A finite difference scheme for option pricing in jump diffusion and exponential Lévy models*

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Abstract

We present a finite difference method for solving parabolic partial integro-differential equations with possibly singular kernels which arise in option pricing theory when the random evolution of the underlying asset is driven by a Lévy process or, more generally, a time-inhomogeneous jump-diffusion process. We discuss localization to a finite domain and provide an estimate for the localization error under an integrability condition on the Lévy measure. We propose an explicit-implicit finite difference scheme to solve the equation and study stability and convergence of the schemes proposed, using the notion of viscosity solution. Our convergence analysis requires neither the smoothness of the solution nor the non-degeneracy of coefficients and applies to European and barrier options in jump-diffusion and pure jump models used in the literature. Numerical tests are performed with smooth and non-smooth initial conditions.

Keywords: parabolic integro-differential equations, finite difference methods, Lévy process, jump-diffusion models, option pricing, viscosity solutions.

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Contents

The shortcomings of diffusion models in representing the risk related to large market movements have led to the development of various option pricing models with jumps, where large returns are represented as discontinuities in prices as a function of time. Models with jumps allow for a more realistic representation of price dynamics and a greater flexibility in modelling and have been the focus of much recent work [11].

Exponential Lévy models, where the market price of an asset is represented as the exponential $S_t = \exp(rt + X_t)$ of a Lévy process X_t, offer analytically tractable examples of positive jump processes which are simple enough to allow a detailed study both in terms of statistical properties and as models for risk-neutral dynamics i.e. option pricing models. Option pricing with exponential Lévy models is discussed in [11, 15, 17, 19, 24]. The flexibility of choice of the Lévy process X allows to calibrate the model to market prices of options and reproduce a wide variety of implied volatility skews/smiles [10]. The Markov property of the price allows us to express prices of European and barrier options in terms of solutions of partial integro-differential equations (PIDEs) which involve, in addition to a (possibly degenerate) second-order differential operator, a non-local integral term which requires specific treatment both at the theoretical and numerical level [12].

In this paper, we propose a finite difference scheme for solving such PIDEs. Our numerical solution is based on splitting the operator into a local and a non-local part: we treat the local term using an implicit step and the non-local term using an explicit step. This idea, previously used for non-linear PDEs [3], allows for an efficient implementation in terms of speed and stability restrictions. Our scheme extends to infinite activity Lévy processes with singular kernels and does not require the diffusion part to be non-degenerate. We study the consistency and stability of this scheme, show its convergence to a viscosity solution of the partial integro-differential equation and study its numerical performance on two examples, the Merton model with Gaussian jumps and the infinite activity Variance Gamma model. Our scheme can be used for European and barrier options and can also be extended to the case of non-constant coefficients.

In section 2 we briefly discuss, following [12], the derivation of integro-differential equations for European and barrier options, first giving conditions under which option values can be described in terms of classical solutions followed by a more general result using the notion of viscosity solution, .

Solving such PIDEs by finite difference methods involves several approximations: localization of the equation to a bounded domain, treatment of the singularity due to small jumps, discretization of the equation in space and iteration in time. We discuss localization errors in section 3.1 and provide an estimate for the localization error under an integrability condition on the Lévy measure. In section 3 we propose an explicit-implicit finite difference scheme and study consistency, stability and convergence of the scheme proposed. To study the convergence of the scheme, we use the notion of *viscosity* solution, which allows to show uniform convergence to the solution without requiring a priori smoothness of the solution or non-degeneracy of coefficients.

In section 5, numerical tests are performed for smooth and non-smooth initial

conditions to assess the effect of various numerical parameters on the accuracy of the scheme.

Various numerical methods for solving such parabolic integro-differential equations have been proposed in the recent literature [2, 20, 14, 27]. In the case where the characteristic function of the log-price is known, Carr & Madan [8] propose a method using the Fast Fourier transform for pricing European options. Our scheme extends this approach to cases where an analytical expression for the characteristic function may not be available and where barriers / boundary conditions may be present. In the case of jump-diffusion models with finite jump intensity, Andersen & Andreasen [2] propose an operator splitting method where the differential part is treated using a Crank-Nicholson step and the jump integral is computed using an explicit time step. Our method applies more generally to models with infinite activity i.e. singular integral kernels; in addition, we propose an analysis of the convergence of our algorithm, which is absent in [2]. Using a variational formulation of the integro-differential equation, Zhang [27, 28] studied a finite difference scheme in the case of jump-diffusion models with finite intensity and possessing all exponential moments. These conditions rule out all models in the literature except the Merton model: our analysis does not require such restrictive conditions. Variations on this scheme are given in [14]. The variational formulation has been recently extended by Matache et al [20] to the infinite activity case using a wavelet Galerkin method. Relation with these methods is further discussed in sections 3–5.

1 Exponential Lévy models

We consider here the class of models where the risk neutral dynamics of the underlying asset is given by $S_t = \exp(rt + X_t)$ where X_t is a time-homogeneous jump-diffusion (Lévy) process.

1.1 Lévy processes: definitions

A Lévy process is a stochastic process X_t with stationary independent increments which is continuous in probability. Without loss of generality we assume that $X_0 = 0$. The characteristic function of X_t has the following form, called the Lévy-Khinchine representation [22]:

$$E[e^{izX_t}] = \exp t\phi(z) = \\ = \exp\{t(-\frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx1_{|x|\leq 1})\nu(dx))\},$$

where $\sigma > 0$ and γ are real constants and ν is a positive measure verifying

$$\int_{-1}^{+1} x^2\nu(dx) < \infty, \qquad \int_{|x|>1} \nu(dx) < \infty. \tag{1.1}$$

The random process X can be interpreted as the superposition of a Brownian motion with drift and an infinite superposition of independent (compensated) Poisson processes with various jump sizes x, $\nu(dx)$ being the intensity of jumps of size x.

In general ν is not a finite measure: $\int \nu(dx)$ need not be finite. In the case where $\lambda = \int \nu(dx) < +\infty$, the measure ν can be normalized to define a *probability measure* μ which can now be interpreted as the distribution of jump sizes:

$$\mu(dx) \quad = \quad \frac{\nu(dx)}{\lambda}.$$

The jumps of X are then described by a *compound Poisson* process with λ as jump intensity (average number of jumps per unit time) and $\mu(.)$ as jump size distribution. In this case the truncation of small jumps is not needed and the Lévy-Khinchin representation reduces to:

$$E[e^{izX_t}] = \exp\{t(-\frac{\sigma^2 z^2}{2} + i\gamma_0(\nu)z + \int_{-\infty}^{\infty}(e^{izx} - 1)\nu(dx))\}.$$

A Lévy process is a Markov process; its infinitesimal generator $L^X : f \rightarrow L^X f$ is an integro-differential operator defined by the expression :

$$\begin{aligned} L^X f(x) \;&=\; \lim_{t\to 0} \frac{E[f(x+X_t)] - f(x)}{t} \;= \\ &=\; \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \gamma\frac{\partial f}{\partial x} + \int \nu(dy)[f(x+y) - f(x) - y1_{\{|y|\leq 1\}}\frac{\partial f}{\partial x}(x)] \end{aligned} \quad (1.2)$$

which is well defined for $f \in C_0^2(\mathbb{R})$.

1.2 Exponential Lévy models

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modelled as a stochastic process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$. Under the hypothesis of absence of arbitrage there exists a measure equivalent to $\mathbb{Q}$ under which (S_t) is a martingale. We will assume in the sequel that $\mathbb{Q}$ is already a martingale measure.

In *exponential Lévy* models, the (risk-neutral) dynamics of S_t under $\mathbb{Q}$ is represented as the exponential of a Lévy process:

$$S_t = S_0 e^{rt+X_t}. \quad (1.3)$$

Here X_t is a Lévy process with characteristic triplet (σ,γ,ν), and the interest rate r is included for ease of notation. Different exponential Lévy models proposed in the financial modelling literature simply correspond to different choices for the Lévy measure ν, see [11, Chap. 3] for a review. The absence of arbitrage then imposes that $\hat{S}_t = S_t e^{-rt} = \exp X_t$ is a martingale, which is equivalent to

the following conditions on the triplet (σ,γ,ν):

$$\int_{|y|>1} \nu(dy)e^y < \infty, \tag{1.4}$$

$$\gamma = \gamma(\sigma,\nu) = -\frac{\sigma^2}{2} - \int (e^y - 1 - y1_{|y|\leq 1})\nu(dy). \tag{1.5}$$

We will assume this relation holds in the sequel. The infinitesimal generator L^X then becomes:

$$L^X f(x) = \frac{\sigma^2}{2}[\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}] + \int_{-\infty}^{\infty} \nu(dy)\ [f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}(x)]. \tag{1.6}$$

We will also use the notation $Y_t = rt + X_t$. The infinitesimal generator of Y_t is

$$Lf \quad = \quad L^X f + r\frac{\partial f}{\partial x}. \tag{1.7}$$

2 Partial integro-differential equation for option prices

The value of an option is defined as a discounted conditional expectation of its terminal payoff H_T under a risk-adjusted martingale measure (sometimes called risk-neutral probability) $\mathbb{Q}$:

$$C_t = E[e^{-r(T-t)}H_T|\mathcal{F}_t].$$

For a European call or put, $H_T = H(S_T)$. From the Markov property, $C_t = C(t,S)$ where

$$C(t,S) \quad = \quad E[e^{-r(T-t)}H(S_T)|S_t = S]. \tag{2.1}$$

Introducing the change of variable $\tau = T - t$, $x = \ln(S/S_0)$, and defining: $h(x) = H(S_0e^x)$ and $u(\tau,x) = e^{r\tau}C(T-\tau, S_0e^x)$, then

$$u(\tau,x) = E[h(x + Y_\tau)]. \tag{2.2}$$

If u is sufficiently smooth –for example, $u \in C^{1,2}$ with bounded derivatives– then by applying Ito's formula to $u(t,X_t)$ between 0 and T one can show [6, 12] that it is a classical solution of the Cauchy problem:

$$\frac{\partial u}{\partial \tau} = Lu, \quad \text{on } (0,T]\times\mathbb{R}; \qquad u(0,x) = h(x), \quad x \in \mathbb{R}. \tag{2.3}$$

Barrier options lead to initial-boundary value problems. Consider for instance an up-and-out call option with maturity T, strike K, and (upper) barrier $U > S_0$. The terminal payoff is given by

$$H_T \quad = \quad (S_T - K)^+ 1_{T<\theta},$$

where $\theta = \inf\{t \geq 0 \mid S_t \geq U\}$, the first moment when the barrier is crossed.

Due to the strong Markov property of Lévy processes, it is possible to express the value of the option $C_t = e^{-r(T-t)}E[H_T|\mathcal{F}_t]$ as a deterministic function of time t and current stock value S_t before the barrier is crossed. Namely, for any $(t, S) \in [0, T] \times (0, \infty)$ we can define

$$C_b(t, S) \quad = \quad e^{-r(T-t)}\ E[H(Se^{Y_{T-t}})1_{T<\theta_t}], \tag{2.4}$$

where $H(S) = (S - K)^+$, $\{Y_{s-t},\ s \geq t\}$ is a Lévy process, and $\theta_t = \inf\{s \geq t \mid Se^{Y_{s-t}} \geq U\}$, the first exit time after t. Then,

$$C_t = C_b(t,\ S_t)1_{t\leq\theta} \tag{2.5}$$

for all $t \leq T$. Note that outside of the set $\{t \leq \theta\}$ the objects C_t and $C_b(t, S_t)$ are different: if the barrier has already been crossed, C_t will always be zero, but $C_b(t, S_t)$ may become positive if the stock returns to the region below the barrier.

