldots, e$, let E_i be the set of the V's for which $b_i = b_V$. Put

- $W_1 = \bigcup_{V \in E_1} V$,
- $W_2 = (\bigcup_{V \in E_2} V) \smallsetminus (\bigcup_{V \in E_1} V)$,

  $\vdots$
- $W_e = (\bigcup_{V \in E_e} V) \smallsetminus (\bigcup_{V \in E_1 \cup \cdots \cup E_{e-1}} V)$.

Note that some of the W_i's may be empty. The set $\{(b_1, W_1), \ldots, (b_e, W_e)\}$ gives a partition $Y = \cup W_i$ in a way that $b_i \in \mathcal{B}^v(f(a_0, x))$ for any $a_0 \in W_i$. ☐

Corollary 3. *Take* $f_1(a, x), \ldots, f_p(a, x) \in \mathcal{O}(U)[x]$ *where* $\mathcal{O}(U)$ *denotes the ring of holomorphic functions on a open subset* U *of* $\mathbb{C}^m$.
Then there exists a finite partition of $U = \cup W$ *where each* W *is an (analytic) locally closed subset of* U *(i.e. each* W *is a difference of two analytic subsets of* U*) such that for any* W*, there exists a rational non zero polynomial* $b(s)$ *which belongs to* $\mathcal{B}^v(f(a_0, x))$ *for any* $a_0 \in W$.

Remark. *As it will appear in the proof, we have the same result if we replace $\mathcal{O}(U)$ by $\mathbb{C}\{a_1, \dots, a_m\}$ or $\mathbf{k}[[a_1, \dots, a_m]]$ ($\mathbf{k}$ being an arbitrary field).*

Proof. Let us write $f_j(a,x) = \sum g_{\alpha,j}(a)x^\alpha$ where $g_{\alpha,j} \in \mathcal{O}(U)$. Let m be the number of the $g_{\alpha,j}$'s and let us introduce m new variables $b_{\alpha,j}$. Consider the (analytic) map $\phi : U \ni a \mapsto (b_{\alpha,j} = g_{\alpha,j}(a))_{\alpha,j} \in \mathbb{C}^m$ where $\mathbb{C}$ is a fixed arbitrary field. Now apply corollary 2 to this situation. Let $\mathbf{k}^m = \cup W$ be the obtained partition and for any W, let $b_W \in \mathbb{Q}[s]$ be the polynomial given in 2. Now apply ϕ^{-1}. This gives a partition $U = \cup \phi^{-1}(W)$. Since ϕ is analytic, the sets $\phi^{-1}(W)$ are locally closed analytic subsets of U. It is then clear that for any W and $a_0 \in \phi^{-1}(W)$, we have $b_W \in \mathcal{B}^v(f(a_0,x))$. □

Proof of the main theorem

In order to prove theorem 1, we shall first prove the following.

Theorem 4. *Let $\mathbf{k}$ be a field and $f \in \mathbf{k}[x]^p$. Then $\mathcal{B}^v(f) \cap \mathbb{Q}[s]$ is not zero.*

Note that in [Br], the author proved (for $p = 1$) that the global Bernstein polynomial has rational roots for any field $\mathbf{k}$ of characteristic zero. The proof of 4 will use the following propositions.

Proposition 5. *Let $\mathbf{K}$ be a subfield of a field $\mathbf{L}$. Suppose that $f \in \mathbf{K}[x]^p$. Let $b(s) \in \mathbf{K}[s]$ be such that $b(s)f^s \in \mathbf{A}_n(\mathbf{L})[s]f^{s+v}$. Then*

