On the Equational Theory of Representable Polyadic Equality Algebras (Extended Abstract)

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Abstract

Among others we will see that the equational theory of ω dimensional representable polyadic equality algebras ($RPEA_\omega$'s) is not schema axiomatizable. This result is in interesting contrast with the Daigneault-Monk representation theorem, which states that the class of representable polyadic algebras is finite schema-axiomatizable (and hence the equational theory of this class is finite schema-axiomatizable, as well). We will also see that the complexity of the equational theory of $RPEA_\omega$ is also extremely high in the recursion theoretic sense. Finally, comparing the present negative results with the positive results of Ildikó Sain and Viktor Gyuris [10], the following methodological conclusions will be drawn: the negative properties of polyadic (equality) algebras can be removed by switching from what we call the "polyadic algebraic paradigm" to the "cylindric algebraic paradigm".[1]

Keywords: algebraic logic, polyadic algebras, cylindric algebras, schema axiomatization, equational theory, computability, degrees of unsolvability

1 Introduction

It is well known that the class $RPEA_\omega$ of (ω dimensional) representable polyadic equality algebras is not closed under taking of ultraproducts, and hence this class is not axiomatizable. This result leaves open the question of axiomatizability of the equational theory of $RPEA_\omega$. The equational theory of $RPEA_\omega$ ($\mathbf{Eq}(RPEA_\omega)$ for short) is not finitely axiomatizable, simply because the language of $RPEA_\omega$ contains uncountably many operation symbols. So the next interesting question is axiomatizability by some kind of schemas. Indeed, "schema axiomatization" has been the traditional approach in polyadic algebra theory, namely, Halmos's axiom schemas, which appeared in 1960's constitute a typical example for schema axiomatization (see [3]). In Section 3 we will see that the equational theory of $RPEA_\omega$ is not axiomatizable by schemas similar to the usual defining schemas of (abstract) polyadic equality algebras. This result is in interesting contrast with the theorem of Daigneault and Monk [6], which states that the class of representable polyadic algebras RPA_ω (without equality) is finite schema axiomatizable. (Independently from Daigneault and Monk, Keisler proved the logical version of this representation theorem.)

Before turning to the goal of Section 3, let us look briefly to the goal of Section

[1] This is a prepublication version of the paper, final version will be prepared later and published elsewhere.

4. (The notational system will be recalled from the textbooks [4], [5] in Section 2, till then we try to use as little notation as we can.) In Section 4 we look at the equational theory of $RPEA_\omega$ from the point of view of recursion theoretic degrees of unsolvability. We have two aims with this:

(i) To shed some more light on the possibility or impossibility of finding some notion of a schema with which $\mathbf{Eq}(RPEA_\omega)$ and related theories would be axiomatizable.

(ii) To study the complexity properties of theories like $\mathbf{Eq}(RPEA_\omega)$ and its finite reducts on their own right.

We will see that $\mathbf{Eq}(RPEA_\omega)$ is highly non-computable, and even its restrictions to finite sublanguages are very far from being recursively enumerable. More concretely, in theorem 4.1 we will claim that

$$\mathbf{Eq}(RPEA_\omega) \text{ is } \Pi^1_1 \text{ hard (in the recursion theoretic sense).}$$

This means the following: let $\mathcal{N}$ denote the standard model of arithmetic. There is a computable function (e.g. by a Turing machine)

$$Tr : (\Pi^1_1 \text{ formulas }^{2} \text{ of arithmetic}) \to (\text{equations of } RPEA_\omega)$$

such that for any Π^1_1 sentence of arithmetic φ:

$$\mathcal{N} \models \varphi \Leftrightarrow RPEA_\omega \models Tr(\varphi).$$

We will see that the above is meaningful because the range of Tr contains only finitely many operation symbols (so the cardinality of $\mathbf{Eq}(RPEA_\omega)$ causes no problem from the point of view of computability of Tr).

On the intuitive consequences of theorem 4.1:

If $\mathbf{Eq}(RPEA_\omega)$ were finitely axiomatizable by some kind of schemas, then the equational consequences of these schemas would form a Π^1_1 hard set (by theorem 4.1). Usually, finite (schema) axiomatizability of some theory implies recursive enumerability of this theory. So we conclude that $\mathbf{Eq}(RPEA_\omega)$ can not be finitely (or recursively) axiomatizable by any kind of reasonable schemas. Further investigations and discussions in this line are in Section 4.