As in the European case, by going to the log variables we define

$$u_b(\tau, x) = e^{r\tau}C_b(T - \tau, S_0e^x). \tag{2.6}$$

Again, if u_b is smooth the Itô formula can be used to show that u_b is a solution of the following initial-boundary-value problem:

$$\frac{\partial u}{\partial \tau} = Lu, \qquad \text{on } (0, T] \times (-\infty, \log(U/S_0)),$$
$$u(0, x) = h(x), \quad x < \log(U/S_0); \qquad u(\tau, x) = 0, \quad x \geq \log(U/S_0).$$

Prices of down-and-out or double barrier options are defined similarly. In the case of pure jump models where $\sigma = 0$ the smoothness is not obvious: the behavior of the option price at the barrier depends on the local behavior of the Lévy process at the barrier. More generally, if smoothness of $C(.,.)$ is not known it should be seen as a viscosity solution of the PIDE, as discussed below.

2.1 Viscosity solutions: definition

Existence and uniqueness of (classical) solutions for the PIDEs considered above in Sobolev / Hölder spaces have been studied in [6, 16] in the case where the diffusion component is non-degenerate: for a Lévy process this simply means $\sigma > 0$ but more generally these results apply to jump diffusion where the diffusion coefficient is bounded away from zero. However many of the models in the financial modelling literature are pure jump models with $\sigma = 0$, for which such results are not available. A notion of solution which yields both existence and uniqueness without requiring non-degeneracy of coefficients or a priori knowledge of smoothness of solutions is the notion of viscosity solution, introduced by Crandall & Lions for PDEs [13] and extended to integro-differential equations of the type considered here in [1, 4, 21, 23, 25].

Denote by USC (respectively LSC) the class of upper semicontinuous (respectively lower semicontinuous) functions $u : [0,T) \times \mathbb{R} \to \mathbb{R}$ and by $C_p^+([0,T] \times \mathbb{R})$ the set of measurable functions on $[0,T] \times \mathbb{R}$ with polynomial growth of degree p at $+\infty$ and bounded on $[0,T] \times \mathbb{R}^-$:

$$\varphi \in C_p^+([0,T] \times \mathbb{R}) \iff \exists C > 0,\ |\varphi(t,x)| \leq C(1 + |x|^p\, 1_{x>0}). \tag{2.7}$$

Let $O = (l,u) \subseteq \mathbb{R}$ be an open interval, $\partial O = \{l,u\}$ its boundary, and $g \in C_p^+([0,T] \times \mathbb{R} \backslash O)$ a continuous function. Consider the following initial-boundary value problem on $[0,T] \times \mathbb{R}$:

$$\frac{\partial u}{\partial \tau} = Lu, \qquad \text{on } (0,T] \times O, \tag{2.8}$$

$$u(0,x) = h(x), \quad x \in O; \qquad u(\tau,x) = g(\tau,x), \quad x \notin O. \tag{2.9}$$

Definition 1 (Viscosity solution). A function $u \in USC$ is a *viscosity subsolution* of (2.8)–(2.9) if for any test function $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$ and any global maximum point $(\tau,x) \in [0,T] \times \mathbb{R}$ of $u - \varphi$, the following properties are verified:

$$\text{if } (\tau,x) \in (0,T] \times O, \qquad \left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x) \leq 0, \tag{2.10}$$

$$\text{if } \tau = 0,\ x \in \overline{O}, \qquad \min\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - h(x)\} \leq 0,$$

$$\text{if } \tau \in (0,T],\ x \in \partial O, \qquad \min\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - g(\tau,x)\} \leq 0,$$

$$\text{if } x \notin \overline{O}, \qquad u(\tau,x) \leq g(\tau,x). \tag{2.11}$$

A function $u \in LSC$ is a *viscosity supersolution* of (2.8)–(2.9) if for any test function $\varphi \in C^2([0,T] \times \mathbb{R}) \cap C_p^+([0,T] \times \mathbb{R})$ and any global minimum point $(\tau,x) \in [0,T] \times \mathbb{R}$ of $u - \varphi$, we have:

$$\text{if } (\tau,x) \in (0,T] \times O, \qquad \left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x) \geq 0,$$

$$\text{if } \tau = 0,\ x \in \overline{O}, \qquad \max\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - h(x)\} \geq 0,$$

$$\text{if } \tau \in (0,T],\ x \in \partial O, \qquad \max\{\left(\frac{\partial \varphi}{\partial \tau} - L\varphi\right)(\tau,x),\ u(\tau,x) - g(\tau,x)\} \geq 0,$$

$$\text{if } x \notin \overline{O}, \qquad u(\tau,x) \geq g(\tau,x).$$

A function $u \in C_p^+([0,T] \times \mathbb{R})$ is called a *viscosity solution* of (2.8)–(2.9) if it is both a subsolution and a supersolution. This function is then continuous on $(0,T] \times \mathbb{R}$.

Note that the initial and boundary conditions are verified in a viscosity sense. The definition also includes the case of initial value problems: $O = \mathbb{R}$.

In the next section, we show that European and barrier option prices are (unique) viscosity solutions of the corresponding integro-differential problems under rather general conditions on the Lévy triplet and the payoff function.

2.2 Option prices as viscosity solutions of PIDE

Existence and uniqueness of viscosity solutions for such parabolic integro-differential equations are discussed in [1] in the case where ν is a finite measure and in [4] and [21] for general Lévy measures. Growth conditions other than $u \in C_p^+$ can be considered (see e.g. [1, 4]) with additional conditions on the Lévy measure ν. The main tool for showing uniqueness is the comparison principle: if u, v are viscosity solutions and $u(0,x) \geq v(0,x)$ then $\forall \tau \in [0,T],\ u(\tau,x) \geq v(\tau,x)$. This property can be extended to subsolutions and supersolutions in the following sense [1, 18]:

Proposition 1 (Comparison principle for semi-continuous solutions[1, 18]). *If $u \in USC$ is a subsolution and $v \in LSC$ is a supersolution then $u \leq v$ on $(0,T] \times \mathbb{R}$.*

Proofs and extensions can be found in [1] for the case where ν is a bounded measure; the case of a general Lévy measure has been recently treated in [18]. The following result, whose proof is given in [12], shows that values of European and barrier options can be expressed as viscosity solutions of (2.8)–(2.9):

Proposition 2 (Option prices as viscosity solutions). *Let the payoff function H verify the Lipschitz condition on its domain of definition:*

$$|H(S_1) - H(S_2)| \leq C|S_1 - S_2|, \qquad \forall S_1, S_2 \in (S_0 e^l, S_0 e^u) \tag{2.12}$$

and let $h(x) = H(S_0 e^x)$ have polynomial growth at infinity. Then:

- *The forward value of a European option $u(\tau,x)$ defined by (2.2) is the unique viscosity solution of the Cauchy problem (2.3) (that is (2.8)–(2.9) with $O = \mathbb{R}$).*
- *Let $u_b(\tau,x)$ be the forward value of a knockout (single or double) barrier option defined by (2.6). If $u_b(\tau,x)$ is continuous then it is the unique viscosity solution of (2.8)–(2.9) (with $g \equiv 0$).*

The hypotheses above on the payoff function apply to put options, single-barrier knockout puts, double barrier knockout options and also to the log-contract. One can then retrieve call options by put-call parity. For barrier options with rebate, the zero boundary condition has to be replaced by the value of the rebate, as in the case of diffusion models. A discussion of sufficient conditions for continuity of value functions for barrier options is given in [12].

3 An explicit-implicit finite difference scheme

In this section we present a numerical procedure for solving the PIDE:

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= Lu, && (0,T] \times O \\ u(0,x) &= h(x), && x \in O \\ u(\tau,x) &= g(\tau,x), && x \notin O \end{aligned} \tag{3.1}$$

where L is defined by (1.7), and $g \in C_p^+([0,T] \times \mathbb{R} \setminus O)$ is a continuous function.

Our method is based on splitting the operator L into two parts:

$$\frac{\partial u}{\partial \tau} = \mathcal{D}u + \mathcal{J}u,$$

where $\mathcal{D}$ and $\mathcal{J}$ stand for the differential and integral parts of L respectively. We replace $\mathcal{D}u$ with a finite difference approximation, $\mathcal{J}u$ with the trapezoidal quadrature approximation and use the following explicit-implicit time stepping scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} = Du^{n+1} + Ju^n.$$

We treat the integral part in an explicit time stepping in order to avoid the inversion of the non-sparse matrix J. We show that this does not affect the stability of the scheme: it is unconditionally stable like the fully implicite one.

Before applying discretization, the equation must be localized to a bounded domain; we first discuss estimates for the localization error using a probabilistic approach. We then describe the space discretization and the time-stepping scheme in the case of a jump-diffusion model where the jump intensity is finite. Finally, we discuss how to deal with the singular case $\nu(\mathbb{R}) = +\infty$.

3.1 Localization to a bounded domain

To solve numerically the initial-boundary-value problem (3.1) in the case of an unbounded domain O, we first truncate the domain to an interval $x \in (-A, A)$. Usually, this leads to define some boundary conditions at $x = -A$ and $x = A$. As noted above, because of the integral term, the operator is non-local. Thus, we need to extend the function $u(\tau, \cdot)$ outside this interval: more precisely, to $\{x + y \mid x \in (-A, A),\ y \in \operatorname{supp} \nu\}$ which in most examples is equal to the whole real line $\mathbb{R}$. In the case of knock-out barrier options, a natural boundary condition is given by the zero extension (or the rebate). In other cases this extension is done by imposing a numerical boundary condition. Choosing $u(\tau, x) = g(\tau, x)$ for some given continuous function g with polynomial growth will lead to a probabilistic interpretation of the solution of the localized problem.

Although many choices are possible for the boundary condition g, we will consider here two cases. The simplest choice is $g = 0$ i.e. extend the solution by zero outside the domain. Another extension is given by the payoff function (the initial condition) itself: $g(\tau, x) = h(x)$ which is asymptotically close to the solution at infinity. We will see that both choices lead to a localization error which exponentially decreases with the domain size. Let us define $u_A(\tau, x)$ as the solution of the localized problem:

$$\begin{aligned} &\frac{\partial u_A}{\partial \tau} = Lu_A, && (0,T] \times (-A, A) \\ u_A(0,x) = h(x), \qquad & x \in (-A, A); \qquad && u_A(\tau, x) = g(\tau, x), \qquad x \notin (-A, A) \end{aligned} \tag{3.2}$$

where $g = 0$ or $g(\tau, x) = h(x)$.

Proposition 3. *Assume that h is bounded ($||h||_\infty < \infty$), and*

$$\exists \alpha > 0, \int_{|x|>1} e^{\alpha|x|}\nu(dx) < \infty. \tag{3.3}$$

Let $u(\tau, x)$ be the solution of (3.1) and $u_A(\tau, x)$ be the (viscosity) solution of (3.2) with boundary condition $g = 0$ or $g(\tau, x) = h(x)$. Then:

$$|u(\tau, x) - u_A(\tau, x)| \leq 2C_{\tau,\alpha}||h||_\infty e^{-\alpha(A-|x|)}, \qquad \forall x \in (-A, A). \tag{3.4}$$

where the constant $C_{\tau,\alpha}$ does not depend on A.

