The Buckling of a Delaminated Sublaminate on an Elastic Foundation

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ABSTRACT

A one-dimensional model of a delamination in a laminated structure is considered in order to evaluate the effective edge fixity coefficient and the load at which the delaminated sublaminate buckles. A Rayleigh-Ritz energy method is applied to the model and the effective boundary fixity coefficient, c', and the relative delamination tip deflection, w_d/w_c, are determined. Appreciable reductions in the buckling load, as compared to classical clamped boundary conditions, and nonzero deflections at the delamination front are predicted. The model also provides a rule of thumb as to the area around a delaminated sublaminate which must be modeled as an elastic foundation in order to properly predict the buckling behavior of the sublaminate.

1. INTRODUCTION

THE UNDERSTANDING OF THE BEHAVIOR OF DELAMINATIONS IN LAMINATED structures is paramount to the design process of composite components. Delaminations rising from initial imperfections or in-service damage can greatly affect the stiffness and strength of the component. Typical analyses of the buckling and growth of delaminated sublaminates have assumed clamped boundary conditions at the delamination tip [1–2]. Even in the case of undelaminated structures, delaminations can arise from natural manufacturing voids and buckling can occur at the ply level leading to gross delamination and failure [3]. The same phenomenon can occur due to events such as impact [4].

This phenomenon can be likened to plates or shells buckling on an elastic foundation. During the cure process, resin flows between the plies leaving an interply region of matrix on the order of one or two fiber diameters thick thus forming the

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elastic foundation [5]. It is this foundation which affects the boundary condition at the delamination tip. Observed scatter in buckling data of delaminated sublaminates has been attributed to the quality of the delamination front [6]. Moreover, growth of delaminations is dependent on the fracture of the interply region via normal and shear stresses imposed by the buckling of the sublaminate. The foundation is thus of paramount importance in the buckling and growth of a delaminated sublaminate, yet is ignored when clamped boundary conditions are theoretically imposed at the delamination front.

Thus, it is important to determine the region of the elastic foundation which has an effect on the overall buckling behavior of the plate or sublaminate as well as the magnitude of this effect. At some distant point away from the delamination tip, clamped boundary conditions do exist. Beyond this point the foundation has no effect on the delaminated sublaminate. The effect of this region between the delamination tip and the clamped conditions is to reduce the compliance and buckling load of the sublaminate and thus the overall compliance of the structure. Additionally, it is in this region where the buckling will cause stresses in the interply matrix layer leading to delamination growth.

The objective of this analysis is to investigate the effect of including an elastic foundation on the buckling of a delaminated sublaminate. The effect will be quantified by looking at the buckling load of the sublaminate, and thus the effective fixity coefficient, and the relative deflection at the delamination tip.

2. ONE-DIMENSIONAL MODEL

The basic model is shown in Figure 1. It is a beam, of unit width, supported partially by an elastic foundation and clamped at the ends. The important parameters are:

L = overall length,
l_f = foundation length,
l_u = unsupported length,
t_f = height of the foundation,
h = thickness of the beam,
k = spring constant of the foundation per unit length,
E = modulus of the beam,

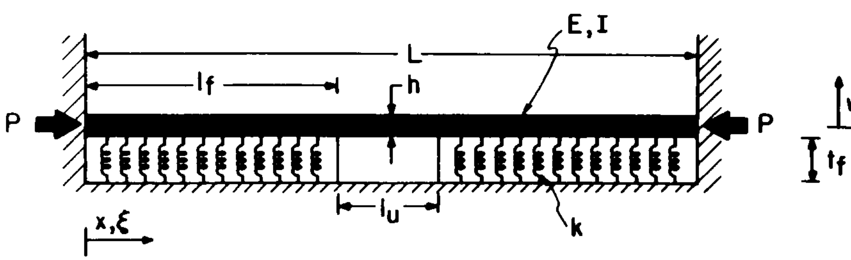

Figure 1. *Sketch of partially delaminated sublaminate model.*

I = moment of inertia of the beam, and
P = applied end load

In applying this to a through-width delaminated sublaminate, l_u corresponds to the delamination length, t_f to the interply matrix layer thickness, and h to the sublaminate thickness. The length is assumed to be much greater than the thickness and the width which reduces the problem to one dimension. Only bending is considered. Although many cases of delamination in composite laminates result in unsymmetric sublaminates which cause coupling between extension and bending, this one-dimensional model is sufficient for the main objective of this paper, that is to evaluate the effect of an elastic foundation on the boundary conditions at the delamination front.

