Progress of Theoretical Physics, Vol. 19, No. 1, January 1958

The Dispersion Relations for the Scattering of Photons from Proton

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(Received October 14, 1957)

The dispersion relations are derived for the non-forward amplitudes for photon-proton scattering. By expanding these relations in momentum transfer and retaining first few terms, the dispersion relations for six independent combinations of the partial wave amplitudes are obtained. Contributions of the unphysical region are calculated to the first order in the fine structure constant, and are found to be consistent with the low energy theorem. Using these relations and photopion production data, the differential cross sections are calculated. In doing this, the amplitudes for the mixtures of dipole and quadrupole waves are taken into account in addition to electric dipole, electric quadrupole and magnetic dipole amplitudes. The results turn out to be fairly improved compared with the dipole model, and roughly explain experiments.

§ 1. Introduction

Gell-Mann, Goldberger and Thirring[1] have derived two dispersion relations for spin-flip and non-spin-flip parts of the forward amplitude for photon-proton scattering. If one assumes that only the partial wave amplitudes for electric and magnetic dipole radiations with total angular momenta $1/2$ and $3/2$, respectively, are enhanced by photopion production, these two dispersion relations are sufficient to determine the real parts of these amplitudes from their imaginary parts. As the imaginary parts can be related to the partial wave amplitudes for photopion production owing to the unitarity of S matrix, one can predict the differential cross sections for photon-proton scatterings using data on photopion production. We refer to this method as the dipole model. Recently, calculations by this method have been made by Capps,[2] predicting too large cross sections compared with experiments below 200 Mev.

In the present paper we will derive the dispersion relations for the non-forward amplitudes. By expanding these relations in powers of momentum transfer, we can obtain more than two dispersion relations. Then one can inquire whether all these dispersion relations can or cannot be satisfied by the above two dipole amplitudes alone. Thus we can examine the validity of the dipole model. We will show in § 4 that the dipole model is sufficient to describe general aspects of the phenomena, but is not so accurate numerically.

Although derivation of the non-forward dispersion relations has been studied in detail for pion-nucleon scattering,[3] it is not so trivial for the present case because the polarization vectors depend on energy and momentum transfer for the non-forward scattering.

However, we can prove necessary symmetry properties and analyticity of the amplitudes by an appropriate choice of four independent combinations of initial and final polarization vectors. As is well known, the unphysical region appears in the non-forward dispersion relations, in spite of the vanishing mass of a photon. For sufficiently small momentum transfer, and to the first order in the fine structure constant, contributions of the unphysical region come only from the single proton intermediate states, and therefore can be easily calculated. They can, however, be determined by another method. By virtue of the low energy theorem,[4] expression of the scattering amplitude is known to the first order in photon energy without entering into details of the theory, and comparison of this expression with the dispersion relations determines the contributions of the unphysical region completely. It is of interest to examine whether these two determinations give the same result, since the direct calculation is not rigorous on account of the fact that it compels us to deal with an unphysical state with imaginary momentum. The affirmative answer to this question will be given in § 2.

Expanding the non-forward dispersion relations in powers of momentum transfer, we obtain from the zeroth order parts two dispersion relations which naturally coincide with those derived in reference 1) and two new relations from the first order parts. Relations obtained from the higher order parts contain, in general, undetermined constants which cannot be expressed in terms of charge and magnetic moment of proton alone. As the present experimental knowledge does not seem accurate enough to determine so many parameters, we derive only two relations from the second order part. The two undetermined constants appearing in these relations are found to be equal to each other to the approximation adopted in calculating the cross sections. We thus obtain six dispersion relations in all, derivation of which will be made in § 2.

In § 3 the cross sections are calculated with the aid of the above dispersion relations, restricting the participating partial wave amplitudes appropriately. The results are found to be fairly improved compared with the dipole model. But a disagreement with experiments still remains in the energy dependence of the cross section. Possibilities of further improvements are discussed in § 4.

§ 2. Dispersion relations

We consider the process in which a photon of momentum $\boldsymbol{k}$ and polarization $\boldsymbol{e}_\tau$ collides with a proton of momentum $\boldsymbol{p}$, and then the photon is scattered into the state of momentum $\boldsymbol{k}'$ and polarization $\boldsymbol{e}_s$, leaving the proton in the state with momentum $\boldsymbol{p}'$. It is convenient for our purpose to take the coordinate system in which the momenta of the colliding particles obey the relations $\boldsymbol{p}=-\boldsymbol{p}'=-\boldsymbol{q}$, $\boldsymbol{k}=\mathbf{Q}+\boldsymbol{q}$, $\boldsymbol{k}'=\mathbf{Q}-\boldsymbol{q}$, where $\mathbf{Q}$ is the total momentum. In this system we define the causal amplitude for the process as in the case of pion-nucleon scattering, which will be denoted by $M(\boldsymbol{k}', \boldsymbol{e}_s, \boldsymbol{k}, \boldsymbol{e}_\tau)$ as a matrix in spin space. The causal amplitude is identical with the Feynman matrix element for the same process when $\nu\equiv|\boldsymbol{k}|=|\boldsymbol{k}'|$ is larger than $\boldsymbol{q}$, and moreover, owing to the requirement of causality, it has the important property of being analytic in the

upper half part of the ν-complex plane, provided that we properly define the ν-dependence of polarization vectors $\mathbf{e}_T$, $\mathbf{e}_S$, as will be shown below.

Table I. Sets of polarization which show simple ν-dependence.

polarization \ λ	I	II	III	IV
initial $\mathbf{e}_r$	$\mathbf{n}\times\boldsymbol{\kappa}$	$\mathbf{n}$	$\mathbf{n}\times\boldsymbol{\kappa}$	$\mathbf{n}$
final $\mathbf{e}_s$	$\mathbf{n}\times\boldsymbol{\kappa}'$	$-\mathbf{n}$	$-\mathbf{n}$	$\mathbf{n}\times\boldsymbol{\kappa}'$

The polarization vectors $\mathbf{e}_T$, $\mathbf{e}_S$ generally change their direction as ν varies, showing indeed that they must be regarded as functions of ν. We give four sets of polarization in Table I, which seem to exhibit the simplest ν-dependences. Here $\mathbf{n}$ is a unit vector perpendicular to the scattered plane, $\boldsymbol{\kappa}=\boldsymbol{k}/k$, and $\boldsymbol{\kappa}'=\boldsymbol{k}'/k'$. We denote the causal amplitude characterized by the λ-th set of polarizations in Table I by $M^{(\lambda)}$, and confine our consideration to the four amplitudes $M^{(1)},\cdots,M^{(4)}$, since a general causal amplitude can be represented in a linear combination of them. To define the ν-dependence of these amplitudes explicitly, we first put

$$\mathbf{Q}=Q(\nu)\boldsymbol{\pi}\ ;\ Q(\nu)=(\nu^2-q^2)^{1/2}, \tag{2·1}$$

where $\boldsymbol{\pi}$ is a unit vector. We introduce cuts along the real axis from $-\infty$ to $-q$ and q to $+\infty$ to make $Q(\nu)$ a one-valued function, and suppose that ν always lies slightly above the real axis; hence $Q(-\nu)=-Q(\nu)$ for $\nu>q$. From (2·1) the polarization vectors in Table I, as functions of ν, take the form

$$\begin{gathered}\mathbf{n}\times\boldsymbol{\kappa}=q^{-1}\mathbf{e}_1(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q}),\ \ \mathbf{n}\times\boldsymbol{\kappa}'=q^{-1}\mathbf{e}_1(\nu,\ -\boldsymbol{\pi},\ \boldsymbol{q})\\ \mathbf{n}=q^{-1}\mathbf{e}_2(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q}),\end{gathered} \tag{2·2}$$

where $\mathbf{e}_1(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=\nu^{-1}(q^2\boldsymbol{\pi}-Q(\nu)\boldsymbol{q})$, $\mathbf{e}_2(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=\boldsymbol{q}\times\boldsymbol{\pi}$. From (2·1) and (2·2) the quantities which occur in the causal amplitudes can be expressed in terms of ν, $\boldsymbol{\pi}$, $\boldsymbol{q}$. Hence the above four amplitudes may be considered as functions of ν, $\boldsymbol{\pi}$ and $\boldsymbol{q}$. They are invariant under spatial rotations, but under reflection $(\boldsymbol{\pi},\ \boldsymbol{q}\to-\boldsymbol{\pi},\ -\boldsymbol{q})$ $M^{(3,4)}$ change the sign, while $M^{(1,2)}$ remain invariant. Consequently $M^{(\lambda)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})$ can be written in the form

$$M^{(1,2)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=\mathcal{M}_N{}^{(1,2)}(\nu,\ q^2)+i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}(\nu)\mathcal{M}_S{}^{(1,2)}(\nu,\ q^2), \tag{2·3a}$$

$$M^{(3,4)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})=i\boldsymbol{\sigma}\cdot\mathbf{Q}(\nu)\mathcal{M}_+{}^{(3,4)}(\nu,\ q^2)+i\boldsymbol{\sigma}\cdot\boldsymbol{q}\mathcal{M}_-{}^{(3,4)}(\nu,\ q^2). \tag{2·3b}$$