Proof. The proof is based on the probabilistic representation of the solutions of (3.1) and (3.2). Let us define $M_\tau^x = \sup_{t\in[0,\tau]} |Y_t + x|$. Then,

$$\begin{aligned} u(\tau, x) &= \mathbb{E}[h(Y_\tau + x)], \\ u_A(\tau, x) &= \mathbb{E}[h(Y_\tau + x)1_{\{M_\tau^x < A\}}], \qquad \text{if } g = 0 \\ \text{or } u_A(\tau, x) &= \mathbb{E}[h(Y_\tau + x)1_{\{M_\tau^x < A\}} + h(Y_{\theta(x)} + x)1_{\{M_\tau^x \geq A\}}] \end{aligned}$$

where $\theta(x) = \inf\{t \geq 0, |Y_t + x| \geq A\}$ is the first exit time of $Y_t + x$ from $[-A, A]$. Subtracting u_A from u gives:

$$\begin{aligned} |u(\tau, x) - u_A(\tau, x)| &= |\mathbb{E}h(Y_\tau + x)1_{\{M_\tau^x \geq A\}}| \\ &\leq ||h||_\infty \mathbb{Q}(M_\tau^x \geq A) \quad \text{for } g = 0 \end{aligned}$$

and in the case $g(\tau, x) = h(x)$ we obtain:

$$\begin{aligned} |u(\tau, x) - u_A(\tau, x)| &\leq \mathbb{E}|h(Y_\tau + x)1_{\{M_\tau^x \geq A\}}| + \mathbb{E}|h(Y_{\theta(x)} + x)1_{\{M_\tau^x \geq A\}}| \\ &\leq 2||h||_\infty \mathbb{Q}(M_\tau^x \geq A). \end{aligned}$$

So, in both cases

$$|u(\tau, x) - u_A(\tau, x)| \leq 2||h||_\infty \mathbb{Q}(M_\tau^x \geq A). \tag{3.5}$$

Theorem 25.18 of [22] together with (3.3) implies

$$C_{\tau,\alpha} = \mathbb{E}e^{\alpha M_\tau^0} < \infty. \tag{3.6}$$

Therefore, Chebyshev's inequality applies and we obtain

$$\mathbb{Q}(M_\tau^0 \geq A) \leq C_{\tau,\alpha} e^{-\alpha A}. \tag{3.7}$$

Now, to pass from M_τ^0 to M_τ^x, we use the following implications:

$$\begin{aligned} & \sup|Y_t + x| \leq \sup|Y_t| + |x| \\ \Rightarrow \quad & (\sup|Y_t + x| \geq A \Rightarrow \sup|Y_t| + |x| \geq A) \\ \Rightarrow \quad & \mathbb{Q}(M_\tau^x \geq A) \leq \mathbb{Q}(M_\tau^0 \geq A - |x|) \\ & \leq C_{\tau,\alpha} e^{-\alpha(A-|x|)} \quad \text{by (3.7)} \end{aligned}$$

Combining the last inequality with (3.5) gives the desired result. □

Remark 1. In the case of a put option $\|h\|_\infty = 1$, and the proposition applies. In other examples h may be unbounded; in this case it is still possible to obtain an exponentially decreasing localisation error under additional restrictions on ν. Note that for the call option, although the payoff grows exponentially, one can transform the problem into pricing a put using put-call parity in order to obtain a smaller localization error.

Remark 2. An exponential bound on localization error in L_2-norm is given in [20] using analytical methods. The advantage of the probabilistic approach is to provide a local (pointwise) estimate. For instance, our estimate (3.4) reflects the intuitive fact that the localization error is more pronounced near the boundary.

The above result implies that the localization error decreases uniformly on each closed subinterval of $(-A, A)$:

$$|u(\tau, x) - u_A(\tau, x)| \leq k e^{-\alpha\delta A}, \qquad \text{for } |x| \leq (1-\delta)A$$

where $0 < \delta < 1$.

The hypothesis (3.3) means that the tails of ν have to decrease exponentially, which is true in all examples considered in the option pricing literature (except Carr & Wu's log-stable model [9]). Note that in an exponential Lévy model we already have $\int_1^{+\infty} e^{\alpha x}\nu(\mathrm{d}x) < \infty$ for all $\alpha \leq 1$, because of the martingale condition, so (3.3) is a condition on the negative jumps.

3.2 Truncation of the integral

To compute numerically the integral term, we need to reduce the region of integration to a bounded interval. In terms of the jump process, this amounts to the truncation of large jumps. We will now give an estimate for the error resulting from this approximation. Recall that the solution of the Cauchy problem (3.1) (in the European case where $O = \mathbb{R}$) is

$$u(\tau, x) = \mathbb{E}[H(S_0 e^{x + r\tau + X_\tau})], \tag{3.8}$$

where X_τ is a Lévy process with the triplet (γ, σ, ν). Let us define a new process $\tilde{X}_\tau$ characterized by the Lévy triplet $(\tilde{\gamma}, \sigma, \nu \mathbf{1}_{x \in [B_l, B_r]})$, where $\tilde{\gamma}$ is determined by the martingale condition:

$$\tilde{\gamma} = -\frac{\sigma^2}{2} - \int_{B_l}^{B_r} (e^y - 1 - y 1_{|y| \leq 1})\nu(dy).$$

We define now:

$$\tilde{u}(\tau, x) = \mathbb{E}[H(S_0 e^{x + r\tau + \tilde{X}_\tau})], \tag{3.9}$$

and we estimate the difference between $\tilde{u}$ and the true solution u.

Proposition 4. *Let H be Lipschitz: $|H(S_1)-H(S_2)| \le c|S_1-S_2|$. Assume that $\exists\ \alpha_r, \alpha_l > 0$, such that $\int_1^\infty e^{(1+\alpha_r)y}\nu(dy) < \infty$, and $\int_{-\infty}^{-1} |y|e^{\alpha_l|y|}\nu(dy) < \infty$.*

If u and $\tilde{u}$ are defined by (3.8) and (3.9) respectively, then

$$|u(\tau,x) - \tilde{u}(\tau,x)| \quad \le \quad 2c\,S_0 e^{x+r\tau}\tau(C_1 e^{-\alpha_l|B_l|} + C_2 e^{-\alpha_r|B_r|}). \qquad (3.10)$$

Proof. Let us denote $R_\tau \stackrel{d}{=} X_\tau - \tilde{X}_\tau,\ R_\tau \perp\!\!\!\perp \tilde{X}_\tau$. We have

$$\begin{aligned} |u(\tau,x) - \tilde{u}(\tau,x)| &= |\mathbb{E}[H(S_0 e^{x+r\tau+\bar{X}_\tau+R_\tau})] - \mathbb{E}[H(S_0 e^{x+r\tau+\bar{X}_\tau})]| \\ &\le c\,S_0 e^{x+r\tau}\mathbb{E}[e^{\bar{X}_\tau}|e^{R_\tau}-1|] = c\,S_0 e^{x+r\tau}\mathbb{E}|e^{R_\tau}-1|. \end{aligned}$$

By construction, $\mathbb{E}[e^{R_\tau}-1] = 0$. Since $|e^{R_\tau}-1| = (e^{R_\tau}-1) + 2(1-e^{R_\tau})^+$, and $(1-e^{R_\tau})^+ \le |R_\tau|$, we obtain

$$|u(\tau,x) - \tilde{u}(\tau,x)| \quad \le \quad 2c\,S_0 e^{x+r\tau}\mathbb{E}|e^{R_\tau}|. \qquad (3.11)$$

The Lévy triplet of R_τ is $(\gamma-\tilde{\gamma}, 0, \nu\mathbf{1}_{x\notin[B_l,B_r]})$ with

$$\gamma - \tilde{\gamma} = -\int_{y\notin[B_l,B_r]} (e^y - 1)\nu(dy).$$

One can write $R_\tau = P_\tau + N_\tau$, where P_τ and N_τ are characterized by $(\int_{-\infty}^{B_l}(1-e^y)\nu(\mathrm{d}y), 0, \nu\mathbf{1}_{x>B_r})$ and $(-\int_{B_r}^{\infty}(e^y-1)\nu(\mathrm{d}y), 0, \nu\mathbf{1}_{x<B_l})$ respectively. We assume without loss of generality that $B_l < -1$, $B_r > 1$[1]. Since P_τ has non-negative drift, no Brownian component, and only positive jumps bounded from below by $B_r > 0$, we have $P_\tau \ge 0$ (recall that $P_0 = 0$). Conversely, N_τ has only negative jumps (bounded from above by $B_l < 0$) and non-positive drift. In consequence, $N_\tau \le 0$. Therefore,

$$\begin{aligned} \mathbb{E}|R_\tau| &\le \mathbb{E}|P_\tau| + \mathbb{E}|N_\tau| \;=\; \mathbb{E}P_\tau - \mathbb{E}N_\tau \\ &= \tau\left[\int_{-\infty}^{B_l}(1-e^y-y)\nu(\mathrm{d}y) + \int_{B_r}^{\infty}(e^y-1+y)\nu(\mathrm{d}y)\right] \\ &\le \tau\left[2\int_{-\infty}^{B_l}|y|\nu(\mathrm{d}y) + 2\int_{B_r}^{\infty}e^y\nu(\mathrm{d}y)\right]. \qquad (3.12) \end{aligned}$$

Using the hypotheses on ν we obtain

$$\begin{aligned} \mathbb{E}|R_\tau| &\le \tau\left(2e^{-\alpha_l|B_l|}\int_{-\infty}^{B_l}|y|e^{\alpha_l|y|}\nu(\mathrm{d}y) + 2e^{-\alpha_r|B_r|}\int_{B_r}^{\infty}e^{(1+\alpha_r)y}\nu(\mathrm{d}y)\right) \\ &\le \tau(C_1 e^{-\alpha_l|B_l|} + C_2 e^{-\alpha_r|B_r|}), \end{aligned}$$

that we substitute into (3.11). □

[1] Clearly, if (3.10) is true for such values, it is true for all B_l, B_r, up to change of the constants. On the other hand, this estimate is not useful if ν has a bounded support. For example, if there are only negative jumps, we will take $B_r = 0$, but in this case $\nu\mathbf{1}_{x\le B_r} = \nu$, and there is no truncation error due to B_r.

Remark 3. The hypotheses on ν in Proposition 4 are a little stronger than (3.3). We require them to obtain an exponential decay of the truncation error. However, we can use directly estimate (3.12). In other words, existence of the integrals in (3.12) suffices to obtain a convergence of $\tilde{u}$ to u as $|B_l|$ and $|B_r|$ grow to infinity, but this convergence does not necessarily occur at an exponential rate.

Remark 4. The requirements are different for the left tail of ν and the right one. For example, in the Variance Gamma model with $\nu(x) = a\exp(-\eta_\pm|x|)/|x|$ one needs η_+ to be greater than 1, and η_- only positive. Proposition 3.10 then applies with $\alpha_l < \eta_-$ and $\alpha_r < \eta_+ - 1$.

Using Propositions 3 and 4 we can fix in advance $[-A, A]$ and $[B_l, B_r]$ to have a given bound on the respective errors. Therefore we will assume this has been done and concentrate ourselves on numerical solving of the localized problem.

3.3 Explicit-implicit scheme: finite activity case

We suppose here that $\nu(\mathbb{R}) = \lambda < +\infty$. Then the integro-differential operator can be written as

$$Lu = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\sigma^2}{2} - r + \alpha\right)\frac{\partial u}{\partial x} - \lambda u + \int_{B_l}^{B_r} \nu(dy)u(\tau, x+y), \qquad (3.13)$$

where $\alpha = \int_{B_l}^{B_r}(e^y - 1)\nu(\mathrm{d}y)$.

We introduce a uniform grid on $[0,T]\times\mathbb{R}$: $\tau_n = n\Delta t$, $n = 0\ldots M$, $x_i = -A + i\Delta x$, $i \in \{0,..,N\}$, with $\Delta t = T/M$, $\Delta x = 2A/N$. Let $\{u_i^n\}$ be the solution of the numerical scheme, to be defined below.

