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Numerical Solution of the Robin Problem by Extrapolated A.D.I. Methods†

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In this paper the Robin problem for elliptic equations is considered and its numerical integration by several variants of the Extrapolated form of the Alternating Direction Implicit (E.A.D.I.) methods is discussed. A comparison among the E.A.D.I. variants included here as well as those studied in an earlier paper is carried out.

1. General Considerations

IN AN earlier paper (Iordanidis, 1971) the author studied the numerical solution of the Robin problem and determined a fixed acceleration parameter as well as a varying accelerating set of Samarskii and Andreyev type. The comparison between the two kinds of accelerating parameters established the superiority of the Samarskii–Andreyev parameter cycle, and thus the results arrived at by Keast (1968) were supplemented.

In this paper two other alternative sets of accelerating parameters are defined, namely the Douglas parameter cycle (Douglas, 1962) and a set with a varying extrapolation parameter, in contrast to the previous sets which assume it to be fixed. Finally, a comparison among the four alternative kinds of accelerating parameters will be carried out.

The problem we shall be dealing with consists of Laplace's equation

$$u_{x_1x_1}(x_1, x_2)+u_{x_2x_2}(x_1, x_2) = 0 \tag{1}$$

considered on the region $R = \{(x_1, x_2): 0 \leqslant x_1 \leqslant s_1, 0 \leqslant x_2 \leqslant s_2\}$, subject to the boundary condition

$$u_n+\alpha(x_1, x_2)u = H(x_1, x_2) \tag{2}$$

along the boundary ∂R of R; u_n stands for the normal derivative directed away from the region R and α, H are given functions properly defined on ∂R.

For the numerical solution of problems (1) and (2), a rectilinear grid is imposed on the region $R \cup \partial R$ of grid spacings h_1 and h_2, and the Laplacian operator is discretized by the five-point difference formula in a similar fashion to the one described in Iordanidis (1971) thus producing $(M_1 + 1)(M_2 + 1)$ linear equations whose matrix formulation is

$$(A_1+A_2)u = K. \tag{3}$$

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The vectors u and K represent the values of the unknown function at grid points and the constant terms respectively;

$$A_1 = I_{(M_2+1)} \otimes U^{(1)}_{(M_1+1)}, \qquad A_2 = \sigma U^{(2)}_{(M_2+1)} \otimes I_{(M_1+1)},$$

with $\sigma = h_1^2/h_2^2$, and $M_i = s_i/h_i$, $i = 1, 2$; I_M the unit matrix of order M and $U^{(i)}_M$ a tridiagonal matrix of order M as given by U_i in Iordanidis (1971).

2. The Iterative Scheme

For the numerical solution of (3) we shall make use of the following E.A.D.I. procedure (see Hadjidimos, 1970)

$$\begin{aligned}(I+rA_1)u^{(*)} &= [(I+rA_1)-\omega r(A_1+A_2)]u^{(n)}+\omega rK,\\ (I+rA_2)u^{(n+1)} &= u^{(*)}+rA_2u^{(n)},\end{aligned} \tag{4}$$

where $u^{(n)}$ is the approximating solution vector at the nth iteration ($u^{(0)}$ is arbitrary); $u^{(*)}$ is an intermediate vector iterate; I is the unit matrix of order $(M_1+1)(M_2+1)$; ω is the extrapolation parameter and r the acceleration parameter. Further, if we assume that the eigenvalue spectra of A_i, $i = 1, 2$ are given by (L_i, U_i), $i = 1, 2$ [more details can be found in Iordanidis (1972]) we shall have the following cases.

Case I. Varying r and Constant ω

A "good" parameter sequence $\{r_m\}$ of Douglas type may be defined on the basis of either spectrum. For the moment, let us proceed on the spectrum of A_1.

So, we start off with $z^{(1)} = L_1$ and define the finite sequences:

$$z^{(m+1)} = \frac{d}{c}z^{(m)}, \qquad r_m = c/z^{(m)}, \qquad m = 1, 2, \ldots, m_0. \tag{5}$$

The common length of the sequences is determined by

$$z^{(m_0)} < U_1 \leqslant z^{(m_0+1)} \tag{6}$$

and the positive parameters c, d satisfy

$$c \leqslant 1 \leqslant d. \tag{7}$$

Apparently, because of (6), we have

$$[z^{(1)}, z^{(m_0+1)}] \supseteq [L_1, U_1]$$

and besides, it can be proved that the sequence $\{r_m\}$ is such that for any

$$z \in [z^{(1)}, z^{(m_0+1)}]$$

there is at least one, and at most two, $i \in \{1, 2, \ldots, m_0\} \equiv N$, say i^*, satisfying $c \leqslant r_{i^*} \cdot z \leqslant d$. For any other $i \in N - \{i^*\}$ we shall have

$$c\frac{L_1}{U_1} \leqslant r_i \cdot z \leqslant c\frac{U_1}{L_1}.$$

Moreover, it can readily be proved that if we iterate m_0 times with the parameter sequence r_m, $m = 1, 2, \ldots, m_0$, then for any $a^{(i)} \in [L_i, U_i]$, $i = 1, 2$ there exists $m^* \in N$ satisfying