The buckling load and shape is determined via a Rayleigh-Ritz energy method. Assumed shapes are used to determine the total potential energy of the system. The minimum point of the total potential energy with nonzero deflections describes the bifurcation point. This minimum point yields both the buckling load and the initial buckled shape.

The problem formulation begins by assuming the deflection in the z direction as a function of the longitudinal distance x

$$w(x) = \sum_{i=1}^{\infty} A_i \phi_i(x) \tag{1}$$

where

$$\phi_i(x) = \cos\,[(2i - 2)\pi x/L] - \cos\,[(2i)\pi x/L] \tag{2}$$

and the A_i are the amplitudes of the individual modes. The ϕ_i modes represent the complete symmetric Fourier series for clamped-clamped boundary conditions. The potential energy of the system is the sum of the potential energies due to the bending in the beam and the deflections in the elastic foundation minus the work done by the applied load at the ends:

$$\Pi_P = U_{Bending} + U_{Spring} - W_{Ends} \tag{3}$$

where

$$U_{Bending} = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{4a}$$

$$U_{Spring} = \frac{1}{2}\int_0^L k(x)w^2 dx \tag{4b}$$

$$W_{Ends} = \frac{P}{2} \int_0^L \left(\frac{dw}{dx} \right)^2 dx \tag{4c}$$

Note that the foundation function $k(x)$ in Equation (4b) can be rewritten as

$$k(x) = k\psi(x) \tag{5}$$

where

$$\psi(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq l_f \\ 0 \text{ if } l_f \leq x \leq L - l_f \\ 1 \text{ if } L - l_f \leq x \leq L \end{cases} \tag{6}$$

The parameters are now nondimensionalized by the total length:

$$\xi = x/L \tag{7a}$$

$$\eta = l_f/L \tag{7b}$$

$$\mu = l_u/L = 1 - 2\eta \tag{7c}$$

The variable η is the foundation ratio. For η equal to zero there is no foundation and the entire beam is unsupported. This represents the case where clamped boundary conditions are assumed at the delamination tips. For η equal to 0.5, the boundary conditions are effectively moved to infinity since the unsupported length remains a finite value with foundation elsewhere. The variable μ is the sublaminate ratio and represents the fractional amount of the beam which is unsupported. The functions ϕ_i and ψ become $\tilde{\phi}_i$ and $\tilde{\psi}$ when Equations (7a–7c) are used in Equations (2) and (6).

The principle of minimum potential energy

$$\frac{\partial \Pi_P}{\partial A_i} = 0 \tag{8}$$

is applied to obtain a system of simultaneous equations for the constants A_i:

$$\left\{ [U_B] + \frac{f}{\mu^4} [U_S] \right\} \{A_i\} = \tilde{P}[W] \{A_i\} \tag{9}$$

where the elements of the matrices are

$$U_{B_{ij}} = \int_0^1 \frac{d^2\tilde{\phi}_i}{d\xi^2} \frac{d^2\tilde{\phi}_j}{d\xi^2} d\xi \tag{10a}$$

$$Us_{ij} = \int_0^1 \tilde{\psi}\tilde{\phi}_i\tilde{\phi}_j d\xi \tag{10b}$$

$$W_{ij} = \int_0^1 \frac{d\tilde{\phi}_i}{d\xi}\frac{d\tilde{\phi}_j}{d\xi} d\xi \tag{10c}$$

and

$$\tilde{P} = \frac{PL^2}{EI} \tag{11}$$

$$f = \frac{kl_u^4}{EI} \tag{12}$$

The variable f is defined as the foundation stiffness parameter. It is a function of the material properties of the system (k and E) and of the geometry of the delamination (I and l_u). The load parameter $\tilde{P}$ is also nondimensional.