To approximate the integral terms we use the trapezoidal quadrature rule with the same step Δx. Let K_l, K_r be such that $[B_l, B_r] \subset [(K_l - 1/2)\Delta x, (K_r + 1/2)\Delta x]$. Then:

$$\int_{B_l}^{B_r} \nu(dy)u(\tau, x_i + y) \approx \sum_{j=K_l}^{K_r} \nu_j u_{i+j}, \qquad \lambda \approx \hat{\lambda} = \sum_{j=K_l}^{K_r} \nu_j,$$
$$\alpha \approx \hat{\alpha} = \sum_{j=K_l}^{K_r} (e^{y_j} - 1)\nu_j, \qquad \text{where} \qquad \nu_j = \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \nu(\mathrm{d}y). \quad (3.14)$$

The space derivatives are discretized using finite differences:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}, \qquad (3.15)$$

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \begin{cases} \frac{u_{i+1}-u_i}{\Delta x}, & \text{if} \quad \sigma^2/2 - r + \hat{\alpha} < 0 \\ \frac{u_i - u_{i-1}}{\Delta x}, & \text{if} \quad \sigma^2/2 - r + \hat{\alpha} \geq 0. \end{cases} \qquad (3.16)$$

The choice of approximation for the first-order derivative is determined by stability requirement and will be discussed later (Section 4.2). Since the two cases are treated similarly, let us suppose without loss of generality that $\sigma^2/2 - r + \hat{\alpha} < 0$.

Using (3.14)–(3.16) we obtain $Lu \approx Du + Ju$, where

$$(Du)_i = \frac{\sigma^2}{2}\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} - \left(\frac{\sigma^2}{2} - r + \hat{\alpha}\right)\frac{u_{i+1} - u_i}{\Delta x} - \hat{\lambda}u_i, \qquad (3.17)$$

$$(Ju)_i = \sum_{j=K_l}^{K_r} \nu_j u_{i+j}. \qquad (3.18)$$

Finally, we replace the problem (3.2) with the following time-stepping scheme:

Initialization :

$$u_i^0 = h(x_i), \qquad i \in \{0, \dots, N\}, \qquad (3.19)$$

$$u_i^0 = g(0, x_i), \qquad \text{otherwise.} \qquad (3.20)$$

(**S**) **For** $\mathbf{n = 0, \dots, M - 1}$:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = (Du^{n+1})_i + (Ju^n)_i, \quad \text{if } i \in \{0, \dots, N\} \qquad (3.21)$$

$$u_i^{n+1} = g(n\Delta t, x_i), \qquad \text{if } i \notin \{0, \dots, N\}. \qquad (3.22)$$

3.4 Explicit-implicit scheme: infinite activity case

If $\nu(\mathbb{R}) = +\infty$ this method cannot be applied directly. The idea is to come down to a non-singular case by approximating the process X_τ by an appropriate finite activity process with a modified diffusion coefficient.

The procedure is similar to the one described in Section 3.2, but this time we deal with small jumps. Given $\varepsilon > 0$ let us define a process X_τ^ε characterized by the Lévy triplet $(\gamma(\varepsilon), \sqrt{\sigma^2 + \sigma^2(\varepsilon)}, \nu \mathbf{1}_{|x| \geq \varepsilon})$, where

$$\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} y^2 \nu(\mathrm{d}y),$$

and $\gamma(\varepsilon)$ is determined by the martingale condition:

$$\gamma(\varepsilon) = -\frac{\sigma^2 + \sigma^2(\varepsilon)}{2} - \int_{|y| \geq \varepsilon} (e^y - 1 - y 1_{|y| \leq 1}) \nu(dy).$$

This means that we replace the jumps of size smaller than ε by a Brownian motion $\sigma(\varepsilon) W_\tau$. Therefore, X_τ^ε has jumps of finite activity.

The function u^ε defined as

$$u^\varepsilon(\tau, x) = \mathbb{E}[h(x + r\tau + X_\tau^\varepsilon)] \equiv \mathbb{E}[h(x + Y_\tau^\varepsilon)] \qquad (3.23)$$

satisfies the following Cauchy problem:

$$\begin{aligned} \frac{\partial u^\varepsilon}{\partial \tau} &= L^\varepsilon u^\varepsilon, \qquad (0,T]\times\mathbb{R} \\ u^\varepsilon(0,x) &= h(x), \qquad x\in\mathbb{R} \end{aligned} \tag{3.24}$$

where

$$L^\varepsilon f = \frac{\sigma^2+\sigma^2(\varepsilon)}{2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\sigma^2+\sigma^2(\varepsilon)}{2} - r + \alpha(\varepsilon)\right)\frac{\partial f}{\partial x} - \lambda(\varepsilon) f(x) + \int_{|y|\geq\varepsilon} \nu(dy) f(x+y), \tag{3.25}$$

and $\alpha(\varepsilon) = \int_{|y|\geq\varepsilon}(e^y-1)\nu(\mathrm{d}y)$, $\lambda(\varepsilon) = \int_{|y|\geq\varepsilon}\nu(\mathrm{d}y)$.

The next proposition gives an estimate of the approximation error.

Proposition 5. *Let h be Lipschitz: $|h(x)-h(y)|\leq c\,|x-y|$. Let u and u^ε be defined by (3.8) and (3.23) respectively. Then*

$$|u(\tau,x) - u^\varepsilon(\tau,x)| \leq C\frac{\int_{-\varepsilon}^{\varepsilon}|y|^3\nu(\mathrm{d}y)}{\sigma^2(\varepsilon)}. \tag{3.26}$$

Proof. We essentially use [11, Proposition 6.2] with the only difference that we also adjust the drift parameter γ to preserve the martingale property. Let us define $Z_\tau = Y_\tau - (\gamma-\gamma(\varepsilon))\tau$. Then,

$$\begin{aligned} |u(\tau,x) - u^\varepsilon(\tau,x)| &= |\mathbb{E}[h(x+Y_\tau)] - \mathbb{E}[h(x+Y^\varepsilon_\tau)]| \\ &\leq |\mathbb{E}[h(x+Z_\tau)] - \mathbb{E}[h(x+Y^\varepsilon_\tau)]| + \\ &\quad + |\mathbb{E}[h(x+Z_\tau+(\gamma-\gamma(\varepsilon))\tau)] - \mathbb{E}[h(x+Z_\tau)]|. \end{aligned} \tag{3.27}$$

Since h is Lipschitz, it is almost everywhere differentiable with $|h'|\leq c$. By [11, Proposition 6.2] we have

$$|\mathbb{E}[h(x+Z_\tau)] - \mathbb{E}[h(x+Y^\varepsilon_\tau)]| \leq K\,c\,\frac{\int_{-\varepsilon}^{\varepsilon}|y|^3\nu(\mathrm{d}y)}{\sigma^2(\varepsilon)} \tag{3.28}$$

with $K<16.5$. The second term may be estimated as follows:

$$|\mathbb{E}[h(x+Z_\tau+(\gamma-\gamma(\varepsilon))\tau)] - \mathbb{E}[h(x+Z_\tau)]| \leq c\,|\gamma-\gamma(\varepsilon)|\tau$$

where

$$\begin{aligned} |\gamma-\gamma(\varepsilon)| &= \left|\frac{\sigma^2(\varepsilon)}{2} - \int_{|y|<\varepsilon}(e^y-1-y)\nu(\mathrm{d}y)\right| \\ &= \left|\frac{1}{2}\int_{|y|<\varepsilon}\nu(\mathrm{d}y)\int_0^y e^s(y-s)^2 ds\right| \leq \frac{e^\varepsilon}{6}\int_{-\varepsilon}^{\varepsilon}|y|^3\nu(\mathrm{d}y). \end{aligned} \tag{3.29}$$

Since $\sigma^2(\varepsilon)\to 0$ as $\varepsilon\to 0$, (3.29) converges faster than (3.28), and therefore may be neglected. □

Remark 5. If $\lim_{x\to 0}\nu(x)|x|^{1+\beta} = a > 0$, with $0 \le \beta < 2$, then (3.26) gives

$$|u(\tau,x) - u^{\varepsilon}(\tau,x)| \le C(\beta)\varepsilon,$$

so the approximation error is proportional to ε. This case includes all practical examples as variance gamma, stable tempered processes and others.

4 Consistency, stability and convergence

We study in this section some properties of the finite difference scheme presented above. First, we show that the scheme is locally consistent, unconditionally stable and monotone. Using these properties, we show that the solution of the scheme converges to the viscosity solution of the PIDE i.e. the option price as $\Delta x,\ \Delta t \to 0$.

4.1 Consistency

Proposition 6 (Consistency). *The finite difference scheme (3.21) is locally consistent with equation (3.2): for all $v \in C_0^\infty([0,T]\times\mathbb{R})$ and for all $(\tau_n, x_i) \in [0,T]\times\mathbb{R}$ we have*

$$\left|\frac{v_i^{n+1}-v_i^n}{\Delta t} - (Dv)_i^{n+1} - (Jv)_i^n - \left(\frac{\partial v}{\partial \tau} - Lv\right)(\tau_n, x_i)\right| = r_i^n(\Delta t, \Delta x) \to 0, \quad (4.1)$$

when $(\Delta t, \Delta x) \to 0$. Moreover:

$$\exists c > 0, \qquad |r_i^n(\Delta t, \Delta x)| \le c(\Delta t + \Delta x)$$

Proof. Using a second order Taylor expansion we obtain:

$$\left|\frac{v_i^{n+1}-v_i^n}{\Delta t} - \frac{\partial v}{\partial \tau}(\tau_n, x_i)\right| = \left|\frac{\Delta t}{2}\frac{\partial^2 v}{\partial \tau^2}(\tilde{\tau}, x_i)\right| \le \frac{\Delta t}{2}\left\|\frac{\partial^2 v}{\partial \tau^2}\right\|_\infty \quad (4.2)$$

Consider now the terms in Dv.

$$\begin{aligned}
|\alpha - \hat{\alpha}| &= \left|\int_{B_l}^{B_r}(e^y - 1)\nu(\mathrm{d}y) - \sum_{j=K_l}^{K_r}(e^{y_j}-1)\nu_j\right| \\
&= \left|\sum_{j=K_l}^{K_r}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}(1-e^{y_j-y})e^y\nu(\mathrm{d}y)\right| \le \Delta x\left|\sum_{j=K_l}^{K_r}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}e^y\nu(\mathrm{d}y)\right| \\
&= \Delta x\int_{B_l}^{B_r}e^y\nu(\mathrm{d}y) \le \Delta x\int e^y\nu(\mathrm{d}y). \quad (4.3)
\end{aligned}$$

We have used that $|1-e^{y_j-y}| \le \Delta x$ as $\Delta x \le 1$, because $|y_j - y| \le \Delta x/2$. So,

$$\left|(Dv)_i^{n+1} - (\mathcal{D}v)(\tau_n, x_i)\right| = \left|(Dv)_i^{n+1} - (\mathcal{D}v)(\tau_{n+1}, x_i) + \Delta t(\mathcal{D}v)(\tilde{\tau}, x_i)\right| =$$
$$\left|\Delta t(\mathcal{D}v)(\tilde{\tau}, x_i) + \frac{\sigma^2}{2}\frac{\Delta x^2}{24}\left[\frac{\partial^4 v}{\partial x^4}(\tau_{n+1}, \xi_1) + \frac{\partial^4 v}{\partial x^4}(\tau_{n+1}, \eta_1)\right] + \right.$$
$$\left.\left(\frac{\sigma^2}{2} - r + \hat{\alpha}\right)\frac{\Delta x^2}{3}\left[\frac{\partial^3 v}{\partial x^3}(\tau_{n+1}, \xi_2) - \frac{\partial^3 v}{\partial x^3}(\tau_{n+1}, \eta_2)\right] + (\alpha - \hat{\alpha})\frac{\partial v}{\partial x}(\tau_{n+1}, x_i)\right|$$
$$\le c_1\Delta t + c_2\Delta x. \quad (4.4)$$

The integral part can be estimated as follows:

$$|(Jv)_i^n - (\mathcal{J}v)(\tau_n, x_i)| = \left|\sum_{j=K_l}^{K_r} v_{i+j}\nu_j - \int_{B_l}^{B_r} v(\tau_n, x_i + y)\nu(\mathrm{d}y)\right|$$
$$= \left|\sum_{j=K_l}^{K_r}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}(v(\tau_n, x_i + y_j) - v(\tau_n, x_i + y))\nu(\mathrm{d}y)\right|$$
$$= \left|\sum_{j=K_l}^{K_r}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x}(y_j - y)\frac{\partial v}{\partial x}(\tau_n, x_i + \xi)\nu(\mathrm{d}y)\right| \le \frac{\Delta x}{2}\lambda\left\|\frac{\partial v}{\partial x}\right\|_\infty. \quad (4.5)$$

Therefore, $r_i^n(\Delta t, \Delta x) \le C_1\Delta t + C_2\Delta x \to 0$. □

Remark 6. (Global error estimation) The precision of the numerical solution is influenced by several parameters: the discretization steps Δx and Δt but also the localization parameters A, B_l, B_r and, in the infinite activity case, ε. Some of the parameteres cannot be chosen independently if we want to minimize the global error.