Now we state the notable properties of $M^{(\lambda)}$ which follow from the requirement of the causality[3] and the invariance under time reflection[5]: (i) $\mathcal{M}^{(\lambda)}(\nu,\ q^2)$ are analytic in the upper half part of the ν-complex plane, (ii) Re $\mathcal{M}_N{}^{(1,2)}(\nu,\ q^2)$, Im$\mathcal{M}_S{}^{(1,2)}(\nu,\ q^2)$, and Re$\mathcal{M}_{+,-}^{(2,4)}(\nu,\ q^2)$ are even, while Im$\mathcal{M}_N{}^{(1,2)}(\nu,\ q^2)$, Re$\mathcal{M}_S{}^{(1,2)}(\nu,\ q^2)$ and Im$\mathcal{M}_{+,-}^{(3,4)}(\nu,\ q^2)$ are odd, as functions of ν, (iii) $\mathcal{M}_+{}^{(3)}(\nu,\ q^2)=\mathcal{M}_+{}^{(4)}(\nu,\ q^2)$ and $\mathcal{M}_-{}^{(3)}(\nu,\ q^2)=-\mathcal{M}_-{}^{(4)}(\nu,\ q^2)$. The last condition means that we may exclude $M^{(4)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})$ from our consideration. The proof of these statements can be done following the familiar pattern, and, perhaps, needs scarcely to be discussed here.

There remains one thing to be done prior to the derivation of dispersion relations, i.e., to examine the behavior of the imaginary part of amplitudes in the unphysical region $\nu<q$. For this purpose we decompose $M^{(\lambda)}(\nu,\ \boldsymbol{\pi},\ \boldsymbol{q})$ into the dispersive and absorptive

part, which are real and imaginary part of $\mathcal{M}^{(\lambda)}(\nu, q^2)$, respectively:

$$M^{(\lambda)}(\nu, \pi, \boldsymbol{q}) = D^{(\lambda)}(\nu, \pi, \boldsymbol{q}) + iA^{(\lambda)}(\nu, \pi, \boldsymbol{q}). \tag{2·4}$$

The absorptive parts can be writren in the form:

$$\begin{aligned} A^{(1,2)}(\nu, \pi, \boldsymbol{q}) = \frac{(2\pi)^6 q^{-2}}{4}\left(\frac{E_q}{M}\right)\sum_{M_n}\{&(\boldsymbol{q}|\mathbf{e}_{1,2}(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=\mathbf{Q}|\mathbf{e}_{1,2}(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q+\nu-E_n) \\ &-(\boldsymbol{q}|\mathbf{e}_{1,2}(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=-\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_{1,2}(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q-\nu-E_n)\}, \end{aligned} \tag{2·5a}$$

$$\begin{aligned} A^{(3)}(\nu, \pi, \boldsymbol{q}) = \frac{(2\pi)^6 q^{-2}}{4}\left(\frac{E_q}{M}\right)\sum_{M_n}\{&(\boldsymbol{q}|\mathbf{e}_{2}(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=\mathbf{Q}|\mathbf{e}_{1}(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q+\nu-E_n) \\ &-(\boldsymbol{q}|\mathbf{e}_{1}(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|M_n, \boldsymbol{p}_n=-\mathbf{Q}) \\ &\times (M_n, \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_{2}(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)|-\boldsymbol{q})\,\delta(E_q-\nu-E_n)\}, \end{aligned} \tag{2·5b}$$

where $\boldsymbol{j}(x)$ is the current operator: $\Box\boldsymbol{A}_\mu(x) = -\boldsymbol{j}_\mu(x)$, $|\pm\boldsymbol{q}\rangle$ is the state vector for the single proton of momentum $\pm\boldsymbol{q}$, and E_q the proton energy. Now the intermediate states in the sum contribute to $A^{(\lambda)}(\nu, \pi, \boldsymbol{q})$ in the unphysical region as follows (we keep $\nu \geq 0$). *We always neglect all but the leading term in powers of* e^2. Under this approximation we can exclude the possibility of photons being in the intermediate states, which means that M_n consists of two spectra: a point spectrum $M_n = M$, which corresponds to the free proton states, and a continuous spectrum $M+\mu \leq M_n < \infty$, μ being the pion mass, which originates from the states of one real nucleon together with a certain number of pions, nucleon pairs and other particles. Inquiring, for a given ν, those values of M_n that make the argument of $\delta(E_q \pm \nu - E_n)$ vanish, we find that the point spectrum gives contributions at the point $\nu = \nu_b \equiv q^2/E_q$ through the terms multiplied by $\delta(E_q-\nu-E_n)$. The continuous spectrum does not contribute in the region $0<\nu<q$ if $q \leq \left(\frac{M+\mu/2}{M+\mu}\right)\mu$, but does over the interval $\nu_a \leq \nu < q$ through the terms multiplied by $\delta(E_q+\nu-E_n)$ if $\left(\frac{M+\mu/2}{M+\mu}\right)\mu < q \leq (M\mu+\mu^2/2)^{1/2}$, where $\nu_a = (M\mu+\mu^2/2-q^2)/E_q$. If $q > (M\mu+\mu^2/2)^{1/2}$, the continuous spectrum yields further contributions over the interval $0 \leq \nu \leq -\nu_a$ through the terms multiplied by $\delta(E_q-\nu-E_n)$. For simplicity, however, we shall leave the last case out of our consideration.

Expressing the contribution of the free proton states, as distinguished from that of the other intermediate states, as

$$A^{(1,2)}(\nu, \pi, \boldsymbol{q}) = \pi\{\Gamma_N^{(1,2)}(q^2) + i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}(\nu_b)\,\Gamma_S^{(1,2)}(q^2)\}\,\delta(\nu-\nu_b), \tag{2·6a}$$

$$A^{(3)}(\nu, \pi, \boldsymbol{q}) = \pi\{i\boldsymbol{\sigma}\cdot\mathbf{Q}(\nu_b)\,\Gamma_+^{(3)}(q^2) + i\boldsymbol{\sigma}\cdot\boldsymbol{q}\Gamma_-^{(3)}(q^2)\}\,\delta(\nu-\nu_b), \tag{2·6b}$$

we obtain the dispersion relations under certain assumptions on the boundedness of the causal amplitudes :

$$\mathrm{Re}\,\mathcal{M}_N^{(\alpha)}(\nu, q^2) - \mathrm{Re}\,\mathcal{M}_N^{(\alpha)}(q, q^2) = \frac{2\Gamma_N^{(\alpha)}(q^2)\,E_q(\nu^2-q^2)}{M^2(\nu^2-\nu_b^2)}$$
$$+\frac{2(\nu^2-q^2)}{\pi}\int_{\nu_a}^{\infty}\frac{\nu'\mathrm{Im}\,\mathcal{M}_N^{(\alpha)}(\nu', q^2)}{(\nu'^2-q^2)(\nu'^2-\nu^2)}\,d\nu', \tag{2·7}$$

$$\mathrm{Re}\,\mathcal{M}_S^{(\alpha)}(\nu, q^2) = -\frac{2\nu\Gamma_S^{(\alpha)}(q^2)}{\nu^2-\nu_b^2}+\frac{2\nu}{\pi}\int_{\nu_a}^{\infty}\frac{\mathrm{Im}\,\mathcal{M}_S^{(\alpha)}(\nu', q^2)}{\nu'^2-\nu^2}\,d\nu' \tag{2·8}$$

with $\alpha=1, 2.$

$\mathcal{M}_{+,-}^{(3)}(\nu, q^2)$ obeys the same dispersion relations as (2·7), except that $\Gamma_{+,-}^{(3)}(q^2)$ occurs now instead of $\Gamma_N^{(\alpha)}(q^2)$. Explicit calculation of $\Gamma^{(\lambda)}(q^2)$ ($\lambda=1, 2, 3$) is given in Appendix I, where it is shown that these can be completely expressed in terms of the proton mass, charge and static magnetic moment.