Equation (9) can be reduced to a classical eigenvalue problem:

$$[F]\{A_i\} = \lambda\{A_i\} \tag{13}$$

where

$$[F] = \left\{[U_B] + \frac{f}{\mu^4}[U_S]\right\}^{-1}[W] \tag{14}$$

and

$$\lambda = 1/\tilde{P} \tag{15}$$

The solution of Equation (13) for the largest eigenvalue yields the lowest buckling load.

The problem has thus been reduced to two parameters: the foundation stiffness parameter, f, and the foundation ratio, η. The matrices U_B and W are independent of these two parameters, and U_S is only dependent on the value of η. The solution is obtained by determining U_B and W for the general case and U_S for various values of η. The matrix F can then be calculated for various values of f and η. The largest eigenvalue of Equation (13) is then determined to obtain the buckling load. Since only the largest eigenvalue and its corresponding eigenvector are desired, they can be found by successive matrix iterations [7]. The error between successive eigenvectors is calculated by the square root of the sum of the squares of the differences between each element normalized by the initial value of the element. Convergence is assumed to be achieved when the error is less than the number of modes times 10^{-5}.

3. RESULTS AND DISCUSSION

The first 20 symmetric cosine modes are used in the assumed shape of w. This is adequate in most cases to assure convergence. The program was implemented on a DEC Pro 350 and calculates the buckling load and the buckled shape in 90 seconds for given values of f and η.

The delamination geometry and the material properties of the system are incorporated into the foundation stiffness parameter, f. For a specific value of f, the foundation ratio, η, is varied from 0 (no foundation, clamped boundary conditions at the delamination tip) to 0.5 (total foundation, clamped boundary conditions at infinity) in order to obtain the buckling behavior for various support lengths. Note that the mathematical limit of this model as η approaches 0.5 for a given delamination size, l_u, is equivalent to an infinitely long clamped beam on an elastic foundation with a finite unsupported length. If the length of the beam were to be kept constant and the delamination length were to be decreased, the results would be erroneous. That mathematical limit is equivalent to a finite beam with an infinitesimal delamination. In that case, the limit would yield the buckling load for a totally supported beam.

Values from 10^0 to 10^8 are used for f. The low values are equivalent to beams constrained by mechanical springs, and the high values are equivalent to plies constrained by an interply matrix region. Using typical values for graphite/epoxy of E_f equal to 3.5 GPa; E equal to 54 GPa, the longitudinal modulus of a quasi-isotropic AS1/3501-6 graphite/epoxy laminate; h equal to 0.134 mm (one ply thickness); and a foundation thickness, t_f, of 0.00675 mm; yields an effective foundation spring constant, k, of 522 GPa per millimeter of beam width and thus gives f approximately equal to 7.6×10^7 for a 6.35 mm long delamination.

Results are presented in terms of the effective coefficient of fixity, c', as based on the unsupported length. This is defined by the classical Euler equation.

$$c' = \frac{P_{cr}}{EI\pi^2/l_u^2} = \frac{\tilde{P}_{cr}\mu^2}{\pi^2} \tag{16}$$

The buckling load is normalized by the unsupported length and not by the total beam length since the fixity coefficient for the unsupported sublaminate is desired. The result of this grouping is that the total effect of the elastic foundation on the apparent buckling load is contained in the effective fixity coefficient. An increase in the value of the effective fixity coefficient corresponds to an increase in the buckling load given the same geometry. However, a change in the geometry will not affect the effective fixity coefficient and the buckling in the same manner. Due to the different geometrical effect on the effective fixity coefficient and the buckling load, it is possible for one to increase while the other decreases.