It is interesting to contrast non-computability of $\mathbf{Eq}(RPEA_\omega)$ (i.e. theorem 4.1) with the fact that $\mathbf{Eq}(RCA_\omega)$ and $\mathbf{Eq}(RQPEA_\omega)$ are recursively enumerable (cf. e.g. Németi, [7]). Here, RCA_ω and $RQPEA_\omega$ denote the class of representable cylindric algebras and the class of representable quasi-polyadic algebras, respectively. For more on this contrast, cf. the end of Section 5 herein. For completeness, we note that

[2] Π^1_1 formulas are second order formulas in which the second order variables are quantified universally (in prenex form). For precise definitions we refer to [9]. We note that Π^1_1 hardness automatically implies non-recursive-enumerability, but Π^1_1 hardness is a much stronger negative property than this.

the main results of this paper are theorems 3.7 and 4.1.

2 Notation

Our notation is mostly standard, but the following list may help.

Sets:
Throughout ω denotes the set of natural numbers. If A, B are sets then ${}^AB$ denotes the set of functions from A to B. If A is a set, then $|A|$ denotes the cardinality of A. If A is a set then $\mathcal{P}(A)$ denotes the power set of A, that is, $\mathcal{P}(A)$ consists of all subsets of A.

Algebras:
If K is a class of algebras, then $\mathbf{H}K$, $\mathbf{S}K$, $\mathbf{P}K$ denote the classes of homomorphic images, subalgebras and direct products of members of K, respectively.

Throughout we use function composition in such a way that the rightmost factor acts first. That is, for functions f, g we define $f \circ g(x) = f(g(x))$.

Let U be a set, and α an ordinal. The *full polyadic equality set algebra* of dimension α with base U is the algebra

$$\langle \mathcal{P}({}^\alpha U), \cap, -, C_{(\Gamma)}, S_\tau, D_{i,j} \rangle_{\Gamma \subseteq \alpha,\ \tau \in {}^\alpha\alpha,\ i,j \in \alpha}$$

where $\cap$ and $-$ are intersection and complementation (with respect to the top element ${}^\alpha U$), and for $X \subseteq {}^\alpha U$, $\Gamma \subseteq \alpha$, $\tau \in {}^\alpha\alpha$ and $i, j \in \alpha$

$$\begin{aligned}
C_{(\Gamma)}(X) &= \{s \in {}^\alpha U : (\exists z \in X)(\forall j \notin \Gamma)(s_j = z_j)\},\\
C_i(X) &= C_{(\{i\})}(X),\\
S_\tau(X) &= \{s \in {}^\alpha U : s \circ \tau \in X\},\\
D_{i,j} &= \{s \in {}^\alpha U : s_i = s_j\}.
\end{aligned}$$

$SPEA_\alpha := \mathbf{S}\{\mathcal{A} : \mathcal{A}$ is a full polyadic equality set algebra of dimension α with base U, for some set $U\}$.

$SPEA_\alpha$ is called the class of <u>s</u>et <u>p</u>olyadic <u>e</u>quality <u>a</u>lgebras of dimension α.

$RPEA_\alpha := \mathbf{SP}\{\mathcal{A} : \mathcal{A}$ is a full polyadic equality set algebra of dimension α with base U, for some set $U\}$.

$RPEA_\alpha$ is called the class of <u>r</u>epresentable <u>p</u>olyadic <u>e</u>quality <u>a</u>lgebras of dimension α.

The class RPA_α of <u>r</u>epresentable <u>p</u>olyadic <u>a</u>lgebras of dimension α is defined to be the class of $D_{i,j}$-free subreducts of members of $RPEA_\alpha$; in symbols:

$$RPA_\alpha = \mathbf{SRd}(RPEA_\alpha).$$

For details see [5].

DEFINITION 2.1
Throughout, α is an ordinal. L_{PEA_α} denotes the language of $RPEA_\alpha$. For completeness we note that a detailed discussion is in [5], p.225.

We will denote the polyadic operations by $\cap, -, C_{(\Gamma)}, S_\tau$ and $D_{i,j}$. The corresponding operation symbols (in L_{PEA_α}) are $\wedge, -, c_{(\Gamma)}, s_\tau$ and $d_{i,j}$, respectively.

3 Non schema-axiomatizability results

DEFINITION 3.1
The class PEA_ω of polyadic equality algebras of dimension ω is an abstract class defined by equation schemas ($P_0 - P_{11},\ E_1 - E_3$) originating from Halmos. Cf. [3],[5] and the Appendix herein.