We employ the notations $M_c^{(\lambda)}$ ($\lambda=1, 2, 3, 4$) for the four Feynman amplitudes in the barycentric system characterized by the sets of polarizations in Table I, and inquire the transformation formulae between $M_c^{(\lambda)}$ and $M^{(\lambda)}$. It can be proved that photons may be treated in this transformation as if they were spinless particles. Therefore the relation between $M_c^{(\lambda)}$ and $M^{(\lambda)}$ is independent of λ, and is same as in the pion-nucleon scattering[3]. The proof is as follows : Referring to the coordinate system so far used as " q-system ", we introduce four-vectors e_μ and e'_μ, whose components in the q-system are respectively given by

$$e_p = (\mathbf{e}_r+\alpha\boldsymbol{k},\ \alpha k),$$
$$e_\mu' = (\mathbf{e}_s+\alpha'\boldsymbol{k}',\ \alpha' k') \tag{2·9}$$

with

$$\alpha=\frac{(\mathbf{e}_r\cdot\boldsymbol{v})}{k-(\boldsymbol{v}\cdot\boldsymbol{k})} \quad\text{and}\quad \alpha'=\frac{(\mathbf{e}_s\cdot\boldsymbol{v})}{k'-(\boldsymbol{v}\cdot\boldsymbol{k}')},$$

where $\boldsymbol{v}$ is the velocity of the CMS in the q-system. Then, $M^{(\lambda)}$ can be *symbolically* written as

$$M^{(\lambda)}\sim(\boldsymbol{q}|(e_\mu' j_\mu')(e_\mu j_\mu)|-\boldsymbol{q}) \tag{2·10}$$

in virtue of the gauge invariance, which asserts that $e_\mu j_\mu=\mathbf{e}_r\cdot\boldsymbol{j}$, $e_\mu' j_\mu=\mathbf{e}_s\cdot\boldsymbol{j}'$ effectively. Here j_μ, j_μ', $\boldsymbol{j}$, $\boldsymbol{j}'$ stand for the current operator. Applying a unitary transformation R corresponding to the Lorentz transformation from q-system to the CMS, (2·10) becomes

$$M^{(\lambda)}\sim(\boldsymbol{q}|R^{-1}(e_{c\mu}'j_{c\mu}')(e_{c\mu}j_{c\mu})R|-\boldsymbol{q}), \tag{2·11}$$

where the subscript c means that quantities with it refer to the C.M.S. From (2·9) we can easily check that $e_{c4}=0$, and $\mathbf{e}_c$ is a unit vector parallel or perpendicular to the plane, through which the photon is scattered, according as $\mathbf{e}_r$ is parallel or perpendicular to the same plane, and that the same is true for $e_{c\mu}'$. Consequently $M_c^{(\lambda)}$ can be written as

$$M_c^{(\lambda)} \sim (\boldsymbol{p}_c' | (e_{c\mu}' j_{c\mu}') (e_{c\mu} j_{c\mu}) | \boldsymbol{p}_c), \tag{2·12}$$

where $|\boldsymbol{p}_c)$ and $|\boldsymbol{p}_c')$ represent the initial and recoiled proton state, respectively. On comparsion of (2·12) with (2·11), it is inferred that $M^{(\lambda)}$ can be obtained from $M_c^{(\lambda)}$ by means of a Lorentz transformation of the free proton states to which the matrix element $M_c^{(\lambda)}$ is referred, irrespective of the photon polarization. This completes the proof.

For practical applications it is convenient to express $M^{(\lambda)}$ in terms of the scattering amplitude $f^{(\lambda)}$ in the C.M.S., which relates to $M_c^{(\lambda)}$ through $M_c^{(\lambda)}=M^{-1}W_c f^{(\lambda)}$, W_c being a total energy in C.M.S. The relation between $M^{(\lambda)}$ and $M_c^{(\lambda)}$ disputed above immediately gives the ones between $M^{(\lambda)}$ and $f^{(\lambda)}$. We use the notation $\boldsymbol{k}_c$, $\boldsymbol{k}_c'$, ν_c, θ_c, and E_c for the momenta of the incident and scattered photon, the photon energy, the scattering angle, and the proton energy respectively, all referring to the CMS. Then $f^{(\lambda)}$ can be written in the form

$$f^{(1,2)}(\nu_c, \theta_c) = f_N^{(1,2)}(\nu_c, \theta_c) + i\boldsymbol{\sigma}\cdot\boldsymbol{k}_c\times\boldsymbol{k}_c'\nu_c^{-2} f_S^{(1,2)}(\nu_c, \theta_c), \tag{2·13a}$$

$$f^{(3,4)}(\nu_c, \theta_c) = \nu_c^{-1}\left\{ f_+^{(3,4)}(\nu_c, \theta_c)\, i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{k}_c+\boldsymbol{k}_c'}{2} + f_-^{(3,4)}(\nu_c, \theta_c)\, i\boldsymbol{\sigma}\cdot\frac{\boldsymbol{k}_c-\boldsymbol{k}_c'}{2}\right\}. \tag{2·13b}$$

The relations of $M^{(\lambda)}$ to $f^{(\lambda)}$ are given by

$$\mathcal{M}_N^{(1,2)}(\nu, q^2) = \frac{W_c}{M}\left\{\frac{E_q^2+ME_c}{E_q(M+E_c)} f_N^{(1,2)}(\nu_c, \theta_c) - 2\left(\frac{q}{\nu_c}\right)^2 \frac{E_q^2-E_c^2}{E_q(M+E_c)} f_S^{(1,2)}(\nu_c, \theta_c)\right\}, \tag{2·14a}$$

$$\mathcal{M}_S^{(1,2)}(\nu, q^2) = \frac{1}{M(M+E_c)}\left\{-f_N^{(1,2)}(\nu_c, \theta_c) + 2\frac{E_q^2+ME_c}{\nu_c^2} f_S^{(1,2)}(\nu_c, \theta_c)\right\}, \tag{2·14b}$$

$$\mathcal{M}_+^{(3,4)}(\nu, q^2) = \frac{E_q}{M\nu_c} f_+^{(3,4)}(\nu_c, \theta_c), \tag{2·14c}$$

$$\mathcal{M}_-^{(3,4)}(\nu, q^2) = \frac{W_c}{M\nu_c} f_-^{(3,4)}(\nu_c, \theta_c), \tag{2·14d}$$

where ν_c, θ_c are connected with ν, q^2 through

$$\nu = W_c E_c E_q^{-1} - E_q, \tag{2·15}$$

$$2q^2 = \nu_c^2(1-\cos\theta_c). \tag{2·16}$$

Eqs. (2·14 c, d) show that $f_+^{(3)}=f_+^{(4)}$, $f_-^{(3)}=-f_-^{(4)}$. Besides, we note that $f_N^{(1)}$, etc., depend on θ_c through $\cos\theta_c$ alone.

By virtue of eqs. (2·13)—(2·16) dispersion relations (2·7), (2·8) and the ones for $\mathcal{M}_{+,-}^{(3)}$ are all expressed in terms of the scattering amplitudes $f^{(\lambda)}$; the form is most convenient for the practical application. For the sake of getting comparatively simple expressions, we do this only in two ways, one simply taking the limit $q\to 0$, and another taking the limit $q\to 0$ after once differentiating dispersion relations with respect to q^2.

In the limit $q\to 0$, $\mathrm{Re}\,\mathcal{M}^{(\lambda)}(q, q^2)$ are fixed from the low energy theorem, and there occurs nothing unknown in the inhomogenous terms of dispersion relations. We have

$$\mathrm{Re}\, f_N^{(1)}(\nu) = -\frac{e^2}{4\pi M}\frac{1}{\sqrt{1+2\nu/M}} + \frac{2\nu^2}{\pi\sqrt{1+2\nu/M}}\int_{\nu_t}^{\infty} d\nu' \frac{\sqrt{1+2\nu'/M}}{\nu'(\nu'^2-\nu^2)}\,\mathrm{Im}\, f_N^{(1)}(\nu'), \tag{DI}$$

$$\mathrm{Re}\{2f_S^{(1)}(\nu) - \epsilon(\nu) f_N^{(1)}(\nu)\} = \frac{e^2}{4\pi M^2}\frac{\nu}{1+2\nu/M} + \frac{2\nu^3}{\pi(1+2\nu/M)}\int_{\nu_t}^{\infty} d\nu' \frac{1+2\nu'/M}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\{2f_S^{(1)}(\nu') - \epsilon(\nu') f_N^{(1)}(\nu')\}, \tag{DII}$$

$$\mathrm{Re}\{2f_S^{(2)}(\nu) - \epsilon(\nu) f_N^{(2)}(\nu)\} = -\frac{e^2 g_p^{\,2}}{4\pi M^2}\frac{\nu}{1+2\nu/M} + \frac{2\nu^3}{\pi(1+2\nu/M)}\int_{\nu_t}^{\infty} d\nu' \frac{1+2\nu'/M}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\{2f_S^{(2)}(\nu') - \epsilon(\nu') f_S^{(2)}(\nu')\}, \tag{DIII}$$

$$\mathrm{Re}\, f_+^{(3)}(\nu) = -\frac{e^2}{8\pi M^2}(g_p-1)^2\frac{\nu}{\sqrt{1+2\nu/M}} + \frac{2\nu^3}{\pi\sqrt{1+2\nu/M}}\int_{\nu_t}^{\infty} \frac{\sqrt{1+2\nu'/M}}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\, f_+^{(3)}(\nu')\, d\nu', \tag{DIV}$$

$$\mathrm{Re}\, f_-^{(3)}(\nu) = \frac{e^2}{8\pi M^2}(g_p^{\,2}-1)\frac{\nu}{1+2\nu/M} + \frac{2\nu^3}{\pi(1+2\nu/M)}\int_{\nu_t}^{\infty} d\nu' \frac{1+2\nu'/M}{\nu'^2(\nu'^2-\nu^2)}\,\mathrm{Im}\, f_-^{(3)}(\nu'), \tag{DV}$$

where

$$\nu_t = \mu + \mu^2/2M,$$

$$\epsilon(\nu) = (\nu/M)^2(1+2\nu/M)^{-1/2}[1+\nu/M+\sqrt{1+2\nu/M}]^{-1},$$

$$f^{(\lambda)}(\nu) \equiv f^{(\lambda)}(\nu_c = \nu/\sqrt{1+2\nu/M},\ \theta_c = 0).$$

Evidently $f_N^{(1)} = -f_N^{(2)}$ in the limit $q \to 0$, and hence the dispersion relation for $f_N^{(2)}$ is equivalent to (DI). It is easily seen that (DI) and (DIV) are identical with the two dispersion relations derived by *GGT*, respectively for the non-spin-flip and spin-flip part of the forward scattering amplitude. It should be noted further that (DII), (DIII) and (DV) are not independent because of the relation

$$f_S^{(1)}(\nu) + f_S^{(2)}(\nu) = -f_-^{(3)}(\nu),$$

which we have proved by means of the partial wave expansion without any approximation.