A plot of the values for c' versus η for various values of f is shown in Figure 2. Figure 3 is a plot of the displacement at the delamination tip, w_d, normalized by the center deflection, w_c, versus η for various values of f. Points after the asymptotic values had been reached are omitted in Figures 2 and 3. When only the asymptotic values of the parameters c' and w_d/w_c are considered, the model

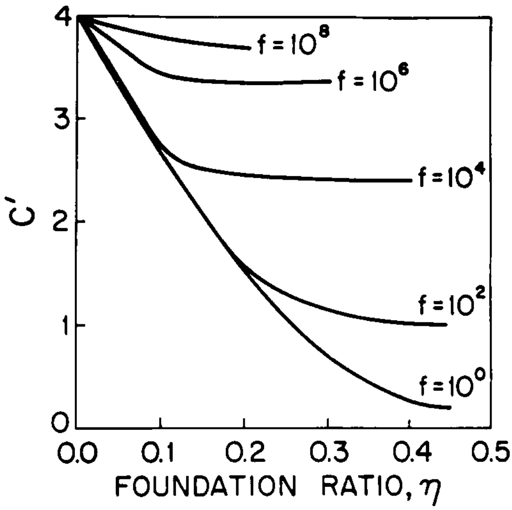

Figure 2. *Effective boundary fixity coefficient versus the foundation ratio for various values of the foundation stiffness parameter.*

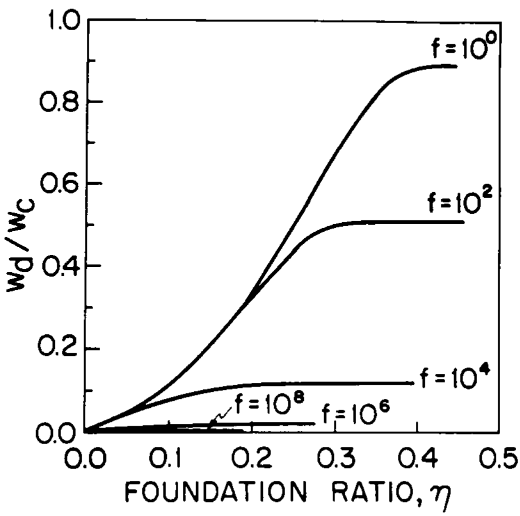

Figure 3. *Normalized delamination tip deflection versus the foundation ratio for various values of the foundation stiffness parameter.*

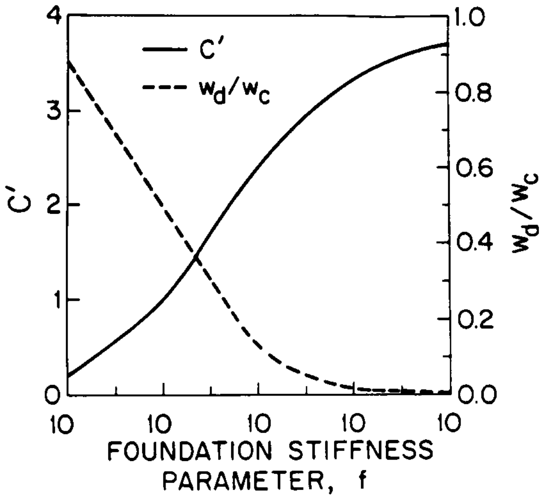

Figure 4. *Combined plot of asymptotic values of the effective boundary fixity coefficient and the normalized delamination tip deflection versus the foundation stiffness parameter.*

can be reduced to a single parameter, the foundation stiffness parameter. Figure 4 is a combined plot of the asymptotic values of c' and w_d/w_c versus f.