$$b(s)f^s \in \mathbf{A}_n(\mathbf{K})[s]f^{s+v}.$$

Proof. The proof is inspired by [Br] in which the case $p = 1$ is treated. As $\mathbf{L}$ is a $\mathbf{K}$-vector space, let us take $\{1\} \cup \{l_\gamma;\, \gamma \in \Gamma\}$ as a basis so that $\mathbf{L}[x, s, \frac{1}{f_1 \cdots f_p}]f^s$ is a free $\mathbf{K}[x, s, \frac{1}{f_1 \cdots f_p}]$-module with $\{f^s\} \cup \{l_\gamma f^s;\, \gamma \in \Gamma\}$ as a basis. Now let P be in $\mathbf{A}_n(\mathbf{L})[s]$ such that $b(s)f^s = Pf^{s+v}$. We decompose $P = P_0 + P'$ where $P_0 \in \mathbf{A}_n(\mathbf{K})[s]$ and P' has its coefficients in $\bigoplus_{\gamma \in \Gamma} \mathbf{K} \cdot l_\gamma$. Now, we have:

$$b(s)f^s = P_0 f^{s+v} + P' f^{s+v},$$

with $b(s)f^s$ and $P_0 f^{s+v}$ in $\mathbf{K}[x, s, \frac{1}{f_1 \cdots f_p}]f^s$ and $P'f^{s+v}$ in $\bigoplus_{\gamma \in \Gamma} \mathbf{K}[x, s, \dfrac{1}{f_1 \cdots f_p}]l_\gamma f^s$. By identification, we obtain:

$$b(s)f^s = P_0 f^{s+v}.$$

□

Proposition 6. *(*[Br] *and* [Br-Mai]*) Given $f \in \mathbb{C}[x]^p$, we have :*

1. *The set $\{\mathcal{B}^v_{x_0}(f);\, x_0 \in \mathbb{C}^n\}$ is finite.*
2. *$\mathcal{B}^v(f)$ is the intersection of all the $\mathcal{B}^v_{x_0}(f)$ where $x_0 \in \mathbb{C}^n$.*

Proof of theorem 4. We shall divide the proof into two steps:
(a) First, suppose that $\mathbf{k} = \mathbb{C}$. By [S1], [S2] and [G], as mentioned in the introduction, each $\mathcal{B}^v_{x_0}(f)$ contains a non zero rational polynomial. By the previous proposition, we can take a finite product of these polynomials and obtain a rational polynomial in $\mathcal{B}^v(f)$.
(b) Now suppose that $\mathbf{k}$ is arbitrary. Let $c_1, \dots, c_N$ be all the coefficients that appear in the writing of the f_j's and consider the field $\mathbf{K} = \mathbb{Q}(c_1, \dots, c_N)$. There exist $e_1, \dots, e_N \in \mathbb{C}$ and an injective morphism of fields $\phi : \mathbf{K} \to \mathbb{C}$ such that $\phi(c_i) = e_i$ for any i. We denote by the same symbol ϕ the natural extension of ϕ from $\mathbf{K}[x]$ to $\mathbb{C}[x]$ and from

$\mathbf{A}_n(\mathbf{K})[s]$ to $\mathbf{A}_n(\mathbb{C})[s]$. Now, consider in $\mathbb{C}[s]$ the Bernstein-Sato ideal $\mathcal{B}^v(\phi(f))$ (where $\phi(f) = (\phi(f_1), \ldots, \phi(f_p))$. Using the result of case (a), there exists $b(s) \in \mathbb{Q}[s] \smallsetminus 0$ that belongs to $\mathcal{B}^v(\phi(f))$. So we have a functional equation:

$$b(s)\phi(f)^s = P \cdot \phi(f)^{s+v},$$

where $P \in \mathbf{A}_n(\mathbb{C})[s]$. By proposition 5, we can suppose $P \in \mathbf{A}_n(\phi(\mathbf{K}))[s]$. Apply ϕ^{-1} to this equation. Since $b(s) \in \mathbb{Q}[s]$, $\phi^{-1}(b(s)) = b(s)$, thus we obtain:

$$b(s)f^s = \phi^{-1}(P) \cdot f^{s+v}.$$

In conclusion $b(s)$ is in $\mathcal{B}^v(f)$. □

Now we dispose of a sufficient material to give the

Proof of theorem 1. By theorem 4, there exists a non zero rational polynomial $b(s)$ in $\mathcal{B}^v((f)_\mathcal{Q})$. Hence, we have the following equation:

