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Total Domination in Lict Graph

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Abstract: For any graph $G = (V, E)$, lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. A dominating set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contains any isolates. The total lict dominating number $\gamma_t(\eta(G))$ of the graph G is a minimum cardinality of total lict dominating set of graph G. In this paper many bounds on $\gamma_t(\eta(G))$ are obtained and its exact values for some standard graphs are found in terms of parameters of G. Also its relationship with other domination parameters is investigated.

Key Words: Smarandachely k-dominating set, total lict domination number, lict graph, edge domination number, total edge domination number, split domination number, non-split domination number.

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§1. Introduction

The graphs considered here are finite, connected, undirected without loops or multiple edges and without isolated vertices. As usual $'p'$ and $'q'$ denote the number of vertices and edges of a graph G. For any undefined term or notation in this paper can be found in Harary [1].

A set $D \subseteq V$ of G is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S and the *Smarandachely k-domination number* $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the *domination number* of G and denoted by $\gamma(G)$ in general.

The lict graph $\eta(G)$ of a graph G is the graph whose vertex set is the union of the set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident. A dominating

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set of a graph $\eta(G)$, is a total lict dominating set if the dominating set does not contain any isolates. The total lict dominating number $\gamma_t(\eta(G))$ of G is a minimum cardinality of total lict dominating set of G.

The vertex independence number $\beta_0(G)$ is the maximum cardinality among the independent set of vertices of G. $L(G)$ is the line graph of G, $\gamma_e^{'}(G)$ is the complementary edge domination number, $\gamma_s(G)$ is the split dominating number, $\gamma_t^{'}(G)$ is the total edge dominating number , $\gamma_{ns}(G)$ is the non-split dominating number, $\chi(G)$is the chromatic number and $\omega(G)$ is the clique number of a graph G. The degree of an edge $e = uv$ of G is $deg(e) = deg(u)+deg(v)-2$. The minimum (maximum) degree of an edge in G is denoted by $\delta^{'}(\Delta^{'})$. A subdivision of an edge $e = uv$ of a graph G is the replacement of an edge e by a path (u, v, w) where $w \ni E(G)$. The graph obtained from G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by $S(G)$. For any real number X,$\lceil X\rceil$ denotes the smallest integer not less than X and $\lfloor X\rfloor$ denotes the greatest integer not greater than X.

In this paper we established the relationship of this concept with the other domination parameters. We use the following theorems for our later results.

Theorem A([2]) *For any graph* G,$\gamma_e(G) \geq \left\lceil \dfrac{q}{\Delta' + 1} \right\rceil$.

Theorem B([2]) *For any graph* G *of order* $p \geq 3$,

(*i*) $\beta_1(G) + \beta_1(\bar{G}) \leqslant 2\left\lceil \dfrac{p}{2} \right\rceil$.

(*ii*) $\beta_1(G) * \beta_1(\bar{G}) \leqslant \left\lceil \dfrac{p}{2} \right\rceil^2$.

Theorem C([3]) *For any graph* G,

(*i*) $\gamma_t^{'}(S(K_p)) = 2\left\lceil \dfrac{p}{2} \right\rceil$.

(*ii*) $\gamma_t^{'}(S(K_{p,q})) = 2q(p \leq q)$.

(*iii*) $\gamma_t^{'}(S(G)) = 2(p - \beta_1)$.

Theorem D([4]) *For every graph* G *of order* p,

(*i*) $\chi(G) \geq \omega(G)$.

(*ii*) $\chi(G) \geq \dfrac{q}{\beta_0}(G)$.

Theorem E([5]) *For any connected graph* G *with* $p \geq 3$ *vertices,* $\gamma_t^{'}(G) \leq \left\lceil \dfrac{2p}{3} \right\rceil$.