Let us turn to the difference (or distance) between PEA_ω and $\mathbf{HSP}(RPEA_\omega)$. We will construct an equation that separates the two classes hence $PEA_\omega \neq \mathbf{HSP}(RPEA_\omega)$. One could hope that adding new, similar schemas to $P_0 - E_3$ might lead to an axiomatization of $\mathbf{HSP}(RPEA_\omega)$. We will see below that this cannot be done.

Motivating Example
Consider the axiom schema of polyadic algebras $s_\tau s_\sigma(x) = s_{\tau\circ\sigma}(x)$ as an example. For simplicity let us write this as $s_\tau s_\sigma = s_{\tau\circ\sigma}$. Let us notice that τ, σ (and ϱ below) are "meta-variables" ranging over ${}^\omega\omega$. We can write the same schema in the form

$$(*) \quad \tau\circ\sigma = \varrho \Rightarrow s_\tau s_\sigma = s_\varrho.$$

When studying schemas like $(*)$, sometimes it will be convenient to treat τ, σ and ϱ in $(*)$ as function symbols of some first order language L_T. At other times we will treat τ, σ and ϱ as second order variables ranging over functions.

DEFINITION 3.2
L_T is the first order language that consists of countably many unary relational symbols, countably many unary function symbols and countably many constants (and nothing more), which are $r_1, r_2, ...$ and $f_1, f_2, ...$ and $n_1, n_2, ...$ respectively.

DEFINITION 3.3
Throughout L_{PT} denotes the similarity type (in the algebraic sense): of Boolean algebras with extra unary operation symbols $c_{r_1}, c_{r_2}, ...$ and $s_{f_1}, s_{f_2}, ...$ and constants $d_{n_1,n_1}, d_{n_1,n_2}, d_{n_2,n_1}....$ Here, the indices of the symbols c, s and d are the same as the corresponding relation, function and constant symbols in the previous definition.

DEFINITION 3.4
By a Halmos schema we mean a pair $\langle s, e\rangle$ where s is a first-order sentence in the language L_T, and e is an equation in L_{PT}.

We will denote $\langle s, e\rangle$ as $s \Rightarrow e$ in order to keep the notation closer to intuition.

Motivating Example, Continued
The form $(*)$ will help us in making certain things explicit about these kinds of schemas. Now let $f_1, f_2, f_3 \in {}^\omega\omega$. Then the equation $s_{f_1}s_{f_2}(x) = s_{f_3}(x)$ will be called an instance of $(*)$ iff indeed $f_1 \circ f_2 = f_3$. Thus, the single schema $(*)$ has continuum many instances (if the meta-variables τ, σ and ϱ are ranging over ${}^\omega\omega$).

Definition 3.5
Let $s \Rightarrow e$ be a Halmos schema, let g be an equation in L_{PEA_α}. We say that g is an (α dimensional) instance of $s \Rightarrow e$ iff there exist sets, functions and constants

$$r_1^M, r_2^M, ..., \subseteq \alpha, \qquad f_1^M, f_2^M, ... \in {}^\alpha\alpha \qquad \text{and} \qquad n_1^M, n_2^M, ... \in \alpha \qquad \text{such that}$$

$$M = \langle \alpha; r_1^M, r_2^M, ..., f_1^M, f_2^M, ... n_1^M, n_2^M, ... \rangle \models s;$$

and g is obtained from e by replacing r_i, f_i and n_i by r_i^M, f_i^M and n_i^M respectively. In this case we say that g is an instance of e with respect to M.

Definition 3.6
Let $\mathcal{A}$ be an L_{PEA_α} type algebra. A Halmos schema h is valid in $\mathcal{A}$ iff every α dimensional instance of h is valid in $\mathcal{A}$.

We note that one can easily translate the usual defining schemas of polyadic (equality) algebras into Halmos schemas. Consider for example $s_{\sigma\circ\tau}(x) = s_\sigma s_\tau(x)$, where $\sigma, \tau \in {}^\alpha\alpha$. This schema is axiom P_7 in [5] on page 225, which is a typical example of the usual defining schemas of polyadic algebras. Let h be the following Halmos schema:

$$(\forall x)(f_1(f_2(x)) = f_3(x)) \quad \Rightarrow \quad s_{f_3}(y) = s_{f_1}s_{f_2}(y).$$

Now one can easily see that the instances of P_7 and the α dimensional instances of h are the same. In the Appendix we give Halmos schemas which correspond to the usual definition of PEA_α. That is, we give a complete list of the translations of the usual defining schemas of polyadic (equality) algebras. Now we can state our first main result.