Indeed, by the Proposition 5, the impact of the truncation of the small jumps vanishes as $\varepsilon \to 0$; at the same time, for a fixed Δx, the discretization error goes to infinity since the constant in (4.5) is of order $\lambda(\varepsilon)$. So, there would be an optimal choice of ε for a given Δx.

Note that the Proposition 6 gives the discretization error for the operator and not for the solution itself. In the classical framework, the Lax theorem states that the solution approximation is of the same order that the discretization error. In the case of viscosity solutions, we have no estimate on the convergence rate. So, properly speaking, the reasoning below does not apply to the non-smooth solutions. However, it highlights the fact that the global convergence rate may be smaller in the infinite activity case, even for smooth solutions.

Let us consider an example. In the variance gamma model, $\lambda(\varepsilon) \sim -\log(\varepsilon)$, and the small jumps truncation error is proportional to ε (see Remark 5). Consider the sum of the discretization and truncation errors:

$$E = -C_1\log(\varepsilon)\Delta x + C_2\varepsilon. \quad (4.6)$$

To estimate an optimal ε for a fixed Δx, let us minimize this expression with respect to ε:

$$\frac{\partial E}{\partial \varepsilon} = -C_1 \frac{\Delta x}{\varepsilon} + C_2 = 0,$$

which implies

$$\varepsilon_{opt} = C\Delta x. \tag{4.7}$$

Substituting this value into (4.6) gives

$$E = C_1 \Delta x \log(1/\Delta x) + C_3 \Delta x = O(\Delta x \log(1/\Delta x)). \tag{4.8}$$

In other examples, where $\nu(x) \sim 1/|x|^{1+\beta}$ near the origin, with $0 < \beta < 2$, we obtain in a similar way that

$$\varepsilon_{opt} = C(\Delta x)^{\frac{1}{1+\beta}}, \qquad E = O((\Delta x)^{\frac{1}{1+\beta}}).$$

So, in all cases, the convergence is slower than $O(\Delta x)$.

4.2 Monotonicity and Stability

Two properties which are important for showing convergence to viscosity solutions are the monotonicity and stability of the scheme. There are various notions of stability for a numerical scheme; we will use here stability in sup norm, which controls the errors on prices of options at all values of the underlying. As we will see below, this is the relevant notion of stability for showing convergence to a viscosity solution.

Definition 2 (Stability). The scheme **(S)** is *stable* iff for a bounded initial condition the solution does exist and is bounded independently in Δt, Δx, uniformly on $[0,T] \times \mathbb{R}$:

$$\exists C > 0,\ \forall\ \Delta t > 0,\ \Delta x > 0,\ i \in \mathbb{Z},\ n \in \{0, \ldots, M\}: \quad |u_i^n| \leq C. \tag{4.9}$$

We will say that vector v (or matrix M) is positive if all its elements are positive. We write $u \geq v$ if $u - v \geq 0$.

Proposition 7 (Stability and discrete comparison principle). *Scheme* **(S)** *is unconditionally stable and monotone (verifies the discrete comparison principle): if u^0, v^0 are two bounded initial conditions then*

$$u^0 \geq v^0 \quad \Rightarrow \quad \forall n \geq 1,\ u^n \geq v^n.$$

Proof. We start with rewriting the equation (3.21) in the following form:

$$-\ c\Delta t u_{i-1}^{n+1} + (1 + a\Delta t) u_i^{n+1} - b\Delta t u_{i+1}^{n+1} = u_i^n + \Delta t \sum_j \nu_j u_{i+j}^n, \tag{4.10}$$

where[2]

$$
\begin{aligned}
a &= \frac{\sigma^2}{(\Delta x)^2} - \left(\frac{\sigma^2}{2} - r + \hat{\alpha}\right)\frac{1}{\Delta x} + \hat{\lambda} \geq 0, \\
b &= \frac{\sigma^2}{2(\Delta x)^2} - \left(\frac{\sigma^2}{2} - r + \hat{\alpha}\right)\frac{1}{\Delta x} \geq 0, \\
c &= \frac{\sigma^2}{2(\Delta x)^2} \geq 0,
\end{aligned} \tag{4.11}
$$

Notice that $a = b + c + \hat{\lambda}$.

Stability. We have a tri-diagonal linear system for $\{u_0^{n+1}, \ldots, u_N^{n+1}\}$. It has a unique solution because the main diagonal is dominant: $1 + a\Delta t \geq c\Delta t + b\Delta t$.

We will show that, if h is a bounded initial condition, $\|h\|_\infty < \infty$, then $\forall\, n$,

$$
\|u^n\|_\infty \leq \|h\|_\infty. \tag{4.12}
$$

Let us proceed by induction. By definition of u^0 we have $\|u^0\|_\infty \leq \|h\|_\infty$. Let (4.12) hold for n. Suppose that $\|u^{n+1}\|_\infty > \|h\|_\infty$. This means that $\exists\, i_0 \in \{0, \ldots, N\}$, such that $|u_{i_0}^{n+1}| = \|u^{n+1}\|_\infty$, and $\forall\, i \in \mathbb{Z}$, $|u_i^{n+1}| \leq |u_{i_0}^{n+1}|$.

Therefore, we have

$$
\begin{aligned}
\|u^{n+1}\|_\infty = |u_{i_0}^{n+1}| &= -c\Delta t|u_{i_0}^{n+1}| + (1 + a\Delta t)|u_{i_0}^{n+1}| - b\Delta t|u_{i_0}^{n+1}| - \hat{\lambda}\Delta t|u_{i_0}^{n+1}| \\
&\leq -c\Delta t|u_{i_0-1}^{n+1}| + (1 + a\Delta t)|u_{i_0}^{n+1}| - b\Delta t|u_{i_0+1}^{n+1}| - \hat{\lambda}\Delta t\|h\|_\infty \\
&\leq |-c\Delta t u_{i_0-1}^{n+1} + (1 + a\Delta t)u_{i_0}^{n+1} - b\Delta t u_{i_0+1}^{n+1}| - \hat{\lambda}\Delta t\|h\|_\infty.
\end{aligned}
$$

Together with (4.10) this gives

$$
\begin{aligned}
\|u^{n+1}\|_\infty &\leq |u_{i_0}^n + \Delta t \sum_j \nu_j u_{i_0+j}^n| - \hat{\lambda}\Delta t\|h\|_\infty \\
&\leq (1 + \hat{\lambda}\Delta t)\|u^n\|_\infty - \hat{\lambda}\Delta t\|h\|_\infty \leq \|h\|_\infty,
\end{aligned} \tag{4.13}
$$

which contradicts our assumption. Thus, $\|u^{n+1}\|_\infty \leq \|h\|_\infty$, and the proof is completed.

Monotonicity. Let u^n and v^n be two solutions of **(S)** corresponding to the initial conditions $h(x)$ and $f(x)$ respectively, and $h(x) \geq f(x)$, $\forall\, x \in \mathbb{R}$. Let us define $w^n = u^n - v^n$. We have to show that $w^n \geq 0$, $\forall\, n \geq 0$.

We proceed similarly as in the previous proof. By construction, we have $w_i^0 = h(x_i) - f(x_i) \geq 0$, $\forall\, i \in \mathbb{Z}$. Let $w^n \geq 0$, and suppose that $\inf_{i\in\mathbb{Z}} w_i^{n+1} < 0$. Since $\forall\, i \in \mathbb{Z}\backslash\{0, \ldots, N\}$, $w_i^{n+1} = h(x_i) - f(x_i) \geq 0$, this implies that $\exists\, i_0 \in$

[2]We recall that the case $\sigma^2/2 - r + \hat{\alpha} < 0$ is considered. If $\sigma^2/2 - r + \hat{\alpha} \geq 0$ we change the approximation of the first-order derivative (see (3.16)) to have $a, b, c \geq 0$, which is needed for stability and monotonicity.

$\{0,\ldots,N\}$, such that $w_{i_0}^{n+1} = \inf_{i\in\mathbb{Z}} w_i^{n+1}$. Using (4.10) we obtain that

$$\begin{aligned}\inf_{i\in\mathbb{Z}} w_i^{n+1} = w_{i_0}^{n+1} &= -c\Delta t w_{i_0}^{n+1} + (1+a\Delta t) w_{i_0}^{n+1} - b\Delta t w_{i_0}^{n+1} - \hat{\lambda}\Delta t w_{i_0}^{n+1} \\ &\geq -c\Delta t w_{i_0-1}^{n+1} + (1+a\Delta t) w_{i_0}^{n+1} - b\Delta t w_{i_0+1}^{n+1} \\ &= w_{i_0}^n + \Delta t \sum_j \nu_j w_{i_0+j}^n \geq 0, \end{aligned} \tag{4.14}$$

which contradicts the assumption. Therefore, $\inf_{i\in\mathbb{Z}} w_i^{n+1} \geq 0$, and, in consequence, $w^{n+1} \geq 0$. □

The discrete comparison principle has an important financial interpretation: it is equivalent to saying that the option values computed using the scheme verify *arbitrage inequalities*: inequalities between payoffs lead to inequalities between values of options. Thus, the discrete comparison principle is a desirable property for our numerical scheme.

Remark 7 (Other notions of stability). Various definitions for the stability of numerical schemes can be found in the literature: for example, the "von-Neumann stability" studied for a similar scheme in [2] represents u^n as a discrete Fourier transform and requires the corresponding Fourier coefficients to be stable. This is essentially equivalent to stability in L^2 norm with periodic boundary conditions and simply controls a global error in the least square sense but does not control the error on the value of a given option. Moreover it does not allow to capture the effect of boundary conditions when they are non-periodic, which is the case here.

4.3 Convergence

In the usual approach to the convergence of finite difference schemes for PDEs, consistency and stability ensure convergence under regularity assumptions on the solution. This approach is not feasible here because, as discussed in sec. 2.1, solutions may be non-smooth and higher order derivatives may not exist. For example, in the Variance Gamma model the value of a call option is C^1 in the price variable but not C^2.

This is where viscosity solutions come to the rescue: in the case of second-order parabolic PDEs verifying a strong uniqueness principle as in proposition (1), Barles & Souganidis [5] show that, for elliptic/parabolic PDEs, any locally consistent, stable and monotone finite difference scheme converges uniformly on each compact subset of $[0,T]\times\mathbb{R}$ to the unique continuous viscosity solution, even when solutions are not smooth. We will now show the convergence of the solution of the scheme considered above to a viscosity solution of (3.2). To make shorter the notations and show the connection with [5] let us rewrite (3.21) in an equivalent form:

$$\frac{1}{\Delta t} S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) = 0, \qquad n \geq 0,\ i \in \{0,\ldots,N\} \tag{4.15}$$

where $\tilde{u} = \{u_{i-1}^{n+1}, u_{i+1}^{n+1}, u^n\}$, and

$$S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) = (1 + a\Delta t)u_i^{n+1} - c\Delta t u_{i-1}^{n+1} - b\Delta t u_{i+1}^{n+1} - \\ - u_i^n - \Delta t \sum_j \nu_j u_{i+j}^n.$$

Coefficients $a, b, c \geq 0$ depend on Δx and are given by (4.11).