Theorem F([5]) *If* G *is a connected graph* G *with* $p \geq 4$ *vertices and* q *edges then* $\dfrac{q}{\Delta'} \leq \gamma_t^{'}(G)$, *further equality holds for every cycle* C_p *where* $p = 4n, n \geq 1$.

§2. Main Results

Theorem 1 *First list out the exact values of* $\gamma_t(\eta(G))$ *for some standard graphs:*

(i) *For any cycle C_p with $p \geq 3$ vertices,*

$$\gamma_t(\eta(C_p)) = \begin{cases} p/2 & \text{if } p \equiv 0(mod4). \\ \left\lfloor \frac{p}{2} \right\rfloor + 1 & otherwise. \end{cases}$$

(ii) *For any path P_p with $p \geq 4$ vertices,$\gamma_t(\eta(P_p)) = \left\lfloor \frac{2q}{3} \right\rfloor$.*

(iii) *For any star graph $K_{1,p}$ with $p \geq 3$ vertices, $\gamma_t(\eta(K_{1,p})) = 2$.*

(iv) *For any wheel graph W_p with $p \geq 4$ vertices, $\gamma_t(\eta(W_p)) = \left\lfloor \frac{p}{2} \right\rfloor$.*

(v) *For any complete graph K_p with $p \geq 3$ vertices, $\gamma_t(\eta(K_p)) = \left\lfloor \frac{2p}{3} \right\rfloor$.*

(vi) *For any friendship graph F_p with k blocks, $\gamma_t(\eta(F_p)) = k$.*

Initially we obtain a lower bound of total lict domination number with edge and total edge domination number.

Theorem 2 *For any graph G,$\gamma_t(\eta(G)) \geq \gamma_e(G)$.*

Proof Let D be a γ_e set of graph G, if D is a total lict dominating set of a graph G, then for every edge $e_1 \in D$ there exists an edge $e_2 \in D, e_1 \neq e_2$ such that e_1 is adjacent to e_2. Hence $\gamma_t(\eta(G)) = \gamma_e(G)$. Otherwise for each isolated edge $e_i \in D$, choose an edge $e_j \in N(e_i)$. Let $E_1 = \{e_j/e_j \in N(e_i)\}$, then $D \cup E_1$ is a total lict dominating set of G and $|D \cup E_1| \geq |D|$. Hence, $\gamma_t(\eta(G)) \geq \gamma_e(G)$. □

Theorem 3 *For any graph G $\gamma_t(\eta(G)) \geq \gamma_t^{'}(G)$, equality holds if G is non-separable.*

Proof Let D be a $\gamma_t^{'}$ set of G, if all the cut vertices of G are incident with at least one edge of D, then $\gamma_t(\eta(G)) = \gamma_t^{'}(G)$. Otherwise there exists at least one cut vertex v_c of graph G which is not incident with any edge of D, then $\gamma_t(\eta(G)) \geq |D \cup e| \geq \gamma_t^{'}(G) + 1$, where e is an edge incident with v_c and $e \in N(D)$. Thus, $\gamma_t(\eta(G)) \geq \gamma_t^{'}(G)$.

For the equality, note that if the graph G is non-separable, then $\eta(G) = L(G)$. Thus $\gamma_t(\eta(G)) = \gamma_t(L(G)) = \gamma_t^{'}(G)$. □

Next we obtain an inequality of total lict domination in terms of number of vertices, number of edges and maximum edge degree of graph G.