Theorem 3.7
(i) The set $\mathbf{Eq}(RPEA_\omega)$ of equations valid in $RPEA_\omega$ is not axiomatizable by any set of Halmos schemas. More formally:

(ii) Let Σ be a set of Halmos schemas valid in $RPEA_\omega$. Then for the class $Mod(\Sigma)$ of all models of all ω dimensional instances of Σ we have

$$Mod(\Sigma) \neq \mathbf{HSP}(RPEA_\omega) \text{ or, equivalently, } Ded(\Sigma) \neq \mathbf{Eq}(RPEA_\omega)$$

where $Ded(\Sigma)$ denotes the set of equational consequences of Σ.

Unless otherwise specified, the proofs of theorems announced are found in [8]. Theorem 3.7 is a corollary of theorem 3.15 at the end of section 3 below. Now we formulate an equivalent version of theorem 3.7.

Definition 3.8
PEA_α^+ denotes the class of all models of all α dimensional instances of those Halmos schemas which are valid in $RPEA_\alpha$.

Note that PEA_α^+ is the smallest variety which contains $RPEA_\alpha$ and is defined by Halmos schemas. The previous theorem stated that the variety generated by $RPEA_\omega$ is not axiomatizable by Halmos schemas, which is equivalent to the following.

Theorem 3.7* $PEA_\omega^+ \neq \mathbf{HSP}(RPEA_\omega)$.

To show this, we will construct an equation e_{CM} which is valid in $RPEA_\omega$ but not in PEA_ω^+. This equation is based on the method of W. Craig and R. McKenzie (cf. [2]), hence the notation.

First we introduce some notation. Throughout, $t_j^i(x)$ abbreviates the term $d_{i,j} \wedge c_j(x)$. As usual, $[i,j] \in {}^\omega\omega$ denotes the transposition which interchanges i and j and leaves every other element fixed. Here i and j are distinct elements of ω. The symbol suc denotes the successor function on ω. To build e_{CM}, we will introduce some simpler equations (see definitions 3.9 and 3.13 below).

Definition 3.9
Throughout, e_1, e_2, e_3 denote the following equations:
e_0 is $x = c_1 x$,
e_1 is $x - d_{3,4} \leq d_{2,3}$,
e_2 is $(x \wedge d_{3,4} - c_0 c_1 t_3^4 t_2^3 s_{suc}(x)) \;\wedge\; (c_0 c_2 s_{[1,2]}(x) - c_0 c_2 t_3^4 t_1^3 s_{suc}(x)) \leq d_{1,2}$,
e_3 is $x \wedge d_{3,4} \leq c_0 c_1 c_2 (x \wedge d_{3,4} - c_0 c_1 t_3^4 t_2^3 s_{suc}(x))$.

The intuitive content of these equations will be illustrated in theorem 3.11.

If s is a finite sequence, and h is another (possibly infinite) sequence, then $s^\frown h$ denotes the concatenation of s and h.

Definition 3.10
Let $\mathcal{A} \in SPEA_\omega$, with base U, let $a \in \mathcal{A}$, and let $h \in {}^\omega U$. Then $(a)_h$ denotes the "projection" of a with respect to h to the second coordinate, that is,
$(a)_h = \{u \in U : \; (\exists v_0, v_1 \in U) \;\; \langle v_0, v_1, u\rangle^\frown h \in a\}$.

Theorem 3.11
Suppose $\mathcal{A} \in SPEA_\omega$, with base U, and $a \in \mathcal{A}$ such that e_0, e_1, e_2 and e_3 hold for a. Then $(\forall h \in {}^\omega U)(|(a)_h| \; < \; \aleph_0)$.

Definition 3.12
Suppose $\mathcal{A} \in SPEA_\omega$, with base U, $a \in \mathcal{A}$ and $h \in {}^\omega U$. Then $f_{a,h}$ denotes the following binary relation:

$$f_{a,h} = \{\langle x, y\rangle \in {}^2U : \; (\exists z)(\langle y, z, x\rangle^\frown h \in a)\}.$$

Of course, $dom(f_{a,h}) = (a)_h$; where $dom(f_{a,h})$ is the domain of $f_{a,h}$ in the usual sense (similarly, $range(f_{a,h})$ denotes the range of $f_{a,h}$).

Definition 3.13
Throughout, e_4, e_5, e_6, e_7 denote the following equations:
e_4 is $s_{[0,1]}(x) \wedge x \leq d_{0,1}$,

e_5 is $s_{[1,2]}(x) \wedge x \leq d_{1,2}$,
e_6 is $s_{[0,2]}(x) \leq c_0(x)$,
e_7 is $s_{[0,2]}(x) \leq c_2(x)$.