$$b(s)\Big(\frac{[f]_\mathcal{Q}}{1}\Big)^s = \frac{[U(s)]_\mathcal{Q}}{[h]_\mathcal{Q}} \cdot \Big(\frac{[f]_\mathcal{Q}}{1}\Big)^{s+v},$$

where $U(s) \in \mathbf{A}_n(\mathcal{C})[s]$ and $h \in \mathcal{C} \smallsetminus \mathcal{Q}$. It follows that:

$$h\, b(s) f^s - U(s) \cdot f^{s+v} \equiv 0 \mod \mathcal{Q}$$

in $\mathcal{C}[x, \frac{1}{f_1,\ldots,f_p}, s]f^s$. Since $f_1 \cdots f_p \notin \mathcal{Q}[x]$ and $\mathcal{Q}$ is prime, we obtain:

$$h\, b(s) f^s - U(s) \cdot f^{s+v} \in \mathcal{Q}[x, \frac{1}{f_1 \cdots f_p}, s]f^s.$$

□

This article is a more general and simplified version of some results of my thesis [Ba].

Acknowledgements
I am grateful to my thesis advisor Michel Granger for having intoduced me to this subject and for valuable remarks.

References

[Ba] R. Bahloul, *Contributions à l'étude des idéaux de Bernstein-Sato d'un point de vue constructif*, thèse de doctorat, Université d'Angers, 2003.

[Be] I. N. Bernstein, *The analytic continuation of generalised functions with respect to a parameter*, Funct. Anal. Appl. 6, 273-285, 1972.

[Bi] H. Biosca, *Polynômes de Bernstein génériques et relatifs associés à une application analytique*, Thèse, Nice Sophia-Antipolis, 1996.

[Bi2] H. Biosca, *Sur l'existence de polynômes de Bernstein génériques associés à une application analytique*, C. R. Acad. Sci. Paris Sér. I Math. 322, no. 7, 659-662, 1996.

[Bj] J.E. Björk, *Dimensions over Algebras of Differential Operators*, preprint, 1973.

[Br] J. Briançon, *Passage du local au global*, notes manuscrites.

[Br-Ge-M] J. Briançon, F. Geandier, Ph. Maisonobe, *Déformation d'une singularité isolée d'hypersurface et polynômes de Bernstein*, Bull. Soc. Math. France 120, no 1, 15-49, 1992.

[Br-Mai] J. Briançon, Ph. Maisonobe, *Remarques sur l'idéal de Bernstein associé à des polynômes*, prépublication Université Nice Sophia-Antipolis n° 650, Mai 2002.

[Br-Mai2] J. Briançon, Ph. Maisonobe, *Examen de passage du local au global pour les polynômes de Bernstein-Sato*, 1990.

[Ge] F. Geandier, *Déformations à nombre de Milnor constant: quelques résultats sur les polynômes de Bernstein*, Compositio Math. 77, no. 2, 131-163, 1991.

[G] A. Gyoja, *Bernstein-Sato's polynomial for several analytic functions*, J. Math. Kyoto Univ. 33-2, 399-411, 1993.

[K] M. Kashiwara, *B-functions and holonomic systems. Rationality of roots of B-functions*, Invent. Math. 38, no 1, 33-53, 1976/1977.

[L] A. Leykin, *Constructibility of the set of polynomials with a fixed Bernstein-Sato Polynomial: an algorithmic approach*, Journal of Symbolic Computation 32, 663-675, 2001.

[S1] C. Sabbah, *Proximité évanescente I. La structure polaire d'un $\mathcal{D}$-Module*, Compositio Math. 62, 283-328, 1987.

[S2] C. Sabbah, *Proximité évanescente II. Equations fonctionnelles pour plusieurs fonctions*, Compositio Math. 64, 213-241, 1987.

Département de Mathématiques, U.M.R. 6093, Université d'Angers, 2 Bd Lavoisier, 49045 Angers cedex 01, France

E-mail address: rouchdi.bahloul@univ-angers.fr