The differentiations with respect to q^2 can be done most conveniently, if the independent variables of (2·7, 8) are changed from ν, ν', q^2 to ν_c, ν_c', q^2 by making use of (2·15, 16). Then, by successive differentiations with respect to q^2, we get dispersion

relations for $[\partial f^{(\lambda)}(\nu_c, \theta_c)/\partial \cos\theta_c]_{\cos\theta_c=1}$, $[\partial^2 f^{(\lambda)}(\nu_c, \theta_c)/\partial(\cos\theta_c)^2]_{\cos\theta_c=1}$, etc. The procedure is rather cumbersome, so we treat $\mathcal{M}_N^{(1,2)}(\nu, q^2)$ alone, expanding all quantities in powers of ν/M, ν'/M, and neglecting all but the lowest and second order terms. The result can be put in the form:

$$\begin{aligned}\mathrm{Re}\,\dot{f}_N^{(\alpha)}(\nu) =& \left(1-\frac{3\nu}{M}\right)\left\{\frac{\Gamma_N^{(\alpha)}(0)}{M}-\nu^2\left(\frac{\Delta_N^{(\alpha)}}{2}+\frac{\dot{\Gamma}_N^{(\alpha)}}{M}\right)\right\}\\ &+\frac{2\nu^4}{\pi}\int_{\nu_t}^{\infty}d\nu'\left(1+3\frac{\nu'-\nu}{M}\right)\frac{\mathrm{Im}\,\dot{f}_N^{(\alpha)}(\nu')}{\nu'^3(\nu'^2-\nu^2)}\\ &+\frac{\nu^2}{\pi}\int_{\nu_t}^{\infty}\frac{d\nu'}{\nu'^3}\left\{1+\frac{2\nu'-3\nu}{M}-\frac{\nu'^3}{M(\nu'+\nu)^2}\right\}\mathrm{Im}\,f_N^{(\alpha)}(\nu')\end{aligned} \tag{DVI}$$

$$\text{with}\quad \alpha=1,\ 2,$$

where

$$\dot{f}_N^{(\alpha)}(\nu)=[\partial f_N^{(\alpha)}(\nu_c=\nu-\nu^2/M,\ \theta_c)/\partial\cos\theta_c]_{\cos\theta_c=1},$$

$$\Delta_N^{(\alpha)}=[\partial\,\mathrm{Re}\,\mathcal{M}_N^{(\alpha)}(q, q^2)/\partial q^2]_{q=0},$$

$$\dot{\Gamma}_N^{(\alpha)}=[\partial\Gamma_N^{(\alpha)}(q^2)/\partial q^2]_{q=0}.$$

$\Gamma_N^{(\alpha)}(0)$, $\dot{\Gamma}_N^{(\alpha)}$ are immediately obtained from Appendix I. The result is that

$$\Gamma_N^{(1)}(0)=-e^2/4\pi, \tag{2·17}$$

and the others are all negligible to the present approximation. However, $\Delta_N^{(\alpha)}$ remain unknown. For the present we have no reliable method to evaluate them theoretically. It is impossible to continue the procedure of q^2-differentiation to obtain new dispersion relations without being worried by further appearance of such unknown constants. This is the main reason why we have ceased to repeat the procedure more than once.

The right-hand side of each of the dispersion relations thus obtained is fixed up to the terms linear in ν, independent of $\Delta_N^{(\alpha)}$ and the imaginary part of amplitudes, while the left-hand side is fixed to the same order from the low energy theorem.[4] Therefore, we can examine the consistency of our dispersion relations to the first order in ν, and thereby get a justification for our method of the evaluation of the unphysical region contributions. In fact, according to the low-energy theorem, we have, to the first order in ν,

$$\begin{aligned}&f_N^{(1)}(\nu_c, \theta_c)=\frac{e^2}{4\pi M}\left\{-\cos\theta_c+\frac{\nu_c}{M}(-1+\cos\theta_c+\cos^2\theta_c)\right\},\\ &f_N^{(2)}(\nu_c, \theta_c)=\frac{e^2}{4\pi M}\left(1-\frac{\nu_c}{M}\right),\\ &f_S^{(1)}(\nu_c, \theta_c)=\frac{e^2}{8\pi M}\frac{\nu_c}{M},\quad f_S^{(2)}(\nu_c, \theta_c)=-\frac{e^2}{8\pi M}\frac{\nu_c}{M}g_p^2,\end{aligned} \tag{2·18}$$

$$f_{+}^{(3)}(\nu_c, \theta_c) = -\frac{e^2}{8\pi M}\frac{\nu_c}{M}(g_p{}^2 - g_p + 1 - g_p \cos\theta_c),$$

$$f_{-}^{(3)}(\nu_c, \theta_c) = \frac{e^2}{8\pi M}\frac{\nu_c}{M}(g_p{}^2 + g_p - 1 - g_p \cos\theta_c).$$

The consistency of these with the dispersion relations can be easily checked.

This section will be concluded with the partial-wave expansion of $f^{(\lambda)}(\nu_c, \theta_c)$. The exact expressions can be obtained from (A II–3) in Appendix II. Here we neglect, however, all the electric multipole amplitudes of the order higher than 2, all the magnetic multipole amplitudes of the order higher than 1, and all the amplitudes mixing electric and magnetic multipole waves except for $a_{3/2}$ $(E1, M1)$ and $a_{3/2}(E2, M1)$, because this approximation will be sufficient for the application in the next section. Then we have from (A II–3)

$$\begin{aligned}
f_N^{(1)}(\nu_c, \theta_c) &= e_N \cos\theta_c + m_N + e_N'(2\cos^2\theta_c - 1),\\
f_N^{(2)}(\nu_c, \theta_c) &= -e_N - (m_N + e_N')\cos\theta_c,\\
f_S^{(1)}(\nu_c, \theta_c) &= (1/2)(e_S + 4e_S'\cos\theta_c - c_1),\\
f_S^{(2)}(\nu_c, \theta_c) &= -(1/2)(m_S + e_S' - c_2),\\
f_{+}^{(3)}(\nu_c, \theta_c) &= -(1/2)[e_S + m_S - e_S' + (2e_S' + c_1 + c_2)\cos\theta_c],\\
f_{-}^{(3)}(\nu_c, \theta_c) &= -(1/2)[e_S - m_S + e_S' + (2e_S' - c_1 + c_2)\cos\theta_c].
\end{aligned} \tag{2·19}$$

The notations used here will be explained in Appendix II. It is found that, to this approximation, the undetermined constants $A_N^{(\alpha)}$ coincide with each other:

$$A_N^{(1)} = A_N^{(2)} \equiv A. \tag{2·20}$$

In fact, on account of (2·15) and (2·16), (2·14a) gives

$$\mathcal{M}_N^{(\alpha)}(q, q^2) = M^{-1}(E_q + q) f_N^{(\alpha)}(q, \pi).$$

Since $f_N^{(1)} = f_N^{(2)}$ for $\theta_c = \pi$ because of (2·19), it is evident from the definition of $A_N^{(\alpha)}$ that (2·20) holds.