We have the following **monotonicity** property:

- If $u_i^{n+1} = v_i^{n+1}$, and $\tilde{u} \leq \tilde{v}$, then

$$S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}) \geq S(\Delta t, \Delta x, n+1, i, v_i^{n+1}, \tilde{v}). \tag{4.16}$$

The **consistency** (4.1) reads as:

- $\forall\ \varphi \in C^\infty([0,T] \times \mathbb{R})$, $\forall\ (\tau, x) \in [0,T] \times [-A, A]$,

$$\frac{1}{\Delta t} S(\Delta t, \Delta x, n+1, i, \varphi_i^{n+1}, \tilde{\varphi}) \to \left(\frac{\partial \varphi}{\partial \tau} - L\varphi \right)(\tau, x), \tag{4.17}$$

  as $\Delta t,\ \Delta x \to 0$, and $(\tau_{n+1}, x_i) \to (\tau, x)$.

- Notice also that if ξ is a constant function on the grid, we have

$$S(\Delta t, \Delta x, n+1, i, u_i^{n+1} + \xi, \tilde{u} + \xi) = S(\Delta t, \Delta x, n+1, i, u_i^{n+1}, \tilde{u}). \tag{4.18}$$

  In fact, one can see from the definition that the scheme is linear and a constant function is always a solution.

Let us define a piecewise interpolation of the solution to the scheme **(S)**:

$$u^{(\Delta t, \Delta x)}(\tau, x) = u_j^n, \quad \text{if } \tau \in [\tau_n, \tau_{n+1}),\ \ x \in [x_{j-1/2}, x_{j+1/2}). \tag{4.19}$$

Proposition 8 (Convergence of the finite difference scheme). *Consider a bounded continuous initial condition h. Then the solution $u^{(\Delta t, \Delta x)}$ of the numerical scheme converges uniformly on each compact subset of $[0,T] \times \mathbb{R}$ to the unique viscosity solution of the problem (3.2).*

Proof. Define

$$\underline{u}(\tau, x) = \liminf_{(\Delta t, \Delta x) \to 0\ (t,y) \to (\tau, x)} u^{(\Delta t, \Delta x)}(t, y) \tag{4.20}$$

$$\overline{u}(\tau, x) = \limsup_{(\Delta t, \Delta x) \to 0\ (t,y) \to (\tau, x)} u^{(\Delta t, \Delta x)}(t, y) \tag{4.21}$$

By construction, $\underline{u} \leq \overline{u}$. We will show that $\overline{u}$ and $\underline{u}$ are respectively viscosity subsolution and supersolution to the problem (3.2). If it is true, then the comparison principle for semi-continuous solutions (Proposition 1) implies that $\underline{u} \geq \overline{u}$. We conclude that $\underline{u} = \overline{u} = u$ is a viscosity solution of (3.2).

We will now show that $\overline{u}$ is a viscosity subsolution of (3.2). First, we remark that $\overline{u}$ is uniformly bounded. In fact, by Proposition 7, $|u^{(\Delta t,\Delta x)}(\tau,x)| \le \|h\|_\infty$ for all positive Δt, Δx, and all $(\tau,x) \in [0,T]\times\mathbb{R}$. Therefore, by construction, $\overline{u}$ is bounded by the same constant.

We formulate the next property as a lemma.

Lemma 1. *$\overline{u}$ is upper semi-continuous:*

$$\forall(\tau,x), \qquad \limsup_{(t,y)\to(\tau,x)} \overline{u}(t,y) \quad \le \quad \overline{u}(\tau,x).$$

The proofs of lemmas used in this proposition are given in the Appendix.

Finally, we will show that $\forall(\tau,x)\in[0,T]\times\mathbb{R}$,

(i) if $x \notin [-A,A]$, we have

$$\overline{u}(\tau,x) = g(\tau,x), \tag{4.22}$$

(ii) if $x\in[-A,A]$, $\varphi \in C^2([0,T]\times\mathbb{R})$ such that

$$\overline{u}(\tau,x) = \varphi(\tau,x) \tag{4.23}$$

and $\overline{u} < \varphi$ on $[0,T]\times\mathbb{R} - \{(\tau,x)\}$ then

$$\frac{\partial\varphi}{\partial\tau}(\tau,x) - L\varphi(\tau,x) \le 0, \quad \text{if } (\tau,x)\in(0,T]\times(-A,A), \tag{4.24}$$

$$\min\{\frac{\partial\varphi}{\partial\tau}(\tau,x) - L\varphi(\tau,x),\ \overline{u}(\tau,x) - h(x)\} \le 0, \quad \text{if } \tau = 0,\ x\in[-A,A] \tag{4.25}$$

$$\min\{\frac{\partial\varphi}{\partial\tau}(\tau,x) - L\varphi(\tau,x),\ \overline{u}(\tau,x) - g(\tau,x)\} \le 0, \quad \text{if } \tau \ne 0,\ x = \pm A, \tag{4.26}$$

which clearly implies (2.10)–(2.11).

Property (i) is an immediate consequence of (3.20) and (3.22). Indeed, if $x \notin [-A,A]$ then, by construction, $\exists\varepsilon > 0$, such that for Δt, $\Delta x < \varepsilon$ and $|(\tau_n,x_i) - (\tau,x)| < \varepsilon$ we have $u^{(\Delta t,\Delta x)}(\tau_n,x_i) = g(\tau_n,x_i)$, so (4.22) follows from the definition of the upper limit.

Let us take (τ,x) and φ as in (ii). Let $\Omega\subset\mathbb{R}$ be a bounded interval containing $[-A+B_l, A+B_r]$.[3]

Lemma 2. *There exist Δt_k, $\Delta x_k \to 0$, and n_k, i_k, such that*

$$(\tau_{n_k}, x_{i_k}) \to (\tau,x), \tag{4.27}$$

$$u^{(\Delta t_k,\Delta x_k)}(\tau_{n_k},x_{i_k}) \to \overline{u}(\tau,x), \qquad \text{as } k\to\infty, \tag{4.28}$$

and for all k, (τ_{n_k},x_{i_k}) is a global maximum point of $u^{(\Delta t_k,\Delta x_k)}(\cdot,\cdot) - \varphi(\cdot,\cdot)$ on the grid inside $[0,T]\times\Omega$:

$$\max_{(\tau_n,x_j)\in[0,T]\times\Omega}\{u^{(\Delta t_k,\Delta x_k)}(\tau_n,x_j) - \varphi(\tau_n,x_j)\}$$
$$= u^{(\Delta t_k,\Delta x_k)}(\tau_{n_k},x_{i_k}) - \varphi(\tau_{n_k},x_{i_k}). \tag{4.29}$$

[3] By the definition of $[-A,A]$ and $[B_l,B_r]$, the values of any functions outside $[0,T]\times\Omega$ are not involved in our computations.

We take a sequence from this lemma. Denoting by ξ_k the right-hand side of (4.29), and $u^{(\Delta t_k, \Delta x_k)}(\tau_n, x_i) \equiv (u^{\Delta_k})_i^n$ we have

$$\begin{aligned} \xi_k &\to 0, \quad \text{as } k \to \infty, && (4.30)\\ (u^{\Delta_k})_{i_k}^{n_k} &= \varphi_{i_k}^{n_k} + \xi_k, && (4.31)\\ (u^{\Delta_k})_i^n &\leq \varphi_i^n + \xi_k, \quad \forall (\tau_n, x_j) \in [0,T] \times \Omega. && (4.32) \end{aligned}$$

If $(\tau, x) \in (0, T] \times (-A, A)$, from (4.27) it follows that $n_k > 0$, $x_{i_k} \in (-A, A)$ for all k greater than some k_0. Therefore, from the definition of $u^{(\Delta t_k, \Delta x_k)}$ as a solution of the scheme, we have

$$\begin{aligned} 0 &= \frac{1}{\Delta t_k} S(\Delta t_k, \Delta x_k, n_k, i_k, (u^{\Delta_k})_{i_k}^{n_k}, \tilde{u}^{\Delta_k})\\ &\geq \frac{1}{\Delta t_k} S(\Delta t_k, \Delta x_k, n_k, i_k, \varphi_{i_k}^{n_k} + \xi_k, \tilde{\varphi} + \xi_k) \quad \text{by (4.31)–(4.32) and (4.16)}\\ &= \frac{1}{\Delta t_k} S(\Delta t_k, \Delta x_k, n_k, i_k, \varphi_{i_k}^{n_k}, \tilde{\varphi}) \quad \text{by (4.18)}\\ &\to \left(\frac{\partial \varphi}{\partial \tau} - L\varphi \right)(\tau, x) \quad \text{by the definition of } \Delta t_k, \Delta x_k, n_k, i_k \text{ and (4.17)} \end{aligned}$$

which is the desired result.

If $\tau = 0$, there are three possibilities:

(1) there exists a subsequence of (4.27) such that $\forall k$, $(\tau_{n_k}, x_{i_k}) \in (0, T] \times (-A, A)$;

(2) $\exists k_0$, such that $\forall k > k_0$, $\tau_{n_k} = 0$, $x_{i_k} \in (-A, A)$;

(3) $\exists k_0$, such that $\forall k > k_0$, $x_{i_k} = \pm A$.

In the first case we apply the same arguments and obtain $\frac{\partial \varphi}{\partial \tau}(\tau, x) - L\varphi(\tau, x) \leq 0$. In the second case we have $(u^{\Delta_k})_{i_k}^{n_k} = h(x_{i_k})$. Taking limits in (4.28) we obtain

$$\overline{u}(\tau, x) = h(x).$$

Finally, in the last case, $(u^{\Delta_k})_{i_k}^{n_k} = g(\tau_{n_k}, x_{i_k})$, so $\overline{u}(\tau, x) = g(0, x)$ (and $\overline{u}(\tau, x) - h(x) \leq 0$ because g equals 0 or h, and h is non-negative). Obviously, in the three cases (4.25) is satisfied.

If $\tau \neq 0$, $x = \pm A$, the alternatives (1) or (3) are only possible, which gives respectively $\frac{\partial \varphi}{\partial \tau}(\tau, x) - L\varphi(\tau, x) \leq 0$ or $\overline{u}(\tau, x) = g(\tau, x)$. Hence, (4.26) is also satisfied.

We have proven that $\overline{u}$ is a viscosity subsolution of (3.2). By exactly the same type of arguments it can be shown that $\underline{u}$ is a viscosity supersolution of this problem. By construction $\overline{u} \geq \underline{u}$ so using the comparison principle for semi-continuous solutions [1] we conclude that $\overline{u} = \underline{u} = u$ is a continuous viscosity solution. Using a variation on the proof of Dini's theorem one can show, as in [5], that the convergence is uniform on all compact subsets of $[0, T] \times \mathbb{R}$. □

Readers used to convergence results for finite difference schemes should note that the requirements on the scheme are quite weak: we have only used *local consistency* whereas the usual conditions for convergence using the Lax theorem require *global* consistency i.e. the convergence in (4.1) must not be pointwise but with respect to some global norm, which requires knowing in advance in what function space the solutions live. This shows the flexibility of the notion of viscosity solution. The price to pay is the loss of information about the order of convergence. Of course in cases where smoothness of the solution can be shown by other means one can discuss the order of convergence using classical methods and higher order schemes can be used. But in absence of information on the smoothness of the solution, the justification of higher-order schemes is not clear.