Suppose $\mathcal{A} \in SPEA_\omega$, with base U, $a \in A$, $h \in {}^\omega U$ and $C_1^A(a) = a$. Let us notice that

$e_4(a)$ expresses that $f_{a,h}$ is a function,
$e_5(a)$ expresses that $f_{a,h}$ is an injective relation,
$e_6(a)$ expresses that $range(f_{a,h}) \subseteq dom(f_{a,h})$,
$e_7(a)$ expresses that $dom(f_{a,h}) \subseteq range(f_{a,h})$.

The variety PEA_ω^+ is a discriminator variety because $c_{(\omega)}$ is a complemented normal closure operator which majorizes all other extra-boolean operation. These properties are expressible by Halmos schemas. It is well known from the theory of discriminator varieties that the quasi-equations over a discriminator variety are translatable into equivalent equations. That is, for any discriminator variety V, and for any quasi-equation q over V, there is a translation $tr(q)$ which is an equation and $V \models q$ iff $tr(q) \in \mathbf{Eq}(V)$. So below we will write quasi-equations and will refer to them as equations.

Definition 3.14
Throughout, e_{CM} is defined as follows:

$$e_{CM} \text{ is the (quasi-)equation } e_0 \& e_1 \& ... \& e_6 \Rightarrow e_7.$$

Theorem 3.15

(i) $RPEA_\omega \models e_{CM}$.
(ii) $PEA_\omega^+ \not\models e_{CM}$.

It is easy to see that theorems 3.7 and 3.7* indeed follow from theorem 3.15.

Proof. [Sketch of the proof of theorem 3.15] Recall that $(a)_h$ is the 2-th projection of the element a with respect to tail h. By theorem 3.11, $e_0 \& e_1 \& e_2 \& e_3(a)$ imply that $(a)_h$ is finite. The equation e_{CM} formulates a combinatorial consequence of $(a)_h$ being finite (injective selfmaps are surjective). Now, e_{CM} is true in the representable algebras simply because it formulates a true combinatorial principle. The proof of the second part consists of showing that the combinatorial principle coded by e_{CM} cannot be derived from the axioms of PEA_ω^+. ∎

4 Complexity of the equational theories

As we mentioned in the introduction, in [6] Daigneault and Monk proved a representation theorem for (ω dimensional) polyadic algebras (PA_ω for short). However, this result does not imply that the equational theory of PA_ω is recursively enumerable. One simple reason for this is that the language of PA_ω contains a continuum of operation symbols. Actually, in [12] it is proved that intuitively, the set $\mathbf{Eq}(PA_\omega)$ is not recursively enumerable for any reasonable generalization (to non-enumerable sets) of the notion of recursive enumerability[3]. More formally this is stated in theorem 5.1 below.

In this section we will see that the set $\mathbf{Eq}(RPEA_\omega)$, i.e., the equational theory of $RPEA_\omega$ is complex in the recursion theoretic sense as well. That is, $\mathbf{Eq}(RPEA_\omega)$ is strictly more complex than the full first order theory of standard arithmetic.

To see this, we will recursively reduce the second order universal theory of standard arithmetic to $\mathbf{Eq}(RPEA_\omega)$. By a Π^1_1 formula of arithmetic we mean a second order formula of arithmetic in which every second order variable is quantified universally (in prenex form). This subject will be further discussed below the formulation of the next theorem. Recall that $\mathcal{N}$ is the standard model of arithmetic.

Theorem 4.1
$\mathbf{Eq}(RPEA_\omega)$ is Π^1_1 hard, by which we mean the following: there is a strictly finite reduct L of the language of $RPEA_\omega$ and a recursive function Tr such that Tr maps the Π^1_1 formulas of arithmetic to equations written in L, and for any Π^1_1 sentence σ of arithmetic

$$\mathcal{N} \models \sigma \Leftrightarrow RPEA_\omega \models Tr(\sigma).$$

(For the second order logical and recursion theoretic details see the next remark and [9].)

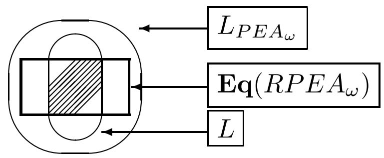
Remark 4.2
We say that a set A of natural numbers is Π^1_1, iff A is definable by a Π^1_1 formula in arithmetic. The set S of all (Gödel numbers of) true Π^1_1 sentences of arithmetic is not Π^1_1, by Tarski's theorem of undefinability of truth. That is, S is more complex than any Π^1_1 set. So theorem 4.1 means that $\mathbf{Eq}(RPEA_\omega)$ is more complex than any Π^1_1 set, this property is called Π^1_1 hardness.