§ 3. Calculation of the cross sections

In this section we calculate the cross sections at energies between 180 and 320 Mev, applying the dispersion relations derived in the preceding section. We assume that the imaginary parts of the amplitudes at such high energies, where multiple photopion production is significant, do not contribute to the integrals appearing in the dispersion relations. In virtue of the unitarity of S matrix, the imaginary part of a partial wave amplitude with definite angular momentum and parity is related to the partial wave amplitudes with the same angular momentum and parity for photopion production, if the multiple photopion production is neglected. The relations will be derived in Appendix II. We further assume that photopions are produced only in S and P states, and that S-wave pion is

produced by electric dipole radiation into positive charge state with amplitude E_d^+, and P-wave pion by magnetic dipole and electric quadrupole radiations into the $(3-3)$ state with amplitudes M_{33} and E_{33} respectively. Then, as is shown in Appendix II, we have

$$\begin{aligned} &\text{Im } e_{1/2}=2\nu_c|E_d^+|^2, \quad \text{Im } m_{3/2}=12\nu_c M_{33}^2, \\ &\text{Im } e'_{3/2}=\nu_c E_{33}^2, \quad \text{Im } c_2=12\nu_c M_{33}E_{33}, \end{aligned} \tag{3·1}$$

all the remaining imaginary parts being zero. For the experimental values of E_d^+ we use those given by Watson et al.[6] These authors have given values of M_{33} and E_{33} also. According to recent investigations,[7] however, it seems that the values of E_{33} are still not decisive. We therefore put

$$E_{33}=cM_{33} \tag{3·2}$$

and make calculations for two choices $c=1$ and $c=0$. If the values in reference 6) were adopted, we would have $c\approx 1$. We determine values of M_{33} by means of

$$\sigma_0=32\pi M_{33}^2, \tag{3·3}$$

where σ_0 denotes the total cross section for neutral photopion production. Strictly speaking, a contribution of E_{33} should be added to the right-hand side, but this is so small that we have neglected it. We neglect Im $e'_{3/2}$ since this is much smaller than either of Im $m_{3/2}$ and Im c_2.

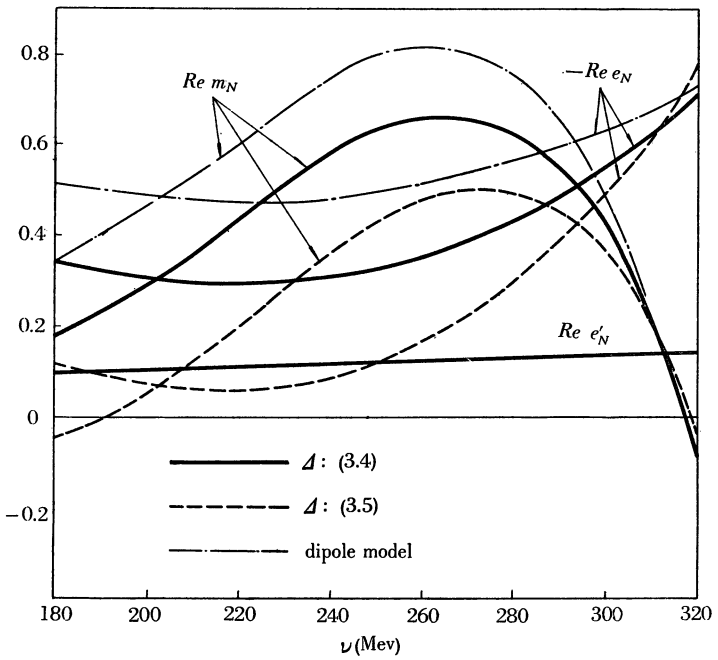

Fig. 1. Real parts of e_N, m_N and e_N' in units of $e^2/4\pi M$

To determine the real parts through the dispersion relations, we neglect the higher partial waves as was mentioned at the end of the preceding section. Then real parts of the non-spin-flip amplitudes e_N, m_N and e_N' are determined by (DI) and (DVI), if the constant Δ is known. It is easily seen that $\mathrm{Re}\,(e_N+m_N)$ and $\mathrm{Re}\,e_N'$ do not depend on Δ, but only $\mathrm{Re}\,(e_N-m_N)$ does. On the other hand, calculations show that $\mathrm{Re}\,(e_N+m_N)$ is very small owing to cancellation of its value at low energy limit by the imaginary part integral. Therefore, for appropriate values of Δ, both $\mathrm{Re}\,e_N$ and $\mathrm{Re}\,m_N$ become very small. This is important, since it will be found later that the smallness of $\mathrm{Re}\,e_N$ and $\mathrm{Re}\,m_N$ is at any rate necessary for getting agreement with experiments. We have estimated Δ by means of perturbation theory,[8] obtaining

$$\Delta = -0.42\,\mu^{-2}(e^2/4\pi M). \tag{3·4}$$

If we determine Δ so as to make the 90° cross section at 190 Mev as small as possible, we obtain

$$\Delta = -1.07\,\mu^{-2}(e^2/4\pi M). \tag{3·5}$$

We calculate $\mathrm{Re}\,e_N$, $\mathrm{Re}\,m_N$ and $\mathrm{Re}\,e_N'$ for these two values of Δ. The results are shown in Fig. 1. We note that contribution of the imaginary part integral to $\mathrm{Re}\,e_N'$ is very small.

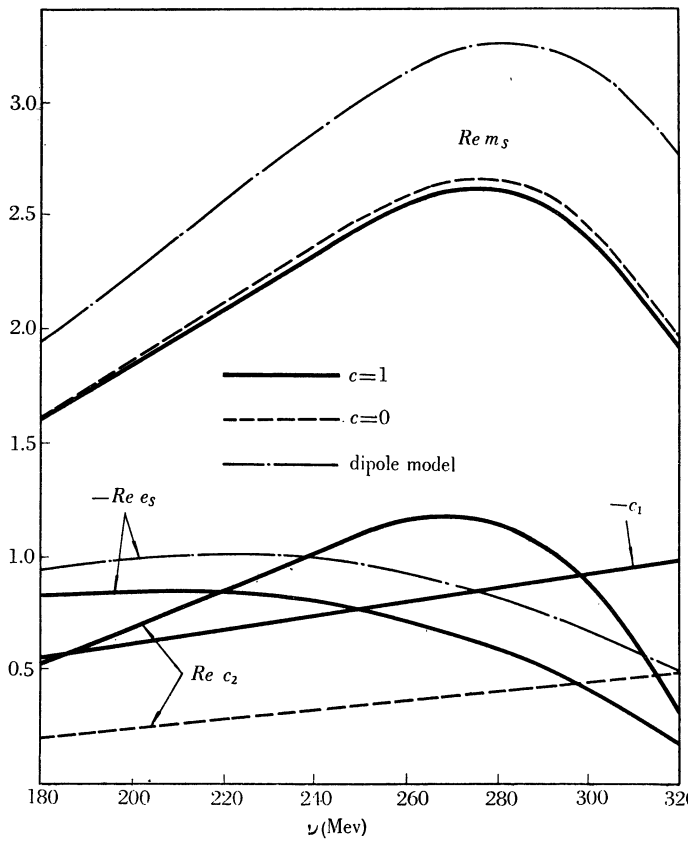

Fig. 2. Real parts of c_1, e_S, m_S and c_2 in units $e^2/4\pi M$

Next we calculate real parts of the spin-flip amplitudes. In view of the smallness of contribution of the imaginary part integral to Re e_N', it is natural to assume that the amplitudes with vanishing imaginary parts are unaffected by the meson effect. Therefore we replace c_1 and e_S' by their respective expressions given by the low energy theorem :

$$c_1 = -g_p(\nu/M)(e^2/4\pi M), \quad e_S' = 0. \tag{3·6}$$

Then Re e_S, Re m_S and Re c_2 are calculated by the use of (DII), (DIII), and (DIV). The results are shown in Fig. 2.

The largeness of Re m_S is mainly due to the large magnetic moment of proton. The peak in Re m_S at 270 Mev is due to the imaginary part integral. The imaginary part integral containing Im $m_{3/2}$ changes its sign at about the resonance energy for photo-pion production, and multiplication of this by ν^3 produces a peak blow the resonance energy. It is likely that experiments on the cross sections at 90° C.M.S. show a peak slightly below the resonance energy, and to explain this the peak in Re m_S is important. From comparison of Fig. 1 with Fig. 2, we observe that the non-spin-flip amplitudes are much smaller than the spin flip ones in the real parts.

In terms of the partial wave amplitudes, the differential cross section in C.M.S. is written as