5 Numerical results

We now illustrate the performance of the scheme proposed above in two examples. The computations were done in variance Gamma models with Lévy density

$$\nu(x) \quad = \quad a\frac{\exp(-\eta_{\pm}|x|)}{|x|}$$

and two sets of parameters: $a = 6.25, \eta_- = 14.4,\ \eta_+ = 60.2$ (VG1) and $a = 0.5,\ \eta_- = 2.7,\ \eta_+ = 5.9$ (VG2), and a Merton model with Gaussian jumps in log price with volatility $\sigma = 15\%$ and Lévy density

$$\nu(x) = 0.1\frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

for a put of maturity of $T = 1$ year. Both of these models satisfy the hypotheses of section 3.1 and 3.2: the Lévy densities have exponentially decreasing tails. The Black-Scholes implied volatilities computed from prices of put options are shown in figure 1: as expected, they display a "smile" (convex) feature as a function of the strike of the option, which flattens out with increasing maturity. These two models have been chosen since in each case there is an alternative method for computing the solution of the PIDE: in the Merton model the solution can be expressed as a series expansion, and in the Variance Gamma model a closed form expression for the characteristic function is available, so one can use Carr & Madan's Fast Fourier method [8]. Comparing our finite difference solution to these alternative solutions (computed with high precision) allows to study the behavior of various error terms in relation with the parameters of the scheme.

The error metric which is relevant from the point of view of financial applications is the error in terms of Black-Scholes implied volatility:

$$\varepsilon(\tau, x) = |\Sigma^{\mathrm{PIDE}}(\tau, x) - \Sigma^{\mathrm{FFT}}(\tau, x)| \qquad \text{in} \quad \%.$$

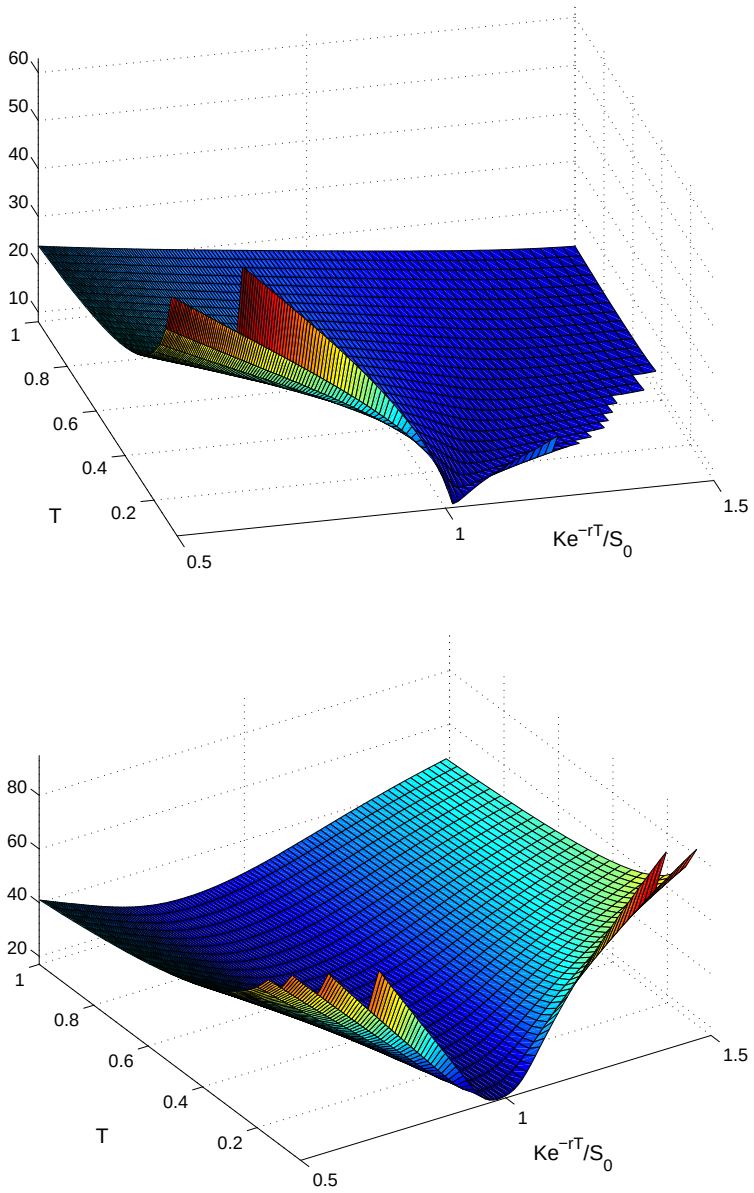

Figure 1: Black-Scholes implied volatilities for put options, as a function of strike and maturity. Above: Variance Gamma model. Below: Merton jump-diffusion model.

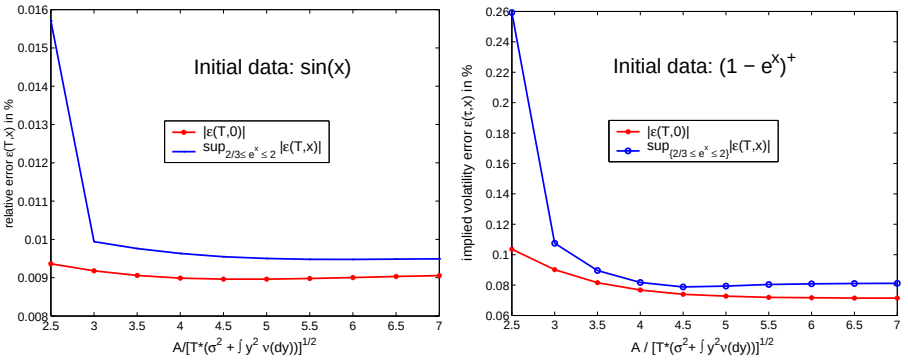

Figure 2: Influence of domain size on localization error for the explicit-implicit finite difference scheme. Left: smooth initial condition $h(x) = \sin(x)$ in Merton jump-diffusion model. Right: put option in Merton jump-diffusion model.

where Σ denotes the Black-Scholes implied volatility computed by inverting the Black-Scholes formula with respect to the volatility parameter and applying it to the computed option price. We have computed both pointwise errors at $x = 0$ (i.e. forward at-the-money options) and uniform errors on the computational range $x \in [\log(2/3), \log(2)]$. This range contains all options prices quoted on the market.

The localization error is shown in figure 2 for the Merton model: domain size A is represented in terms of its ratio to the standard deviation of X_T. An acceptable level is obtained for values of order $\simeq 5$. Notice that as soon as this ratio is ≥ 3, the uniform and pointwise errors are quite close to each other, indicating that we are out of the zone of influence of the numerical boundary conditions. Figure 3 shows the same analysis for the Variance Gamma model.

Figure 4 illustrates the decay of numerical error when $\Delta t, \Delta x \to 0$ i.e. when the number of time / space steps is increased. The behavior is quite similar to the case of the Black-Scholes model.

Figure 5 illustrates the behavior of the error (for a fixed grid size) as a function of maturity for two initial conditions: a smooth one (forward contract) and a non-smooth one (put option). We observe that a non-smooth initial condition leads to a lack of precision for small T. This phenomenon, which is not specific to models with jumps, can be overcome using an irregular time-stepping scheme which exploits the smoothness in time of the solution. Matache et al. [20] have suggested to use irregularly (logarithmically) spaced time stepping, more refined near maturity, in order to improve this convergence.

In the case of infinite activity models an additional parameter which influences the solution is the truncation parameter ε for the small jumps. Choosing this parameter as small as possible is not necessarily a good idea, as discussed in the Remark 6. Figure 6 confirms that, for a given $\Delta x > 0$, the minimal error is obtained for a finite ε which in this case is *larger* than Δx. The optimal

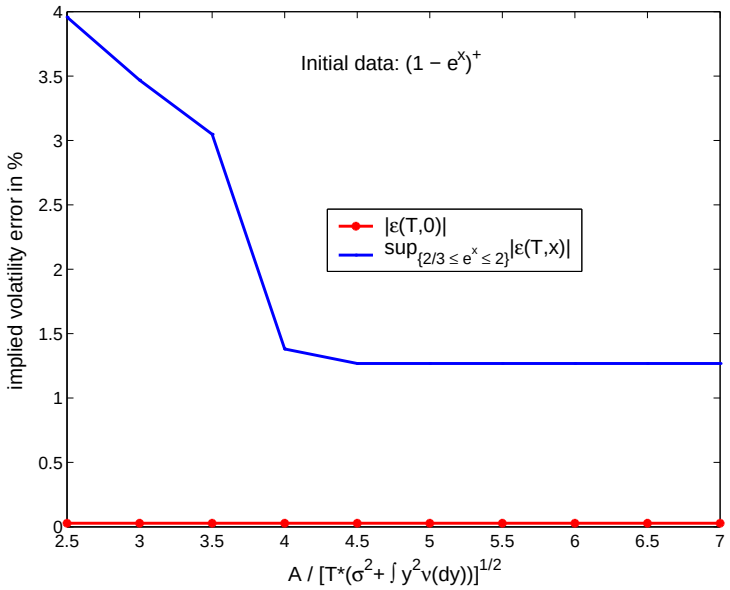

Figure 3: Influence of domain size on localization error for the explicit-implicit finite difference scheme: put option in Variance Gamma model.

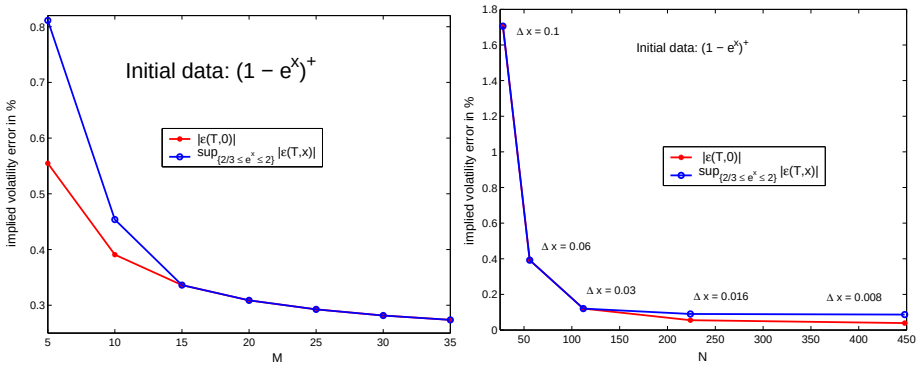

Figure 4: Numerical precision for a put option in the Merton model. Left: Influence of number of time steps M. $\Delta x = 0.05$, $\Delta t = T/M$. Right: influence of number of space steps N. $\Delta x = 2A/N$, $\Delta t = 0.02$.

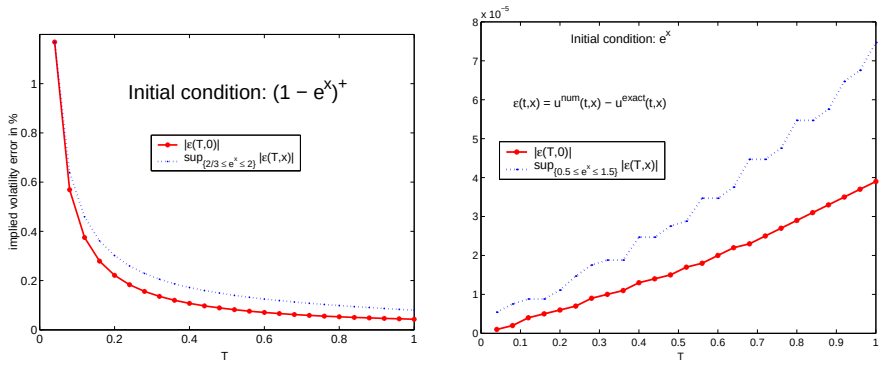

Figure 5: Decrease of error with maturity in the Merton model. Left: non-smooth initial condition (put option) $h(x) = (1 - e^x)^+$. Right: smooth initial condition $h(x) = e^x$.