The analysis shows that the behavior of the delaminated sublaminate can be characterized by the foundation stiffness parameter, f, which is a function of the geometry and material properties of the system. Closer examination of the nondimensional parameter f reveals that it is made up of several other nondimensional parameters. Substituting $E_f b/t_f$ for k and $bh^3/12$ for I, where b is the unit width of the beam (the dimension not shown in Figure 1) and E_f is the modulus of the foundation, in Equation (9), the foundation parameter can be written as

$$f = 12\left(\frac{E_f}{E}\right)\left(\frac{l_u}{h}\right)^4\left(\frac{t_f}{h}\right)^{-1} \tag{17}$$

where E_f/E is the ratio of the moduli of the foundation and the sublaminate, l_u/h is the sublaminate slenderness ratio, and t_f/h is the interply thickness ratio.

The area around the delamination where the foundation has a finite effect on its buckling behavior can be determined from Figure 2. The value of η at which the asymptote is reached determines the amount of this area. Further increment in the amount of the foundation considered has no further effect on the buckling behavior of the delaminated sublaminate. In physical terms, the ratio of the foundation length, l_f, to the length of the delamination, l_u, is given by

$$\frac{\eta}{1 - 2\eta} = \frac{1 - \mu}{2\mu} \tag{18}$$

For example, for values of f greater than 10^6 the asymptote is reached at η approximately equal to 0.1. Therefore, the foundation from the delamination tip to a distance approximately one eighth of the length of the delamination affects the buckling behavior of the delaminated sublaminate.

4. APPLICATION TO QUASI-ISOTROPIC CASE

In order to understand the importance of this effect, the model is applied to a quasi-isotropic case. It is assumed that each ply is quasi-isotropic. Typical values of 54 GPa for the modulus of the sublaminate and 3.5 GPa for the modulus of the foundation are used as in the previous section. The combined effect of delamination length and depth can be seen by varying the length at different depths and determining the effective fixity coefficient. These results are shown in Figure 5. It is clear from this figure that the effect is significant, ranging from a 5% to 50% reduction of the fixity coefficient and, therefore, the buckling load as compared to a clamped-clamped beam of the same length as the undelaminated section.

The individual effect of the delamination length and depth on c' and the buckling load is assessed by varying one while maintaining the other constant. A variable length delamination is assumed to exist below one ply of a laminated structure. As the delamination length is decreased, the buckling load of a clamped beam increases. However, a decrease in the delamination length correlates to a decrease in the foundation stiffness parameter, and thus a decrease in the effective fixity coefficient as shown in Figure 6 as the difference in the normalized buckling loads between the clamped and elastic foundation boundary conditions. Thus the buckling load increases with decreasing length, but at a lesser rate than that

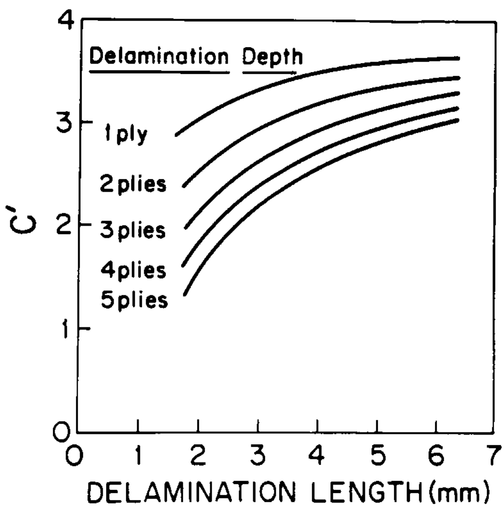

Figure 5. Asymptotic values of the effective boundary fixity coefficient versus delamination length at different depth locations.