$$c \leqslant r_{m^*}a^{(1)} \leqslant d, \qquad c\frac{L_2}{U_1} \leqslant r_{m^*}a^{(2)} \leqslant c\frac{U_2}{L_1}, \tag{8}$$

while for any $m \in N-\{m^*\}$ the inequalities

$$c\frac{L_1}{U_1} \leqslant r_m a^{(1)} \leqslant c\frac{U_1}{L_1}, \qquad c\frac{L_2}{U_1} \leqslant r_m a^{(2)} \leqslant c\frac{U_2}{L_1} \tag{9}$$

hold. Now, the reduction matrix for (4) at the $(n+1)$th iteration will be given by

$$T_{n+1} = I-\omega r_{n+1}(I+r_{m+1}A_1)^{-1}(I+r_{n+1}A_2)^{-1}(A_1+A_2)$$

and therefore its amplification factor will be

$$p_{i,j} = 1-\omega f, \quad \text{with} \quad f(r_{n+1}a_i^{(1)}, r_{n+1}a_j^{(2)}) = \frac{r_{n+1}(a_i^{(1)}+a_j^{(2)})}{(1+r_{n+1}a_i^{(1)})(1+r_{n+1}a_j^{(2)})}, \tag{10}$$

with $a^{(i)}$ denoting the eigenvalues of A_i, $i = 1, 2$. On the other hand, the error vectors are associated with the equation

$$e^{(n+1)} = \left[\prod_{i=1}^{n+1} T_i\right] e^{(0)}$$

and hence the error decay within a whole cycle of iterations with the parameters r_m, $m = 1, 2, \ldots, m_0$ will be

$$\|e^{(m_0)}\|/\|e^{(0)}\| \leqslant \left\|\prod_{i=1}^{m_0} T_i\right\| = \max_{i,j}\left|\prod_{m=1}^{m_0} p_{i,j}(\omega, r_m)\right| \leqslant$$
$$\max\{|1-\omega f_M^*|, |1-\omega f_m^*|\} \prod_{m\in N-\{m^*\}} \max\{|1-\omega f_M|, |1-\omega f_m|\},$$

where f_M^*, f_m^* stand for the extrema of f under (8) and f_M, f_m for the extrema of f under (9).

Further if we set $p_1^*(c, d) = \max\{|1-\omega f_M^*|, |1-\omega f_m^*|\}$ and require that

$$\max\{|1-\omega f_M|, |1-\omega f_m|\} \leqslant 1,$$

then the above error decay becomes

$$\|e^{(m_0)}\|/\|e^{(0)}\| \leqslant p_1^*(c, d). \tag{11}$$

Now because of (6) and (11) we easily obtain the number of iterations for attaining an error decay ε as

$$I_1 \sim \frac{\ln(L_1/U_1)\ln(\varepsilon)}{\ln(c/d)\ln p_1^*(c, d)}. \tag{12}$$

Apparently, the same process could be carried out on the basis of the second spectrum, which would give the number of iterations for the attainment of the same error decay as

$$I_2 \sim \frac{\ln(L_2/U_2)\ln(\varepsilon)}{\ln(c/d)\ln p_2^*(c, d)}. \tag{13}$$

Case II. Varying ω and Constant r

Let us now assume that the extrapolation parameter varies cyclically and that it takes on values from the finite sequence $\{\omega_k\}$ with length k_0.

The iterative matrix for scheme (4) this time will be

$$T_{k+1} = I-\omega_{k+1}r(I+rA_1)^{-1}(I+rA_2)^{-1}(A_1+A_2) \tag{14}$$

while the error vector will become

$$e^{(k_0)} = \left(\prod_{k=1}^{k_0} T_k\right) e^{(0)}.$$

Accordingly, the error decay within a cycle will be

$$\|e^{(k_0)}\|/\|e^{(0)}\| \leqslant \left\|\prod_{k=1}^{k_0} T_k\right\| \equiv \|F_k(A)\|, \tag{15}$$

where $F_k(x) = \prod_{k=1}^{k_0} (1-\omega_k x)$ denotes a polynomial of degree k_0, and

$$A = r(I+rA_1)^{-1}(I+rA_2)^{-1}(A_1+A_2).$$

Obviously, the matrix A is symmetric and therefore

$$\|F_k(A)\| = \max_{a \leqslant x \leqslant b} |F_k(x)| \tag{16}$$

with

$$a = \min_{i,j} \left\{ \frac{r(a_i^{(1)}+a_j^{(2)})}{(1+ra_i^{(1)})(1+ra_j^{(2)})} \right\}, \qquad b = \max_{i,j} \left\{ \frac{r(a_i^{(1)}+a_j^{(2)})}{(1+ra_i^{(1)})(1+ra_j^{(2)})} \right\}.$$

Further, by definition, we have

$$F_k(0) = 1 \quad \text{and} \quad F_k\left(\frac{1}{\omega_i}\right) = 0, \qquad i = 1, 2, \ldots, k_0, \tag{17}$$

and from (15), (16) and (17) we easily conclude that the problem becomes the following minimax one within the set of all real polynomials $F_k(x)$ of degree k_0 satisfying $F_k(0) = 1$:

$$\min_{F_k \in P_{k_0}} \{ \max_{a \leqslant x \leqslant b} |F_k(x)| \}. \tag{18}$$