Theorem 4 *For any connected graph G with $p \geq 3$ vertices, then $\gamma_t(\eta(G)) \leq 2\left\lceil \frac{q}{3} \right\rceil$.*

Proof Let $E(G) = \{e_1, e_2, e_3, \cdots, e_l\}$ and let $D = \{e_l/1 \leq i \leq l$ and $i \neq 0(mod3)\} \cup \{e_{l-1}\}$. Then D is total lict dominating set of G and $|D| = 2\left\lceil \frac{q}{3} \right\rceil$. Hence, $\gamma_t(\eta(G)) \leq 2\left\lceil \frac{q}{3} \right\rceil$. □

Theorem 5 *For any non-separable graph G,*

(i) $\gamma_t(\eta(G)) \leq \left\lceil \frac{2p}{3} \right\rceil$, *$p \geq 3$.*

(ii) $\frac{q}{\Delta'} \leq \gamma_t(\gamma(G))$, *$p \geq 4$ vertices, equality holds for every cycle C_p, where $p = 4n, n \geq 1$.*

Proof Let G be a non-separable graph,then $\gamma_t(\eta(G)) = \gamma_t^{'}(G)$. Using Theorems E and F, the result follows. □

Theorem 6 *For any connected graph G,$\gamma_t(\eta(G)) \leq q - \Delta^{'}(G) + 1$, where $\Delta^{'}$ is a maximum degree of an edge.*

Proof Let e be an edge with degree $\Delta^{'}$ and let S be a set of edges adjacent to e in G. Then $E(G) - S$ is the lict dominating set of graph G. We consider the following two cases.

Case 1 If $\langle E(G) - S \rangle$ contains at least one isolate in $\eta(G)$ other than the vertex corresponding to e in $\eta(G)$.

Let E_1 be the set of all such isolates, then for each isolate $e_i \in E_1$, let $E_2 = \{e_j / e_j \in (N(e_i) \cap N(e)\}$, then $F = [\{(E(G) - S) - E_1\} \cup E_2]$ is a total lict dominating set of graph G. Thus, $\gamma_t(\eta(G)) \leq q - \Delta^{'}(G)$.

Case 2 If $\langle E(G) - S \rangle$ contains only e as an isolate in $\eta(G)$.

Then for an edge $e_i \in N(e), \{(E(G) - S) \cup e_i\}$ is a total lict dominating set of a graph G. Thus, $\gamma_t(\eta(G)) \leq |(E(G) - S) \cup e_i| = q - \Delta^{'}(G) + 1$.

From Cases 1 and 2, the result follows. □

Theorem 7 *For any connected graph G,$\gamma_t(\eta(G)) \geq \left\lceil \dfrac{q}{\Delta' + 1} \right\rceil$.*

Proof Using Theorem 2 and Theorem A, the result follows. □

Theorem 8 *For any connected graph G,$\gamma_t(\eta(G)) \leq p - 1$.*

Proof Let T be a spanning tree of a graph G. Let $A = \{e_1, e_2, e_3, \cdots, e_k\}$ be the set of edges of spanning tree T, A covers all the vertices and cut vertices of a graph $\eta(G)$. Hence,$\gamma_t(\eta(G)) \leq A = p - 1$. □

Now we obtain the relationship between total lict domination and total domination of a line graph.

Theorem 9 *For any graph G,with k number of cut vertices,*

$$\gamma_t(\eta(G)) \leq \gamma_t(L(G)) + k.$$

Proof We consider the following two cases.

Case 1 $k = 0$.

Then the graph G is non-separable, and in that case $\eta(G) = L(G)$. Hence, $\gamma_t(\eta(G)) = \gamma_t(L(G))$.

Case 2 $k \neq 0$.

Let D be a total dominating set of $L(G)$ and let S be the set of cut vertices which is not incident with any edge of D, then for each cut vertex $v_c \in S$, choose exactly one edge in

E_1, where $E_1 = \{e_j \in E(G)/e_j$ is incident with v_c and $e_j \in N(D)\}$ with $|E_1| = |v_c|$. Hence, $\gamma_t(\eta(G)) \leq \gamma_t(L(G)) + |E_1| = \gamma_t(L(G)) + |v_c| = \gamma_t(L(G)) + k$.

From Cases 1 and 2, the result follows. □

In the following theorems we obtain total lict domination of any tree in terms of different parameters of G.