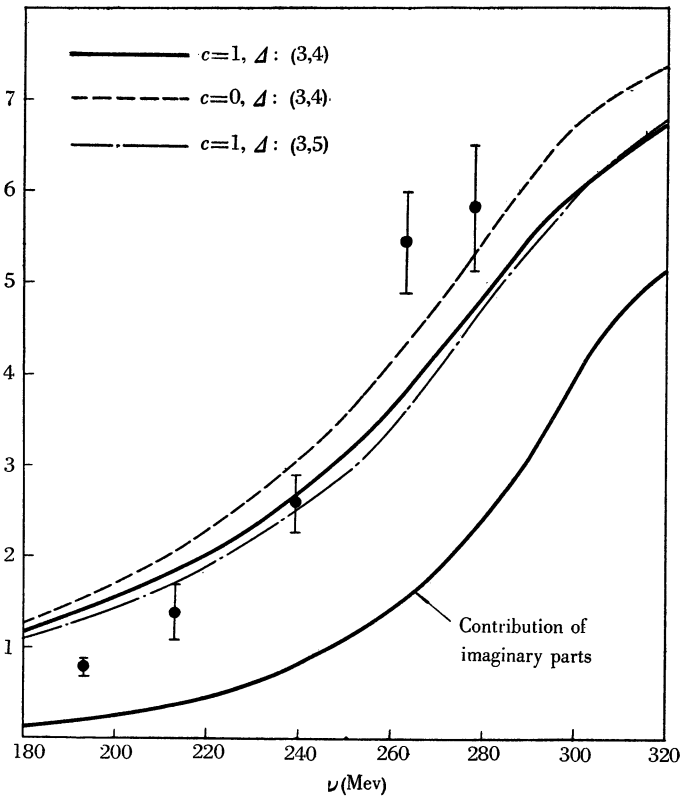

Fig. 3. Cross sections at 90° C.M.S. in units of $(e^2/4\pi M)^2$/sterad

$$\frac{d\sigma}{d\Omega}=\tfrac{1}{2}|e_N|^2+\tfrac{1}{2}|m_N-e_N'|^2+\tfrac{1}{4}|e_S|^2+\tfrac{1}{4}|m_S|^2+\tfrac{1}{8}|e_S-c_1|^2+\tfrac{1}{8}|m_S-c_2|^2$$
$$+\tfrac{1}{2}\mathrm{Re}\{m_S^*(e_S+c_1)+c_2{}^*e_S+4e_N^*m_N\}\cos\theta_c$$
$$+\{\tfrac{1}{8}|e_S+c_1|^2+\tfrac{1}{8}|m_S+c_2|^2+\tfrac{1}{4}|c_1|^2+\tfrac{1}{4}|c_2|^2-\tfrac{1}{4}|e_S-c_1|^2 \qquad (3\cdot7)$$
$$-\tfrac{1}{4}|m_S-c_2|^2+\tfrac{1}{2}|e_N|^2+\tfrac{1}{2}|m_N|^2-\tfrac{3}{2}|e_N'|^2+3\mathrm{Re}\,m_N^*e_N'\}\cos^2\theta_c$$
$$+\tfrac{1}{2}\mathrm{Re}\,(c_1{}^*c_2+4e_N^*e_N')\cos^3\theta_c+2|e_N'|^2\cos^4\theta_c\,.$$

Using this, we calculate the differential cross section at 90° C.M.S. as a function of energy, and angular distributions at 200, 260 and 320 Mev. The results are shown in Fig. 3 and Fig. 4. In Fig. 3 we also show experimental data given by Yamagata et al.[9] From Fig. 3 we observe that the two values of Δ (3·4) and (3·5) yield almost same results. This means that real parts of the non-spin-flip amplitudes have been made so small already by the choice of (3·4) that their contribution to the cross section is very small. Indeed main contributions to the results in Fig. 3 come from Re m_S and the imaginary parts. In Fig. 4, main contributions to the results come from the real parts of the amplitudes at 200 and 260 Mev. We observe that, at these energies, the cross section is larger in backward directions than in forward. This is due to the opposite signs of Re m_S and Re (e_S+c_1). For $c=1$, the largeness of Re c_2 makes the term in $\cos^2\theta_c$

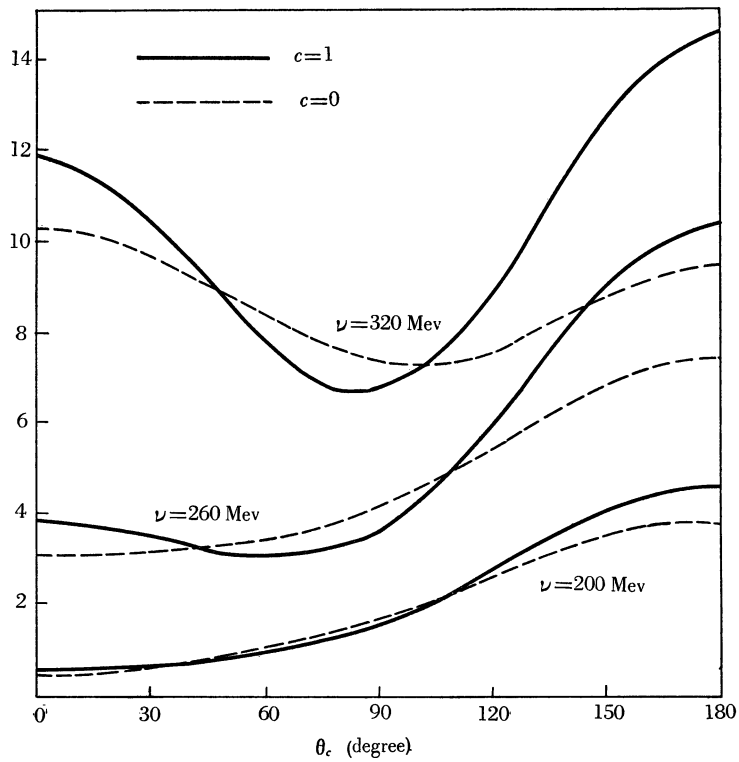

Fig. 4. Angular distributions in units of $(e^2/4\pi M)^2$/sterad calculated with $\Delta=-0.42\ \mu^{-2}(e^2/4\pi M)$

in (3·7) large, and results in the angular distributions rather indented in the neighbourhood of 90°. At 320 Mev, main contributions come from the imaginary parts. Dropping the real parts entirely, (3·7) becomes

$$d\sigma/d\Omega = \begin{cases} \frac{1}{2}\mathrm{Im}\, e_{1/2}^{\,2} + \frac{3}{4}\mathrm{Im}\, m_{1/2}^{\,2} + \frac{5}{4}\mathrm{Im}\, m_{3/2}^{\,2} \cos^2\theta_c & \text{for } c=1, \\ \frac{1}{2}\mathrm{Im}\, e_{1/2}^{\,2} + \frac{7}{8}\mathrm{Im}\, m_{3/2}^{\,2} + \frac{1}{2}\mathrm{Im}\, e_{1/2} m_{3/2} \cos\theta_c + \frac{3}{8}\mathrm{Im}\, m_{3/2}^{\,2} \cos^2\theta_c & \text{for } c=0. \end{cases}$$

These expressions explain general features of the results at 320 Mev in Fig. 4.

Yamagata et al. have given a few data on the cross sections at 70° and 130° also. From these data, the ratio of the cross section at 130° to the one at 90° is estimated to be about 1.7 at 200 Mev and about 2 at 240 Mev. Our results are in good agreement with these values for $c=1$, but the agreement is rather bad for $c=0$. The data at 70° shows a cross section at this angle larger than at 90° at 226 Mev. This cannot be understood by our calculation.

§ 4. Discussions

We consider the difference between the cross section at 90° and contribution of the imaginary parts to it. Experimentally, this is considerably smaller than $(e^2/4\pi M)^2$ at energies near 200 Mev, then increases with energy, and reaches a maximum or a plateau at about 270 Mev. These features are explained roughly by the smallness of $\mathrm{Re}\, e_N$ and $\mathrm{Re}\, m_N$ and the behavior of $\mathrm{Re}\, m_S$ as a function of energy mentioned in the preceding section. However, there remain considerable discrepancies between theory and experiment, namely, theory predicts too large cross sections in the neighborhood of 200 Mev, and rather small ones at energies near 270 Mev, as compared with experiments. In the following, we will look for a possible origin of these discrepancies. First, errors in the data on photopion production come into mind. Indeed, if we consider the cross section at a particular energy, the errors produce a considerable ambiguity in the calculated value of it. However, in view of the rather systematic character of the above discrepancies, it is unlikely that these discrepancies are entirely attributable to these errors over the whole energy range considered. Second, we have neglected imaginary parts of the partial wave amplitudes in the high energy region. Nowadays, data on photon-proton collisions in Bev region are not available, and even if they become available, the partial wave analysis of them will be extremely difficult. Therefore there is no possibility of obtaining experimental values of the imaginary parts at such high energies. However, contributions of such a high energy region to the imaginary part integrals will be almost independent of energy ν if $\nu \lesssim 300$ Mev. Therefore the effect of including the high energy contributions is that a constant is added to each of the imaginary part integrals. We have attempted to fit the energy dependence of the 90° cross section to experiments by adjusting the above constants, and found that the fit cannot be achieved unless some of the constants take unreasonably large values. Thirdly, we have neglected the amplitudes for photopion other than E_d^+, M_{33} and E_{33}. We have calculated the amplitude $M_d(1/2)$ (See (A II–12).), using the meson theory to the lowest order perturbation approximation, and estimated

Im $m_{1/2}$. We have, however, found that the inclusion of this only makes the 90° cross section slightly smaller over the whole energy range considered, making the discrepancies in the neighborhood of 270 Mev larger, and that, even at energies near 200 Mev, the decrement is too small to explain the discrepancies. Finally, we have used the approximate expressions (3·6) for c_1 and e_S'. It should be noted that these expressions are correct only to the first order in ν/M, and that changes in c_1 and e_S' produce the one in Re m_S, the largest real part. A quantity of the order $(\nu/M)^2$ cannot be disregarded even in the neighborhood of 200 Mev, when this is multiplied by a rather large numerical factor, and interferes with Re m_S. To obtain the expressions for c_1 and e_S' exact within the approximation to which the partial wave amplitudes other than those considered so far are all neglected, we must derive two new dispersion relations for the spin-flip amplitudes. As was mentioned in § 2, these dispersion relations will contain new adjustable parameters of the same kind as Δ. It is not unexpected that the use of the expressions for c_1 and e_S' thus obtained, on adjusting the parameters appropriately, remove the discrepancies mentioned above.