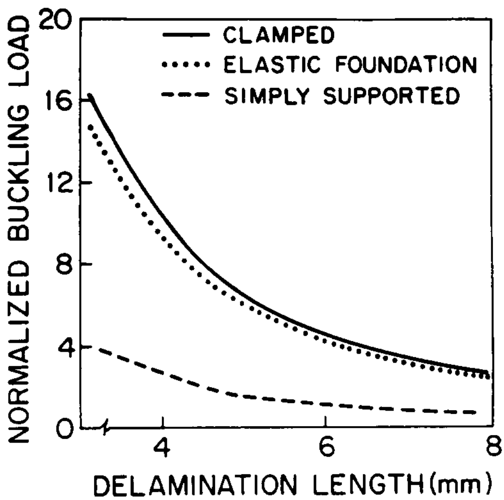

Figure 6. *Normalized buckling loads versus delamination length at one ply depth (loads normalized by simply-supported buckling load for 6.35 mm delamination at one-ply depth).*

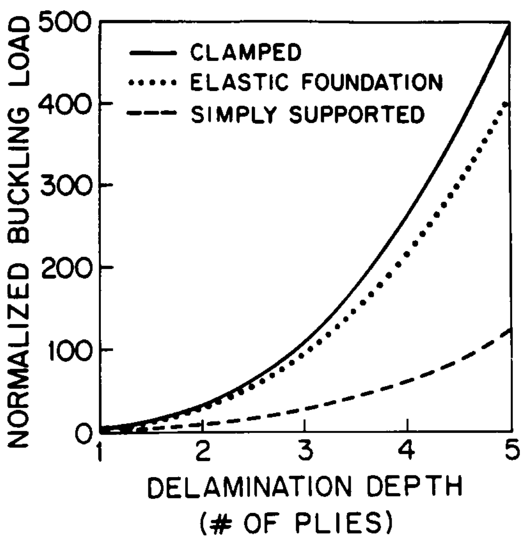

Figure 7. *Normalized buckling loads versus delamination depth for a 6.35 mm delamination (loads normalized by simply-supported buckling load for 6.35 mm delamination at one-ply depth).*

of a clamped beam. In the plot, the buckling loads are normalized by the buckling load for a simply-supported 6.35 mm delamination.

If the delamination size is kept constant at 6.35 mm and the depth is varied, the buckling load of a clamped beam increases with increasing depth. Again, however, an increase in the delamination depth correlates to a decrease in the foundation stiffness parameter, and thus a decrease in the effective fixity coefficient as shown in Figure 7. Thus the buckling load increases with increasing depth, but at a lesser rate than that of a clamped beam. Again the buckling loads are normalized by that for a simply-supported 6.35 mm delamination one ply thickness down. Also plotted in Figures 6 and 7, for comparison, are the normalized buckling loads for simply supported and clamped boundary conditions.

The importance of the effect of the foundation on the buckling behavior of a delaminated sublaminate is minimized at large values of f. This occurs when the depth of the delamination is small or the delamination length is large. Thus, in Figures 6 and 7 a geometry which corresponds to a large value of f yields a buckling load approximately equal to that of a beam with clamped boundary conditions. Relatively small values of f result in effective coefficients considerably less than 4 and buckling loads significantly less than that of a clamped beam. However, even in cases where the effect of the foundation on the buckling load is minimized, the inclusion of the matrix layer in the analysis allows out-of-plane deflections at the delamination tip. Interlaminar stresses will arise due to these deflections and eventually cause the failure of the matrix and subsequent delamination growth. Even for large values of f, the tip deflections can give rise to significant stresses in the foundation region and therefore should not be omitted when determining the failure of a delaminated sublaminate.

5. SUMMARY

When a partial elastic foundation is incorporated into a buckling analysis, the effect is to relax the boundary condition at the tip of the unsupported sublaminate. This results in a decrease of the buckling load and in displacements at the delamination tip which give rise to interlaminar normal stresses in the foundation layer. The one-dimensional model provides a rule of thumb as to the area around a delamination which should be considered in any attempt to determine its buckling, as well as postbuckling, behavior.

Furthermore, an increase in the importance of the foundation coincides with an increase in the buckling load. Thus, in the postbuckling regime of the sublaminate, the elastic foundation probably will have a greater effect provided that the postbuckling loads are of greater magnitude than the buckling load. Therefore, elements of this analysis should be included in future work on the postbuckling behavior of delaminated sublaminates.

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