Model	Put	t sec.	Up-and-out call $H = 120$	t sec.	Double-barrier put $L = 80$, $H = 120$	t sec.
VG1	6.72	0.5	2.73	0.2	2.42	0.1
VG2	8.38	0.9	3.34	0.5	1.68	0.1
Merton	11.04	1.2	1.17	0.5	3.35	4

Table 1: Examples of numerical values for at-the-money option prices. $S = K = 100$, $T = 1$, $A = 5\sqrt{\sigma^2 + \int y^2 \nu(dy)}$, $dt = 0.02$, $dx = 0.01$. Parameter ε is chosen according to Figure 6.

choice of ε is not universal and depends on the growth of the Lévy density near zero. In all cases, the numerical diffusion term $\sigma(\varepsilon)$ significantly improves the convergence by introducing a "numerical viscosity", so even when the underlying model is a pure jump Lévy process, this numerical diffusion term plays a regularizing effect.

The last two figures illustrate the numerical tests for barrier options in the Merton model. Figure 7 shows the price of an up-and-out call. Figure 8 illustrates the numerical convergence of a double-barrier put price as the number N of space steps increases.

In Table 1, we give some examples of option values obtained with our numerical scheme as well as the corresponding computation time in seconds.

6 Discussion and extensions

We have discussed in this work the computation of option prices in jump-diffusion and exponential Lévy models via the solution of integro-differential

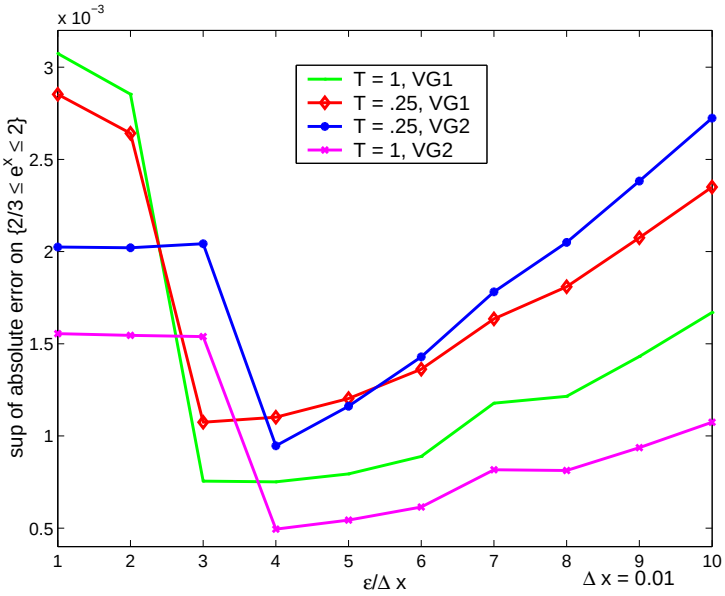

Figure 6: Influence of truncation of small jumps on numerical error in various Variance Gamma models. Put option.

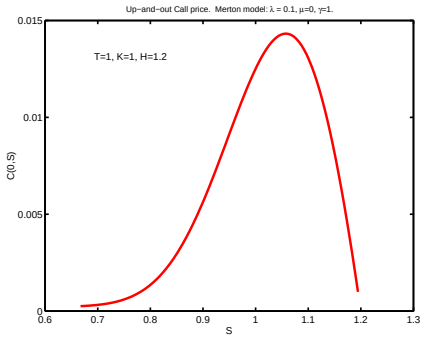

Figure 7: An up-and-out call price in the Merton model. Barrier level: $H = 1.2$.

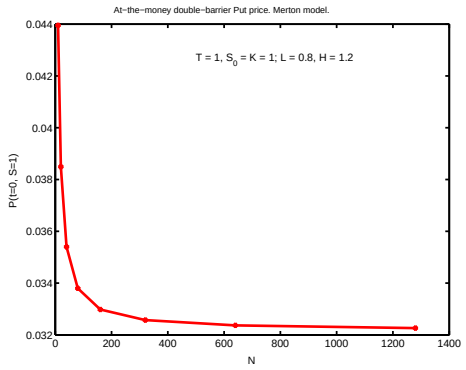

Figure 8: At-the-money double-barrier put price as a function of the number of space steps. Barrier levels: $L = 0.8$, $H = 1.2$.

equations involving an integral term with a possibly singular kernel (Lévy density). After establishing a precise relation between the option prices and the solutions of such equations, we have proposed an explicit-implicit finite difference scheme for solving them. Some difficulties arise due to the non-local character of the integral operator, the non-smoothness of initial conditions, the singularity at zero of the integral kernel and the possible degeneracy of the diffusion coefficient. We have proposed approximation which resolve these difficulties and provided error estimates in each case, under hypotheses which are easily verified on the Lévy density.

For European call options, the finite difference method proposed here required more operations than the Fast Fourier transform method of Carr & Madan [8]; however, our method is applicable in a more general context: it does not require a closed form expression for the characteristic function of the log-price and can handle barrier options.

While other finite difference methods have been proposed in the literature for such equations [2, 14, 26] a comparable analysis of consistency, stability and convergence is absent from these studies. Moreover, by appealing to the formalism of viscosity solutions, our analysis allows fairly general hypotheses on the model. In particular it does not require the diffusion part to be non-degenerate and thus includes models based on pure-jump Lévy processes such as the Variance Gamma model, hyperbolic models, NIG models etc.

Another approach is based on the variational formulation of the PIDE [6]: studied by Zhang [27] for the Merton model, this approach has been recently extended by Matache et al [20] to the infinite activity case using a wavelet Galerkin method. This method requires to know the analytical form of the Lévy density ν, whereas our finite difference method also accomodates a numerical representation of the Lévy measure; for example, ν can be the result of a calibration to option prices as in [10]. The finite difference scheme is also simpler to implement while retaining the flexibility of separating the local and the non-local part of

the operator.

The finite difference method described here is also applicable when the local characteristics $\sigma(t), \nu_t$ of the process are time-dependent, as is the case in time-inhomogeneous jump-diffusion models [11]; it also applies in the case where the diffusion coefficient depends on the price as in [2], without any non-degeneracy requirement on the diffusion coefficient. However empirical studies on performance of risk neutral jump-diffusion models [10] seem to indicate that this feature is redundant with jumps and leads to highly ill-posed calibration problems so the usefulness of combining local volatility models with jumps is far from being clear.

It may be interesting to investigate generalizations of this method to cases where non-linearities are present in the equation such as the valuation of American options or problems with transaction costs.

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A Appendix

Proof of Lemma 1.

Proof. We need to show that $\forall \varepsilon > 0$, $\exists\ U(\tau, x)$ a neighbourhood of (τ, x), such that $\forall (t, y) \in U(\tau, x)$, $\overline{u}(t, y) < \overline{u}(\tau, x) + \varepsilon$. Suppose this is not true, that is

$$\exists \varepsilon > 0,\ \exists (t_k, y_k) \to (\tau, x) : \quad \forall k,\ \overline{u}(t_k, y_k) \geq \overline{u}(\tau, x) + \varepsilon. \tag{A.1}$$

For a fixed k we have by construction:

$$\overline{u}(t_k, y_k) \quad = \quad \limsup_{(\Delta t, \Delta x) \to 0\ (s,z) \to (t_k, y_k)} u^{(\Delta t, \Delta x)}(s, z).$$

Therefore, we can find Δt_k, Δx_k, s_k, z_k in the $1/k$-neighborhood of $(0, 0, t_k, y_k)$, such that

$$u^{(\Delta t_k, \Delta x_k)}(s_k, z_k) \quad > \quad \overline{u}(t_k, y_k) - \frac{1}{k}. \tag{A.2}$$

Combining (A.2) with (A.1) we obtain

$$u^{(\Delta t_k, \Delta x_k)}(s_k, z_k) \quad > \quad \overline{u}(\tau, x) + \varepsilon - \frac{1}{k}. \tag{A.3}$$

By construction, we have $(\Delta t_k, \Delta x_k, s_k, z_k) \to (0, 0, \tau, x)$, therefore

$$\limsup_{k \to \infty} u^{(\Delta t_k, \Delta x_k)}(s_k, z_k) \quad \leq \quad \limsup_{(\Delta t, \Delta x) \to 0\ (t,y) \to (\tau, x)} u^{(\Delta t, \Delta x)}(t, y) = \overline{u}(\tau, x).$$

Taking the upper limit in (A.3) as $k \to \infty$ gives

$$\overline{u}(\tau, x) \quad \geq \quad \overline{u}(\tau, x) + \varepsilon,$$

which is impossible. This contradiction implies that (A.1) is not verified and $\overline{u} \in USC$. □

Proof of Lemma 2.

Proof. The existence of Δt_k, Δx_k, n_k, i_k satisfying (4.27)–(4.28) follows from the definition of the upper limit. Let us take such a sequence and consider for each k the function $u^{(\Delta t_k, \Delta x_k)} - \varphi$. Since there is a finite number of grid points on $[0,T] \times \Omega$, the maximum in (4.29) is reached at a point (τ_{m_k}, x_{j_k}).

Sequence $\{(\tau_{m_k}, x_{j_k})\}_{k \in \mathbb{N}}$ is bounded, therefore one can extract a converging subsequence (for simplicity we will take the same notation): $(\tau_{m_k}, x_{j_k}) \to (\tau^*, x^*)$ as $k \to \infty$. Let us show that $(\tau^*, x^*) = (\tau, x)$. Suppose this is not true. Then, using the definition of $\overline{u}$, the continuity of φ and the fact that $\overline{u} < \varphi$ on on $[0,T] \times \mathbb{R} - \{(\tau, x)\}$ we obtain

$$\limsup_{k \to \infty} \left(u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) - \varphi(\tau_{m_k}, x_{j_k}) \right) \leq$$
$$\leq \limsup_{\substack{(\Delta t, \Delta x) \to 0 \\ (t,y) \to (\tau^*, x^*)}} u^{(\Delta t, \Delta x)}(t,y) - \lim_{k \to \infty} \varphi(\tau_{m_k}, x_{j_k}) =$$
$$= \overline{u}(\tau^*, x^*) - \varphi(\tau^*, x^*) < 0. \quad \text{(A.4)}$$

On the other hand, from the definition of (τ_{m_k}, x_{j_k}) as a maximum point, and properties (4.28) and (4.23) it follows that

$$\limsup_{k \to \infty} \left(u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) - \varphi(\tau_{m_k}, x_{j_k}) \right) \geq$$
$$\geq \limsup_{k \to \infty} \left(u^{(\Delta t_k, \Delta x_k)}(\tau_{n_k}, x_{i_k}) - \varphi(\tau_{n_k}, x_{i_k}) \right) = \overline{u}(\tau, x) - \varphi(\tau, x) = 0,$$

which contradicts (A.4). Therefore, $(\tau^*, x^*) = (\tau, x)$.

From the last inequality we obtain, in particular, that

$$\limsup_{k \to \infty} u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) \geq \varphi(\tau, x) = \overline{u}(\tau, x).$$

From the definition of $\overline{u}$ we have

$$\limsup_{k \to \infty} u^{(\Delta t_k, \Delta x_k)}(\tau_{m_k}, x_{j_k}) \leq \overline{u}(\tau, x).$$

Therefore, this upper limit is equal to $\overline{u}(\tau, x)$, and there exists a subsequence $u^{(\Delta t_k, \Delta x_k)}(\tau_{m'_k}, x_{j'_k}) \to \overline{u}(\tau, x)$, which satisfies by construction the desired properties (4.27)–(4.29). ∎