Finally we will examine the validity of the dipole model. Following the assumptions of this model, we replace the partial wave amplitudes other than $e_{1/2}$ and $m_{3/2}$ by their respective expressions given by the low energy theorem. Thus we have, in addition to (3·6),

$$2e_N+e_S=\frac{e^2}{4\pi M}\left\{-2+\frac{\nu}{M}(3-g_p)\right\},$$

$$m_N-m_S=-\frac{e^2}{4\pi M}\frac{\nu}{M}(g_p{}^2+\tfrac{1}{2}),$$

$$e_N'=\frac{e^2}{4\pi M}\cdot\frac{\nu}{2M}, \quad c_2=0.$$

Then we calculate real parts of e_N, e_S, m_N and m_S, using only two dispersion relations (DI) and (DIV), neglecting Im c_2. The results are plotted in Fig. 1 and Fig. 2. We observe that individual amplitude in the dipole model is considerably larger than that calculated in § 3, while relative magnitudes of the amplitudes, as well as their energy dependences, are not so different from the results in § 3. Therefore the dipole model will predict energy and angular dependences of the cross section similar to the results for $c=0$ in the preceding section. But it predicts cross sections much larger than the results in the preceding section, making the agreement with experiment worse.

In conclusion, the authors express their thanks to Professor K. Nakabayasi for his kind interest taken in this work.

Appendix I

Calculation of contributions of the unphysical region

Here we concern ourselves with the derivation of explicit expressions for $I'^{(\lambda)}(q^2)$ defined in (2·6a, b). The contributions to $A^{(\lambda)}(\nu, \pi, \boldsymbol{q})$ of the intermediate free proton

state, which are separated off from the sum (2·5a, b), are given by

$$\begin{aligned} A^{(1,2)}(\nu, \pi, \boldsymbol{q}) &= -\frac{(2\pi)^6 q^{-2}}{4}\left(\frac{M}{E_q}\right)(\boldsymbol{q}|\mathbf{e}_{1,2}(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)\,|M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}) \\ &\quad\times(M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_{1,2}(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)\,|-\boldsymbol{q})\,\delta(\nu-\nu_b), \\ A^{(3)}(\nu, \pi, \boldsymbol{q}) &= -\frac{(2\pi)^6 q^{-2}}{4}\left(\frac{M}{E_q}\right)(\boldsymbol{q}|\mathbf{e}_1(\nu, \pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)\,|M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}) \\ &\quad\times(M_n=M,\ \boldsymbol{p}_n=-\mathbf{Q}|\mathbf{e}_2(\nu, -\pi, \boldsymbol{q})\cdot\boldsymbol{j}(0)\,|-\boldsymbol{q})\,\delta(\nu-\nu_b), \end{aligned} \tag{AI–I}$$

where $\mathbf{Q}$ depends on ν through eq. (2·1). Calculation can be performed with the aid of the well-known expression for the matrix elements of the current operator referred to the free proton states:

$$(\boldsymbol{p}'|j_\mu(0)\,|\boldsymbol{p}) = (2\pi)^{-3}\bar{u}(\boldsymbol{p}')\left\{\{ie\gamma_\mu F[(\Delta p)^2]-\frac{ie}{2M}G[(\Delta p)^2]\sigma_{\mu\nu}(\Delta p)_\nu\right\}u(\boldsymbol{p}). \tag{AI–2}$$

Here $\bar{u}(\boldsymbol{p}')$ and $u(\boldsymbol{p})$ denote the Dirac spinors which are normalized by the condition $\bar{u}(\boldsymbol{p}')\beta u(\boldsymbol{p})=\boldsymbol{l}$, where $\boldsymbol{l}$ is a unit matrix in spin space. Besides, $(\Delta p)_\nu=p_\nu'-p_\nu$; $p_\nu=(\boldsymbol{p}, E_p)$, $p_\nu'=(\boldsymbol{p}', E_{p'})$, and $F(0)=1$, $G(0)=g_\mathrm{p}$, the proton magnetic moment in units of the nuclear magneton.

(AI–2) is now substituted in (AI–I) with $\{\boldsymbol{p}', \boldsymbol{p}\}$ identified in turn with $\{\boldsymbol{q}, -\mathbf{Q}_\mathrm{b}\}$, $\{-\mathbf{Q}_\mathrm{b}, -\boldsymbol{q}\}$, $\{\boldsymbol{q}, -\mathbf{Q}_\mathrm{b}\}$, $\{-\mathbf{Q}_\mathrm{b}, -\boldsymbol{q}\}$, where $\mathbf{Q}_\mathrm{b}=i(qM/E_q)\pi$, *an imaginary momentum*. Regardless of the assignment of $\{\boldsymbol{p}', \boldsymbol{p}\}$, it is found that

$$(\Delta p)^2=0. \tag{AI–3}$$

Therefore, $F[(\Delta p)^2]$, $G[(\Delta p)^2]$ in (AI–2) may be replaced by 1, g_p respectively. This shows that the contributions under consideration can be expressed in terms of the proton mass charge and static magnetic moment only.

After rather lengthy calculations we find

$$\begin{aligned} A^{(1)}(\nu, \pi, \boldsymbol{q}) &= -\frac{\pi}{2}\left(\frac{e^2}{4\pi}\right)\left\{1+\left(\frac{M}{E_q}\right)^2+\frac{i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}_\mathrm{b}}{ME_q}\right\}\delta(\nu-\nu_b), \\ A^{(2)}(\nu, \pi, \boldsymbol{q}) &= -\frac{\pi}{2}\left(\frac{e^2}{4\pi}\right)g_\mathrm{p}^2\left\{\left(\frac{q}{E_q}\right)^2-\frac{i\boldsymbol{\sigma}\cdot\boldsymbol{q}\times\mathbf{Q}_\mathrm{b}}{ME_q}\right\}\delta(\nu-\nu_b), \\ A^{(3)}(\nu, \pi, \boldsymbol{q}) &= \frac{\pi}{2}\left(\frac{e^2}{4\pi}\right)\frac{g_\mathrm{p}}{M}\left\{i\boldsymbol{\sigma}\cdot\mathbf{Q}_\mathrm{b}-\frac{M}{E_q}i\boldsymbol{\sigma}\cdot\boldsymbol{q}\right\}\delta(\nu-\nu_b). \end{aligned} \tag{AI–4}$$

On comparison with (2·6a, b) we arrive at the final results:

$$\begin{aligned} &\Gamma_N^{(1)}(q^2) = -\frac{e^2}{8\pi}\frac{E_q^2+M^2}{E_q^2}, \qquad \Gamma_N^{(2)}(q^2) = -\frac{(eg_\mathrm{p})^2}{8\pi}\left(\frac{q}{E_q}\right)^2, \\ &\Gamma_S^{(1)}(q^2) = -\frac{e^2}{8\pi}\frac{1}{ME_q}, \qquad \Gamma_S^{(2)}(q^2) = \frac{(eg_\mathrm{p})^2}{8\pi}\frac{1}{ME_q}, \\ &\Gamma_+^{(3)}(q^2) = \frac{e^2}{8\pi}\frac{g_\mathrm{p}}{M}, \qquad \Gamma_-^{(3)}(q^2) = -\frac{e^2}{8\pi}\frac{g_\mathrm{p}}{E_q}. \end{aligned} \tag{AI–5}$$

Appendix II

Properties of the partial wave amplitudes resulting from the invariance under time reflection and the unitarity of S matrix

Throughout this appendix, we take the barycentric system, and consider the total energy to be fixed to a definite value. Let $|\boldsymbol{\kappa}, \mathbf{e}, \alpha)$ denote the state vector representing the linear momentum state of a free photon propagating in the direction of unit vector $\boldsymbol{\kappa}$ with polarization $\mathbf{e}$ and a free proton in the spin state α, and let $|j, m, El)$ ($|j, m, Ml)$) represent the state with total angular momentum j and its z-component m in which the photon is in the electric (magnetic) multipole state of order l. Inner products of these state vectors are written as

$$\begin{aligned}(j, m, El|\boldsymbol{\kappa}, \mathbf{e}, \alpha) &= \sum_{m'=\pm 1/2} C_{l,\,1/2}(j, m\,;\, m-m', m') \\ &\quad \times (\boldsymbol{\kappa}\times\mathbf{e})\cdot X^*_{l,m-m'}(\boldsymbol{\kappa})\,(\chi^*_{m'}\,\chi_\alpha), \\ (j, m, Ml|\boldsymbol{\kappa}, \mathbf{e}, \alpha) &= i\sum_{m'=\pm 1/2} C_{l,\,1/2}(j, m\,;\, m-m', m') \\ &\quad \times \mathbf{e}\cdot X^*_{l,m-m'}(\boldsymbol{\kappa})\,(\chi^*_{m'}\,\chi_\alpha),\end{aligned} \tag{AII–1}$$