Theorem 10 *For any tree T with k number of cut vertices,$\gamma_t(\eta(G)) \leq k + 1$, further equality holds if $T = K_{1,p}$, $p \geq 3$.*

Proof Let $A = \{v_1, v_2, v_3, \cdots, v_k\} \subset V(G)$ be the set of all cut vertices of a tree T with $|A| = k$. Since every edge in T is incident with at least one element of A, A covers all the edges and cut vertices of $\eta(G)$, if for every cut vertex $v \in A$ there exists a vertex $u \in A$,$u \neq v$, such that v is adjacent to u. Otherwise let $e_1 \in E(G)$ such that e_1 is incident with A, so that $\gamma_t(\eta(G)) \leq \{A \cup e_1\} = |A| + 1 = k + 1$.

To prove the equality, let $K_{1,p}$ be a star and C be the cut vertex and e be any edge of $K_{1,p}$. Then $D = \{C \cup e\}$ is the γ_t set of $\eta(G)$ with cardinality $k + 1$. □

Theorem 11 *For any tree T,$\gamma_t(\eta(T)) \geq \chi(T)$ and equality holds for all star graph $K_{1,p}$.*

Proof $\chi(T) = 2$ and $2 \leq \gamma_t(T) \leq p$. Hence, $\gamma_t(\eta(T)) \geq \chi(T)$. For $T = K_{1,p}$, clearly $\chi(T) = 2$. Using Theorem 1(iii), the equality follows. □

Theorem 12 *For any tree T ,$\gamma_t(\eta(T)) \geq \omega(T)$.*

proof The result follows from Theorem 11 and Theorem D. □

Theorem 13 *For any tree T, $\gamma_t(\eta(T)) \geq \dfrac{q}{\beta_0(T)}$.*

Proof The result follows from Theorem 11 and Theorem D. □

Theorem 14 *For any tree T, $\gamma_t(\eta(T)) \leq \gamma_t(T)$.*

Proof Let T be a tree and D be γ_t of T. Let E_1 denotes the edge set of the induced graph $\langle D \rangle$. Let F be the set of cut vertices which are not incident with any edge of E_1. we consider the following two cases.

Case 1 If $F = \Phi$, and in $\eta(T)$ if E_1 does not contains any isolates then E_1 is a total lict dominating set of T. Otherwise for each isolated edge $e_i \in E_1$, choose exactly one edge in E_2, where $E_2 = \{e_j \in E(T)/e_j \in N(e_i)\}$. Then $D^* = E_1 \cup E_2$ is a total lict dominating set of tree T. Hence, $\gamma_t(\eta(T)) \leq |D^*| \leq |D| = \gamma_t(T)$.

Case 2 If $F \neq \Phi$, then for each cut vertex $v_c \in F$. Let $E_2 = \{e_j \in E(T)/e_j \in N(e_i)$ and incident with $v_c\}$.Then $D^* = E_1 \cup E_2$ is a total lict dominating set of tree T. Hence, $\gamma_t(\eta(T)) \leq |D^*| \leq |D| = \gamma_t(T)$.

From Cases 1 and 2, the result follows. □

Theorem 15 *For any tree T with $p \geq 3$, in which every non-end vertex is incident with an end vertex, then $\gamma_t(\eta(T)) \leq \beta_0(T)$.*

Proof We consider the following two cases.

Case 1 T=$K_{1,p}$.

Noticing that $\beta_0(T) = p - 1 \geq 2$ for $p \geq 3$, and using Theorem 1(iii), the result follows. Hence,$\gamma_t(\eta(T)) \leq \beta_0(T)$.

Case 2 $T \neq K_{1,p}$.