Proof. [Sketch of proof of theorem 4.1] The proof is based on a systematic translation of the Π^1_1 formulas of arithmetic into polyadic equations. For all $n \in \omega$ we can construct a polyadic equation e_8 with one variable such that if an element y satisfies it (i.e. if $e_8(y)$ holds), then one can identify (or simulate) ω with the first n projections ("columns") of y. A further equation $e_9(z)$ ensures that the element z codes the successor function "*suc*" on this ω. Then, using this y and z, we construct a translation mapping which associates to each arithmetical term τ the number theoretic meaning of τ in terms of y, z and the polyadic operations. This translation of τ is a polyadic term of y and z and the translation procedure can be generalized to translate even the Π^1_1 formulas of arithmetic. The so obtained translation of Π^1_1 arithmetic into the equational language of $RPEA_\omega$ (in some sense) is used to prove the theorem. ∎

As we explained close to the end of section 1, theorem 4.1 implies that $\mathbf{Eq}(RPEA_\omega)$ can not be finitely (or recursively) axiomatizable by any kind of reasonable schemas. (From the point of view of certain applications this seems to be an inconvenient feature.)

[3] The reason for this intuitive interpretation was given above definition 4.3 for an analogous situation.

An intuitive consequence of the above considerations is that $\mathbf{Eq}(RPEA_\omega)$ is not computable. This is not a precise mathematical statement because the cardinality of this set is the continuum. But suppose one generalizes the concept of computability in a meaningful way to continuum big sets. Cf. the figure.



The only thing we assume about the new, generalized concept of computability is that the intersection of two computable set is computable, and countable sets are not computable if and only if they are non-computable in the old sense. Now $\mathbf{Eq}(RPEA_\omega)$ is not computable in the new, generalized sense, because its intersection with a countable decidable (in the old sense) set L forms a Π^1_1 hard (hence non-computable) set, as theorem 4.1 states.

Concluding this section we briefly summarize some of the results concerning upper bounds for the complexity of $\mathbf{Eq}(RPEA_\omega)$.

Definition 4.3
If Σ is a set of Halmos schemas then

$$Ded(\Sigma) = \{e : e \text{ is an equation derivable from the } \omega \text{ dimensional instances of } \Sigma\}.$$

Let us call $A \subseteq \omega$ Σ^1_1 *enumerable* if there exists an algorithm which uses an oracle deciding the Σ^1_1 theory of standard arithmetic and enumerates A. Let L be a reduct of the language of $RPEA_\omega$. In [8] we show that if L is of certain form and the basic operations of L are neither too numerous nor too complicated, then the set $Ded(\Sigma) \cap L$ of the L consequences of a finite schema Σ of axioms is Σ^1_1 enumerable, i.e., "of relatively low complexity".

Remark 4.4
We do not claim that the set $Ded(\Sigma) \cap L$ is Σ^1_1 i.e. is definable by some Σ^1_1 formula of arithmetic.

5 Related results

Here we summarize some results about the possibility of axiomatizing of some reducts of $RPEA_\omega$. The complexities of the equational theories of these reducts are also discussed.

5.1 Case of algebras without equality

By the theorem of Daigneault and Monk, every PA_ω is representable, therefore the "complexity" of the equational theory of PA_ω is the same as the "complexity" of the set of all equational consequences of the PA_ω axiom schemas. But this representation theorem does not imply recursive enumerability, as the following theorem claims.

Theorem 5.1
There is a strictly finite reduct L_G of the language of PA_ω, such that the set $\mathbf{Eq}(PA_\omega) \cap L_G$ of equations valid in PA_ω is not recursively enumerable. Moreover, the indices of the substitutions and cylindrifications occurring in L_G are recursive functions and sets, respectively.

A proof can be found in [12].

Definition 5.2
Let *pred* denote the usual predecessor function on ω, with $pred(0) = 0$. Throughout, $L_{suc,pred}$ denotes the language that consists of the Boolean operations and $c_0, s_{suc}, s_{pred}, s_{[0,1]}, s_{[i/j]},\ i, j \in \omega$,

Note that in $L_{suc,pred}$ the operations $c_i, s_{[i,j]},\ i, j \in \omega$ are term definable. The following theorem is due to I. Sain and V. Gyuris, (see [10]) which is a possible solution of (some variant of) the *finitization problem* (in the sense of [7]).