where χ_m denotes the spin eigenstate with magnetic quantum number m, $C_{l,\,1/2}(j, m\,;\, m-m', m')$ the Clebsch-Gordon coefficient, and $X_{l,m}(\boldsymbol{\kappa})$ the vector spherical harmonics.[10] One can easily verify the orthogonality condition

$$\begin{aligned}\int d\Omega_{\boldsymbol{\kappa}} \sum_{\alpha,\mathbf{e}} (j, m, Pl|\boldsymbol{\kappa}, \mathbf{e}, \alpha)\,(\boldsymbol{\kappa}, \mathbf{e}, \alpha|j', m', P'l') \\ = \delta_{jj'}\delta_{mm'}\delta_{ll'}\delta_{PP'},\end{aligned} \tag{AII–2}$$

where the integration is to be performed over all directions of κ, and P and P' represent either of E and M. Owing to the invariance under spatial rotation and reflection, the scattering amplitude for the transition from state $|\boldsymbol{\kappa}, \mathbf{e}, \alpha)$ to $|\boldsymbol{\kappa}', \mathbf{e}', \beta)$ is written, in terms of the partial wave amplitudes $a_j(P'l', Pl)$ in the following form:

$$\begin{aligned}&f(\boldsymbol{\kappa}', \mathbf{e}', \beta\,;\, \boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &= 4\pi\sum_{l=1}^{\infty}\Big\{\sum_{j=l\pm 1/2}\sum_{m=-j}^{j}\sum_{P} a_j(Pl, Pl) \\ &\quad \times (\boldsymbol{\kappa}', \mathbf{e}', \beta|j, m, Pl)\,(j, m, Pl|\boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &\quad + \sum_{l'=l\pm 1}\sum_{m=-(l+l')/2}^{(l+l')/2}\sum_{P\neq P'} a_{(l+l')/2}(P'l', Pl) \\ &\quad \times (\boldsymbol{\kappa}', \mathbf{e}', \beta|(l+l')/2, m, P'l')\,((l+l')/2, m, Pl|\boldsymbol{\kappa}, \mathbf{e}, \alpha)\Big\}.\end{aligned} \tag{AII–3}$$

Now, from the invariance under time reflection, it follows that

$$\begin{aligned}&f(\boldsymbol{\kappa}', \mathbf{e}', \beta\,;\, \boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &= \sum_{\alpha'\beta'}\sum_{\mathbf{e}''\mathbf{e}'''} (\chi^*_\beta\sigma_y\chi^*_{\beta'})\,(\mathbf{e}'\cdot\mathbf{e}''') \\ &\quad \times f(-\boldsymbol{\kappa}, \mathbf{e}'', \alpha'\,;\, -\boldsymbol{\kappa}', \mathbf{e}''', \beta')\,(\mathbf{e}''\cdot\mathbf{e})\,(\chi_{\alpha'}\sigma_y\chi_\alpha).\end{aligned} \tag{AII–4}$$

On the other hand, it is shown by an elementary calculation that from (AII–1) follows

$$\sum_{\beta'\mathbf{e}'''} (\chi^*_\beta \sigma_y \chi^*_{\beta'}) (\mathbf{e}'\cdot\mathbf{e}''') (j, m, Pl|-\boldsymbol{\kappa}', \mathbf{e}''', \beta') = -i(-1)^{j+m}(\boldsymbol{\kappa}', \mathbf{e}', \beta|j, m, Pl). \tag{AII–5}$$

Substituting (AII–3) in (AII–4), and using (AII–5) and its complex conjugate, we obtain

$$a_j(Ej\pm 1/2, Mj\mp 1/2) = a_j(Mj\mp 1/2, Ej\pm 1/2). \tag{AII–6}$$

Next, we consider a system of a free pion and a free nucleon with total charge unity. Let the state of this system, in which the pion propagates in the direction of unit vector $\boldsymbol{l}$ and the nucleon is in the spin state α, be represented by the state vector $|\boldsymbol{l}, Q, \alpha)$, and let the state in which the pion has orbital angular momentum l and total angular momentum and its z-component are j and m respectively be represented by $|j, m, Ql)$, where Q is charge of the pion. Then, similarly to (AII–1), we have

$$(j, m, Ql|\boldsymbol{l}, Q, \alpha) = \sum_{m'=\pm 1/2} C_{l,\,1/2}(j, m: m-m', m') \times Y^*_{l,m-m'}(\boldsymbol{l}) (\chi^*_{m'}, \chi_\alpha), \tag{AII–7}$$

which satisfies an orthogonality condition analogous to (AII–2). The photopion production amplitude for the transition from $|\boldsymbol{\kappa}, \mathbf{e}, \alpha)$ to $|\boldsymbol{l}, Q, \beta)$ is written in the form

$$\begin{aligned} &g(\boldsymbol{l}, Q, \beta\,;\, \boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &= 4\pi \sum_{l=1}^{\infty} \Big\{ \sum_{j=l\pm 1/2} \sum_{m=-j}^{j} b_j(Ql, Ml) \\ &\quad \times (\boldsymbol{l}, Q, \beta|j, m, Ql) (j, m, Ml|\boldsymbol{\kappa}, \mathbf{e}, \alpha) \\ &\quad + \sum_{l'=l\pm 1} \sum_{m=-(l+l')/2}^{(l+l')/2} b_{(l+l')/2}(Ql', El) \\ &\quad \times (\boldsymbol{l}, Q, \beta|(l+l')/2, m, Ql') ((l+l')/2, m, El|\boldsymbol{\kappa}, \mathbf{e}, \alpha) \Big\}. \end{aligned} \tag{AII–8}$$

Now, it follows from the unitarity of S matrix that

$$\begin{aligned} &f(\boldsymbol{\kappa}', \mathbf{e}', \beta: \kappa, \mathbf{e}, \alpha) - f^*(\boldsymbol{\kappa}, \mathbf{e}, \alpha\,;\, \boldsymbol{\kappa}', \mathbf{e}', \beta) \\ &= \frac{i\nu_c}{2\pi} \int d\Omega_l \sum_{\beta' Q} g^*(\boldsymbol{l}, Q, \beta'\,;\, \boldsymbol{\kappa}', \mathbf{e}'\,\beta) g(\boldsymbol{l}, Q, \beta'\,;\, \boldsymbol{\kappa}, \mathbf{e}, \alpha) \end{aligned} \tag{AII–9}$$

provided that multiple photopion production is disregarded and terms of the order higher than e^2 are neglected. Substituting (AII–3) and (AII–8) in this relation, and carrying out the integration with the aid of the orthogonality condition, we have

$$\begin{aligned} &\mathrm{Im}\, a_{l\pm 1/2}(El, El) = \nu_c \sum_{Q} |b_{l\pm 1/2}(Ql\pm 1, El)|^2, \\ &\mathrm{Im}\, a_{l\pm 1/2}(Ml, Ml) = \nu_c \sum_{Q} |b_{l\pm 1/2}(Ql, Ml)|^2, \end{aligned} \tag{AII–10}$$

$$\operatorname{Im} a_{l\pm 1/2}(Ml\pm 1,\ El) = \nu_c \sum_Q b_{l\pm 1/2}(Ql\pm 1,\ El) \times b^*_{l\pm 1/2}(Ql\pm 1,\ Ml\pm 1).$$

In the text we have made the following abbreviations:

$$\begin{aligned}
&e_j = a_j(E1,\ E1), \quad m_j = a_j(M1,\ M1), \quad e_j' = a_j(E2,\ E2),\\
&c_1 = 2\sqrt{3}\, a_{3/2}(E1,\ M2) = 2\sqrt{3}\, a_{3/2}(M2,\ E1),\\
&c_2 = 2\sqrt{3}\, a_{3/2}(E2,\ M1) = 2\sqrt{3}\, a_{3/2}(M1,\ E2),\\
&e_N = e_{3/2} + \tfrac{1}{2} e_{1/2}, \qquad e_S = e_{3/2} - e_{1/2},\\
&m_N = m_{3/2} + \tfrac{1}{2} m_{1/2}, \qquad m_S = m_{3/2} - m_{1/2},\\
&e_N' = e_{3/2}' + \tfrac{3}{2} e_{5/2}', \qquad e_S' = e_{5/2}' - e_{3/2}'.
\end{aligned} \tag{AII–11}$$

If all other amplitudes are neglected, (AII–3) yields (2·19) in the text. As for photo-pion production, the partial wave amplitudes used by Watson et al.[6] are related to ours through

$$\begin{aligned}
&E_d^Q = -\frac{1}{\sqrt{2}} b_{1/2}(Q0,\ E1), \quad E_q^Q = -\sqrt{6}\, b_{3/2}(Q1,\ E2),\\
&M_d^Q(j) = -\frac{1}{\sqrt{2}} b_j(Q1,\ M1) \quad (j = 1/2,\ 3/2).
\end{aligned} \tag{AII–12}$$

If production of P-wave pion into all states other than the (3–3) state and production of neutral S-wave pion are neglected, eqs. (3·1) in the text follow from (AII–10), definitions of E_{33} and M_{33} being found in reference 6).

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