Let $B = \{v_1, v_2, v_3, \cdots, v_m\} \subset V(G)$ such that $|B| = \beta_0(T)$. Let $S \subseteq B$ be the set of k end vertices of T and $N \subseteq B$ be the set of l non-end vertices of T such that $S \cup N = B$. In T, for each vertex $v_i \in S$ there exists cut vertex $C_i \in N(v_i)$. Then in $\eta(T)$ the cut vertex C_i covers the edges incident with cut vertex C_i of T where $i = 1, 2, 3, 4, 5, ...k$ and for each vertex $v_i \in N$ in T, a vertex $v_j \in \eta(T)$ which is a cut vertex of T covers all the edges incident with v_j where $j = 1, 2, 3, 4, 5.......l$. Thus $\{C_i\}_{i=1}^{k} \cup \{v_j\}_{j=1}^{l}$ forms a total lict dominating set of T. Hence $\gamma_t(\eta(T)) \leq |S \cup N| \leq |B| = \beta_0(T)$.

From case(1) and case(2) the result follows. □

Theorem 16 *Let T be any order $p \geq 3$ and n be the number of pendent edges of T, then $n \leq \gamma_t(\eta(S(T))) \leq 2(p-1) - n$ and equality holds for all $K_{1,p}$.*

Proof Let $u_1v_1, u_2v_2, u_3v_3, u_4v_4, \cdots, u_nv_n$ be the pendent edges of T. Let w_i be the vertex set of $S(T)$ that subdivides the edges $u_iv_i, i = 1, 2, 3, 4, \cdots, n$. Any total lict dominating set of $S(T)$ contains the edges $u_iw_i, i = 1, 2, 3, 4, \cdots, n$ and hence $\gamma_t(\eta(S(T))) \geq n$. Further $E(S(T)) - S$, where S is the set of all pendent edges of $S(T)$ forms a total lict dominating set of $S(T)$. Hence,$\gamma_t(\eta(S(T))) \leq 2(p-1) - n$.

Notice that the edges of $D = \{u_iw_i\}, i = 1, 2, 3, \cdots, n$ will forms a γ_t of $\eta(S(T))$ for $K_{1,p}$. Thus, the equality $\gamma_t(\eta(S(T))) = n$. Similarly, the set $\{E(S(T) - S\}$ will forms a γ_t of $\eta(S(T))$ for $K_{1,p}$. So $\gamma_t(\eta(S(T))) = 2(p-1) - n$. □

Now we obtain the relation between total lict domination in terms of complimentary edge domination, total domination and split domination and non-split domination.

Theorem 17 *For any graph G if $\gamma_e(G) = \gamma_e^{'}(G)$, then $\gamma_t(\eta(G)) \geq \gamma_e^{'}(G)$.*

Proof Let us consider the graph G, with $\gamma_e(G) = \gamma_e^{'}(G)$ and using Theorem 2.2, the result follows. □

Corollary 1 *Let D be the γ_e set of a non-separable graph G then, $\gamma_t(\eta(G)) \geq \gamma_e^{'}(G)$.*

Proof Since every complementary edge dominating set is an edge dominating set, the follows from Theorem 2. □

Theorem 18 *For any non-separable graph G with $p \geq 3$, then $\gamma_t(G) \leq \gamma_t(\eta(G))$, equality holds for all cycle C_p.*

Proof Let $D = \{v_1, v_2, v_3, \cdots, v_k\}$ be a γ_t set of a graph G. Let $E^* = \{e_i \in E(G)/e_i$ is incident with $v_i\}, i = 1, 2, 3, 4, \cdots, k$. Then every edge in $\langle E(G) - E^* \rangle$ is adjacent to at least one edge in E^*. Clearly E^* covers all the vertices in $\eta(G)$, and $\langle E^* \rangle$ does not contain any isolates, E^* is a total lict dominating set of graph G and $|D| \leq |E^*|$. Hence,$\gamma_t(G) \leq \gamma_t(\eta(G))$.

For any cycle C_p, $\eta(G) = L(G)$,$\gamma_t(L(G)) = \gamma_t(G)$. Hence,$\gamma_t(G) = \gamma_t(\eta(G))$. □

Theorem 19 *For any cycle C_p $p \geq 3$, $\gamma_s(C_p) \leq \gamma_t(\eta(C_p)) \leq \gamma_{ns}(C_p)$.*

Proof We consider the following two cases.