Theorem 5.3
Let $L_{suc,pred,\omega} = L_{suc,pred} \cup \{c_{(\omega)}\}$. Then the variety generated by the reduct of RPA_ω to $L_{suc,pred,\omega}$ is axiomatizable by a recursive set $\Sigma_{suc,pred,\omega}$ of equations. This set is explicitly given by finitely many schemas (these schemas are essentially simpler than Halmos schemas in general).

This theorem is similar to the Daigneault-Monk theorem in the sense that it is a representation theorem. But now, in addition to this, the axiom system is recursive[4], and therefore, $\mathbf{Eq}(RPA_\omega) \cap L_{suc,pred,\omega} = Ded(\Sigma_{suc,pred,\omega}) \cap L_{suc,pred,\omega}$ is recursively enumerable.

5.2 Case of algebras with equality

The following theorem is also in [10].

Theorem 5.4
Let $L_{suc,pred,D} = L_{suc,pred} \cup \{d_{0,1}\}$. Note that in this language all the $d_{i,j}$'s are term definable for $i, j \in \omega$. Then the variety generated by the reduct of $RPEA_\omega$ to $L_{suc,pred,D}$ is axiomatizable by a recursive set $\Sigma_{suc,pred,D}$ of equations. This set is explicitly given by finitely many schemas (these schemas are essentially simpler than Halmos schemas in general).

All the remarks standing below theorem 5.3 apply to theorem 5.4 too. The following theorem is due to R. McKenzie.

Theorem 5.5
Let $L_{suc,pred,\omega,D} = L_{suc,pred} \cup \{c_{(\omega)}, d_{i,j},\ i, j \in \omega\}$. Then the set $\mathbf{Eq}(RPEA_\omega) \cap L_{suc,pred,\omega,D}$ is not recursively enumerable.

Theorems 5.3, 5.4 and 5.5 together seem to indicate that a kind of division-line has been found between the area where positive results dominate (like theorems 5.3, 5.4)

[4] That is, the set of all instances of the axiom schemas is recursive.

and where negative results dominate (like theorems 5.5, 4.1). More precisely, theorem 5.5 implies that theorems 5.3, 5.4 are best possible in the sense that adding $c_{(\omega)}$ to the language of theorem 5.4 then it becomes false. Actually theorems 5.3 and 5.4 do not admit a common generalization in the sense that the negative theorem 5.5 already applies to the union of their languages.

Finally, let us turn to the complexity of the first order theory of $RPEA_\omega$. One can extend the proof of theorem 4.1 to this theory as follows: Tr is extendable in such a way that Tr maps the full second order theory of arithmetic into the first order theory of (a finite reduct of) $RPEA_\omega$. This means that, for some finite language L, $\mathbf{Th}(RPEA_\omega) \cap L$ does not form an analytical set (so it is at least as complex as the full second order theory of standard arithmetic). Here $\mathbf{Th}(RPEA_\omega)$ denotes the first order theory of $RPEA_\omega$.

5.3 Methodological conclusions

Comparing the negative and positive results above, we found that there are two paradigms. In the first paradigm (called polyadic algebra oriented) the indices or names of the operation symbols have a complex and fixed structure. For example the indices of the substitutions are functions, and one can compose them. The properties and the complexity of these objects are implicitly mirrored in the equational theories that we are studying, and possibly this is the reason for our negative results.

The other paradigm (called cylindric algebra oriented) is the opposite. In this case the indices of operation symbols have basically no structure. There are neither operations nor relations defined between the names of the operation symbols. Particular examples for the cylindric paradigm are: cylindric algebras; the languages (or algebras) in theorems 5.3 and 5.4; quasi-polyadic algebras in the sense of [11]; and $SCA_\alpha, QPA_\alpha, QPEA_\alpha$ in [7]. We note that all the positive results quoted in this paper follow the cylindric paradigm.

These paradigms are not rigid categories. For example, the results about quasi-polyadic algebras in [11] follow the cylindric paradigm (and, indeed, these results are positive).

The main methodological conclusion is the following: suppose one wants to apply polyadic algebras to a certain area, but the negative complexity-properties of polyadic algebras are undesirable for the applications. Then it is possible to remove these properties by switching from the polyadic paradigm to the cylindric paradigm.