The solution of the above problem (18) is unique (Forsythe & Wasow, 1960) and given by

$$F_k(x) = C_{k_0}\left(\frac{b+a-2x}{b-a}\right) \Big/ C_{k_0}\left(\frac{b+a}{b-a}\right),$$

with $C_{k_0}(x)$ signifying the ordinary normalized Chebyshev polynomial of degree k_0. Finally, the error decay attained within a cycle of k_0 iterations will be

$$\left[C_{k_0}\left(\frac{b+a}{b-a}\right)\right]^{-1}$$

and for obtaining a predetermined accuracy ε, the required number of iterations will be

$$I_{CHk_0} \sim -k_0 \frac{\ln(\varepsilon)}{\ln C_{k_0}\left(\frac{b+a}{b-a}\right)}. \tag{19}$$

3. Comparison of the Alternative Settings

To compare the efficiency of the four E.A.D.I. settings discussed here and in Iordanidis (1971), we first derived from quantities (12) (13) and (19) the measures of calculation CVD and CH (for the Douglas parameter cycle and the varying extra-

polation parameter respectively) which are compatible to measures CF and CVS produced in Iordanidis (1971) (for the cases of constant r and the Samarskii and Andreyev set respectively), and then we compiled Tables 1 and 2 by incorporating all the above measures for numbers of subdivisions in the range 3(1)10, 25, 50. In particular, comparing the varying extrapolation parameter to the set with fixed ω and r, we can easily establish the superiority of the first set even in the case with a cycle length of only two. Furthermore, this supremacy of the varying extrapolation parameter becomes rather pronounced when sequences of larger lengths are used. So, for the case of 50 subdivisions, when we employ two extrapolation parameters, we get a convergence which is twice as fast for both problems, while if we use 9 extrapolation parameters, we achieve a convergence which is five- and six-times faster than that obtained with fixed ω and r.

TABLE 1

Problem 1. *Parameters* $p_i = 1{\cdot}0$, $q_i = 1{\cdot}0$, $i = 1, 2$

S/D	CF	CVS	CVD	CH $k_0 = 2$	CH $k_0 = 4$	CH $k_0 = 9$
3	1·1600	1·4796	1·6517	0·8605	0·7495	0·6991
4	1·5404	1·7440	1·9750	1·0840	0·9152	0·8411
5	1·9218	1·9522	2·2260	1·2988	1·0668	0·9675
6	2·3037	2·1236	2·4307	1·5080	1·2085	1·0827
7	2·6847	2·2687	2·6033	1·7135	1·3432	1·1896
8	3·0670	2·3952	2·7526	1·9165	1·4726	1·2902
9	3·4508	2·5073	2·8839	2·1173	1·5976	1·3854
10	3·8312	2·6018	3·0013	2·3165	1·7191	1·4763
25	9·5721	3·4797	4·0176	5·2400	3·3580	2·5766
50	19·1758	4·1427	4·7850	10·0635	5·8725	4·0150

TABLE 2

Problem 2. *Parameters* $p_i = -0{\cdot}25$, $q_i = 1{\cdot}0$, $i = 1, 2$

S/D	CF	CVS	CVD	CH $k_0 = 2$	CH $k_0 = 4$	CH $k_0 = 9$	CH $k_0 = 37$
3	2·4138	2·1678	2·4835	1·5678	1·2482	1·1144	1·0464
4	3·1990	2·4352	2·7994	1·9855	1·5158	1·3237	1·2286
5	3·9880	2·6449	3·0455	2·3969	1·7676	1·5121	1·3896
6	4·7765	2·8168	3·2468	2·8048	2·0087	1·6865	1·5356
7	5·5721	2·9636	3·4171	3·2100	2·2422	1·8502	1·6701
8	6·3596	3·0897	3·5645	3·6132	2·4998	2·0053	1·7955
9	7·1558	3·2021	3·6942	4·0140	2·6925	2·1533	1·9132
10	7·9480	3·3023	3·8101	4·8144	3·1298	2·4339	2·1316
25	19·7688	4·1718	4·8186	10·3606	6·0248	4·0969	3·3110
50	39·5313	4·8332	5·5831	20·2534	11·0391	6·6334	4·8276

Finally, as regards the performance of the varying extrapolation parameter against the Douglas and the Samarskii and Andreyev parameter cycle, we would like to stress two points. First, that a varying extrapolation parameter sequence of appropriate cycle length can theoretically produce, for any number of mesh subdivisions, a better convergence rate than any of the other alternatives. Second, that for a fixed varying extrapolation cycle length, its superiority over the other two cycles recedes as we move on to finer grids and a rather long sequence of extrapolation parameters is required for restoring its supremacy. Thus, if we assume a grid of 50 subdivisions in each direction, then sequences of 9 and 37 extrapolation parameters are called upon for establishing better convergence rates for problems 1 and 2 respectively, as is shown in Tables 1 and 2.

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