Case 1 $\gamma_s(C_p) \leq \gamma_t(\eta(C_p))$.

Let $A = \{v_1, v_2, v_3, \cdots, v_k\}$ be a γ_s dominating set of cycle C_p. For any cycle C_p, $\eta(G) = L(G)$, the corresponding edges $B = \{e_1, e_2, e_3, \cdots, e_k\}$ will be a split dominating set of $\eta(G)$. Since $\langle B \rangle$ is disconnected, $\gamma_t(\eta(C_p)) \leq \gamma_s(C_p) + 1$. Hence,$\gamma_s(C_p) \leq \gamma_t(\eta(C_p))$.

Case 2 $\gamma_t(\eta(C_p)) \leq \gamma_{ns}(C_p)$.

Let $A = \{v_1, v_2, v_3, \cdots, v_k\}$ be a γ_{ns} dominating set of cycle C_p. For any cycle C_p, $\eta(G) = L(G)$, the corresponding edges $B = \{e_1, e_2, e_3, \cdots, e_k\}$ will be a split dominating set of $\eta(G)$. Since $\langle B \rangle$ is connected. Hence, $\gamma_t(\eta(C_p)) \leq \gamma_{ns}(C_p)$.

The result follows from Cases 1 and 2. □

Now we obtain the total lict dominating number in terms of independence number and edge covering number.

Theorem 20 *For any graph G,$\gamma_t(\eta(G)) \leq 2\beta_1(G)$.*

Proof Let S be a maximum independent edge set in a graph G. Then every edge in $E(G) - S$ is adjacent to at least one edge in S. Let D be the set of cut vertices that is not incident with any edge of S and let $E_1 = \{e_i \in E(G) - S/e_i \in N(S)\}$. We consider the following two cases.

Case 1 If $D = \phi$, then for each edge $e_j \in S$, pick exactly one edge $e_i \in E_1$, such that $e_i \in N(e_j)$. Let D_1 be the set of all such edges with $|D_1| \leq |S|$. Then $F = S \cup D_1$ is a total lict dominating set of G. Hence, $\gamma_t(\eta(G)) \leq |S \cup D_1| = |S| + |D_1| \leq |S| + |S| = 2\beta 1(G)$.

Case 2 If $D \neq \phi$, then for each cut vertex $v_c \in D$. Let $E_2 = \{e_i \in E(G) - S/e_j \in N(S)$ and incident with $v_c\}$, $E_3 = \{e_k \in S/e_k \in N(E_2)\}$ and $D_2 = S - E_3$. Now for each edge $e_l \in D_2$, pick exactly one edge in $e_i \in E_1$, such that e_l is adjacent to e_i. Let D_3 be the set of all such edges. Then $F = D_2 \cup D_3 \cup E_2 \cup E_3$ is a total lict dominating set of G. Hence,

$$\begin{aligned}
\gamma_t(\eta(G)) &\leq |F| = |D_2 \cup E_3 \cup D_3 \cup E_2| \\
&\leq |D_2 \cup E_3| + |D_3 \cup E_2| \\
&= |S| + |S| = 2|S| = 2\beta_0(G)
\end{aligned}$$

From Cases 1 and 2,the result follows. □

Theorem 21 *For any graph G,$\gamma_t(\eta(G)) \leq 2\alpha_0(G)$.*

Proof Let $S = \{v_1, v_2, v_3, v_4, \cdots, v_k\} \subset V(G)$ such that $|S| = \alpha_0(G)$. Then for each vertex v_i, choose exactly one edge in E_1 where $E_1 = \{e_i \in E(G)/e_i$ is incident with $v_i\}$ such that $|E_1| \leq |S|$. Let D be the set of cut vertices that is not incident with any edge of E_1 and let $E_2 = \{e_j \in E(G) - E_1/e_j \in N(E_1)\}$. We consider the following two cases.