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Appendix

Here we give the translations of the usual defining schemas of α dimensional polyadic (equality) algebras into Halmos schemas. First of all we recall from [5] the original defining schemas:

(P0) The Boolean Algebra axioms,
(P1) $c_{(\Gamma)}(0) = 0$,
(P2) $x \leq c_{(\Gamma)}(x)$,
(P3) $c_{(\Gamma)}(x \wedge c_{(\Gamma)}(y)) = c_{(\Gamma)}(x) \wedge c_{(\Gamma)}(y)$,
(P4) $c_0(x) = x$,
(P5) $c_{(\Gamma)}c_{(\Delta)}(x) = c_{(\Gamma\cup\Delta)}(x)$,
(P6) $s_{Id_\alpha}(x) = x$,
(P7) $s_{\sigma\circ\tau}(x) = s_\sigma s_\tau(x)$,
(P8) $s_\sigma(x \vee y) = s_\sigma(x) \vee s_\sigma(y)$,
(P9) $s_\sigma(-x) = -s_\sigma(x)$,

(P10) If $(\alpha - \Gamma) \mid \sigma = (\alpha - \Gamma) \mid \tau$, then $s_\sigma c_{(\Gamma)}(x) = s_\tau c_{(\Gamma)}(x)$,
(P11) If $(\tau^{-1}(\Gamma)) \mid \tau$ is one-one and $\Delta = \tau^{-1}(\Gamma)$ then $c_{(\Gamma)}s_\tau(x) = s_\tau c_{(\Delta)}(x)$,

(E1) $d_{\kappa,\kappa} = 1$,
(E2) $x \wedge d_{\kappa,\lambda} \leq s_{[\kappa/\lambda]}(x)$,
(E3) $s_\tau(d_{\kappa,\lambda}) = d_{\tau(\kappa),\tau(\lambda)}$.

Here $\Gamma, \Delta \subseteq \alpha, \quad \sigma, \tau \in {}^\alpha\alpha, \quad \kappa, \lambda \in \alpha$.

The translations are:
(P0') $\langle true,$ The Boolean Algebra axioms $\rangle$,
(P1') $\langle true,\ c_{r_1}(0) = 0\rangle$,
(P2') $\langle true,\ x \leq c_{r_1}(x)\rangle$,
(P3') $\langle true,\ c_{r_1}(x \wedge c_{r_1}(y)) = c_{r_1}(x) \wedge c_{r_1}(y)\rangle$,
(P4') $\langle (\forall y)(\neg r_1(y)),\ c_{r_1}(x) = x\rangle$,
(P5') $\langle (\forall y)(r_1(y) \vee r_2(y) \leftrightarrow r_3(y)),\ c_{r_1}c_{r_2}(x) = c_{r_3}(x)\rangle$,
(P6') $\langle (\forall y)(f_1(y) = y),\ s_{f_1}(x) = x\rangle$,
(P7') $\langle (\forall y)(f_1(f_2(y)) = f_3(y)),\ s_{f_3}(x) = s_{f_1}s_{f_2}(x)\rangle$,
(P8') $\langle true,\ s_{f_1}(x \vee y) = s_{f_1}(x) \vee s_{f_1}(y)\rangle$,
(P9') $\langle true,\ s_{f_1}(-x) = -s_{f_1}(x)\rangle$,

(P10') $\langle (\forall y)(\neg r_1(y) \rightarrow f_1(y) = f_2(y)),\ s_{f_1}c_{r_1}(x) = s_{f_2}c_{r_1}(x)\rangle$,
(P11') $\langle (\forall y)((r_2(y) \leftrightarrow\ (\exists z)(r_1(z)\ \&\ y = f_1(z))\ \&\ (\forall y, z)(r_2(y)\ \&\ r_2(z)\ \&\ y \neq z \rightarrow$
$f_1(y) \neq f_1(z))),\ c_{r_1}s_{f_1}(x) = s_{f_1}c_{r_2}(x)\rangle$,

(E1') $\langle true,\ d_{n_1,n_1} = 1\rangle$,
(E2') $\langle (\forall y)(y \neq n_1 \rightarrow f_1(y) = y\ \&\ f_1(n_1) = n_2),\ x \wedge d_{n_1,n_2} \leq s_{f_1}(x)\rangle$,
(E3') $\langle (f_1(n_1) = n_3\ \&\ f_1(n_2) = n_4),\ s_{f_1}(d_{n_1,n_2}) = d_{n_3,n_4}\rangle$.

It is easy to see that the instances of (P0)-P(11), (E1)-(E3) are the same as the α dimensional instances of (P0')-P(11'), (E1')-(E3').

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