Case 1 If $D = \phi$, then for each edge $e_i \in E_1$, pick exactly one edge $e_j \in E_2$,such that $e_j \in N(e_i)$. Let D_1 be the set of all such edges with $|D_1| \leq |E_1| = |S|$. Then $F = E_1 \cup D_1$ is a total lict dominating set of G. Hence,$\gamma_t(\eta(G)) \leq |E_1 \cup D_1| = |E_1| + |D_1| \leq |S| + |S| = 2\alpha_0(G)$.

Case 2 If $D \neq \phi$, then for each cut vertex $v_c \in D$. Let $E_3 = \{e_l \in E(G) - E_1/e_l \in N(E_1)$ and incident with $v_c\}$,

$E_4 = \{e_k \in E_1/e_k \in N(E_3)\}$ and $D_3 = E_1 - E_4$. Now for each edge $e_r \in D_2$, pick exactly one edge in $e_j \in E_2$, such that e_r is adjacent to e_j. Let D_3 be the set of all such edges. Then $F = D_2 \cup D_3 \cup E_3 \cup E_4$ is a total lict dominating set of G. Hence,

$$\begin{aligned}
\gamma_t(\eta(G)) &\leq |F| = |D_2 \cup E_4 \cup D_3 \cup E_3| \\
&\leq |D_2 \cup E_4| + |D_2 \cup E_4| \\
&= |E_1| + |E_1| = |S| = 2\alpha_0(G)
\end{aligned}$$

From Cases 1 and 2, the result follows. □

Now we obtain the total lict dominating number of a subdivision graph of a graph G in terms of edge independence number and number of vertices of a graph G.

Theorem 22 *For any graph G, $\gamma_t(\eta(S(G))) \leq 2q - 2\beta_1 + p_0$, where p_0 is the number of vertices that subdivides β_1.*

Proof Let $A = \{u_iv_i/1 \leq i \leq n\}$ be the edge set of a graph G. Let $X = \{u_iv_i/1 \leq i \leq n\}$ be a maximum independent edge set of graph G. Then X is edge dominating set of a graph G. Let w_i be the vertex set of $S(G)$ and let $p_0 \in w_i$ be the set of vertices that subdivides X. Then for each vertex p_0, choose exactly one edge in E_1,where $E_1 = \{u_iw_i$ or $w_iv_i \in S(G)/u_iw_i$ or w_iv_i is incident with p_0 and adjacent to $A - X\}$. Let $F = \{\{\{A - \{X\}\} - \{E_1\}\}$ covers all the edges and cut vertices of $S(G)$. Hence,$\gamma_t(\eta(S(G))) \leq F = |A - X - E_1| = 2q - 2\beta_1 + p_0$.□

Theorem 23 *For any non-separable graph G,*

(i) $\gamma_t(\eta(S(K_p)) = 2\lceil \frac{p}{2} \rceil$.
(ii) $\gamma_t(\eta(S(K_{p,q})) = 2q(p \leq q)$.
(iii) $\gamma_t(\eta(S(G)) = 2(p - \beta_1)$.

Proof Using the definitions of total lict dominating set and total edge dominating set of a graph, the result follows from Theorem C. □

Next, we obtain the Nordhus-Gaddam results for a total domination number of a lict graph.

Theorem 24 *For any connected graph G of order $p \geq 3$ vertices,*

(*i*) $\gamma_t(\eta(G)) + \gamma_t(\eta(\bar{G})) \leq 4\lceil \frac{p}{2} \rceil$.

(*ii*) $\gamma_t(\eta(G)) * \gamma_t(\eta(\bar{G})) \leq 4\lceil \frac{p}{2} \rceil^2$.

Proof The result follows from Theorem B and Theorem 20. □

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