Fractional Laplace Model for Hydraulic Conductivity

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Based on an examination of K data from four different sites, a new stochastic fractal model, fractional Laplace motion, is proposed. This model is based on the assumption of spatially stationary $\ln(K)$ increments governed by the Laplace PDF, with the increments named fractional Laplace noise. Similar behavior has been reported for other increment processes (often called fluctuations) in the fields of finance and turbulence. The Laplace PDF serves as the basis for a stochastic fractal as a result of the geometric central limit theorem. All Laplace processes reduce to their Gaussian analogs for sufficiently large lags, which may explain the apparent contradiction between large-scale models based on fractional Brownian motion and non-Gaussian behavior on smaller scales.

1. Introduction

The stochastic theory of non-stationary processes with stationary increments began to be applied to detailed hydraulic conductivity (K) measurements during the early 1990's [Molz and Bowman, 1993; Painter, 1996]. Numerous additional applications followed [Molz et al., 2003]. Initial studies assumed that $\ln(K)$ increments or fluctuations (the stationary process) would follow Gaussian probability density functions (PDFs). However, careful analysis of a variety of measurements soon showed that the increment PDFs were strongly non-Gaussian with a distinct resemblance to the Lévy-stable PDF [Painter and Paterson, 1994]. This PDF was attractive, because like the Gaussian PDF it served as the natural mathematical basis for a stochastic fractal. Still further analysis of measurements, and simulations, led researchers to realize that the tails of the empirical PDFs do not have a power-law decay [Painter, 1996; Lu and Molz, 2001]. This led to the proposal of stochastic models that can be varied between Gaussian and Lévy behavior [Painter, 2001].

Based on recent and ongoing studies of existing and new data sets, with the new data being carefully measured in two well-defined sandstone facies [Castle et al., 2003], we report that the increment PDFs, including the full tails, appear to follow the double exponential (Laplace) PDF or stretched exponential PDF, exemplified by the so-called Bessel PDF [Kotz et al., 2001]. Plotted on semi-log scale, the Laplace PDF has a distinct "tent" shape. This behavior of the PDF has been observed in a variety of fields, and for some time, the PDF (and its related cousin, the log-Laplace distribution) has been used for modeling various random processes in archaeology, biology, economics and physics [Kozubowski and Podgórski, 2003] as well as seismic reflection coefficients [Walden and Hosken, 1986; Painter, 2003]. Of particular interest to us is the appearance of the Laplace and related PDFs in studies of turbulence, since turbulence is often involved in sedimentation [Heslot et al., 1987; Ching and Tu, 1994; Sparling and Bacmeister, 2001; Stepanova et al., 2003]. Here we describe a stochastic process based on the Laplace PDF, similar to fractional Brownian motion, and offer an initial hypothesis for its appearance in sedimentation processes.

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2. Statistical analysis

It is common to model $\ln(K)$ as a stochastic process with long-range dependence, characterized by the Hurst coefficient. Figure 1 indicates a Hurst coefficient of 0.24 for $\ln(K)$ measurements from three horizontal transects of sandstone facies at a site in Utah [Castle et al., 2003], based on a dispersional analysis [Caccia, et al., 1997, Lu, et al., 2002], similar to rescaled range analysis. This suggests a fractional Brownian motion or fractional Lévy motion model for $\ln(K)$, models that have been previously applied to other sites [Molz and Bowman, 1993; Painter, 1996]. In a fractional Brownian motion, increments are normally distributed. In a fractional Lévy motion, increments follow a stable distribution [Feller, 1971; Samorodnitsky and Taqqu, 1994], a bell-shaped curve with a heavier, power-law tail. A careful analysis shows that neither of these models is a good approximation for the Utah $\ln(K)$ data, because the probability tails of these measurements fall off at an exponential rate.

Taking increments reduces statistical dependence and clarifies the underlying distribution. Increments of $\ln(K)$ at both the MADE and Cape Cod sites are nearly uncorrelated with a similar shape, fairly symmetric with a sharp peak at zero and long tails. A Laplace PDF $f(x) = (\lambda/2)\exp(-\lambda|x|)$ gives a good fit to each of these data sets (see Figure 2). The peak in the Laplace distribution at zero means that values of hydraulic conductivity at nearby locations are more likely to be closer together than the Gaussian model or the Lévy model predicts. The tails are heavier than Gaussian but lighter then Lévy. This was verified by plotting absolute increments of $\ln(K)$ against tail probability on a semilog scale (similar to Figure 4).

Increments in $\ln(K)$ at the Utah site and the Borden site appear more bell-shaped (see Figure 3). In order to distinguish between different bell-shaped distributional models, we examine the probability tails. For both sites we find that the α power of the absolute increments in $\ln(K)$ ($\alpha = 1.18$ at the Utah site and $\alpha = 1.23$ at the Borden site) has a tail probability that falls off exponentially. This corresponds to the so-called "stretched exponential" distribution recently observed in turbulent flow-fields. This was verified by plotting the α power of the absolute increments of $\ln(K)$ against tail probability on a semilog scale, illustrated in Figure 4 for the Utah data.

Figure 5 shows that the distribution of increments at the MADE site for different scales $n = 2, 3, \ldots, 9$ converge to a normal distribution as n gets larger, consistent with the central limit theorem. Standard statistical tests also indicate convergence to normality. Increments of $\ln(K)$ at the other three sites examined show a similar behavior, converging to a normal distribution as the scale increases.

Presently, it is not clear why a pure Laplace PDF is associated with measurements at some sites and a stretched Laplace at others, although the same type of mixed behavior has been reported in turbulence studies [Ching and Tu, 1994]. In the present case, there is a possibility that the differences are related, at least in part, to the K measurement process utilized in the various studies. It is well known that in sedimentary materials we almost always see a pronounced anisotropy in property values. Thus property values should always be measured along a principal axis of the anisotropy. The MADE and Cape Cod data are based on a horizontal flow field set up by pumping a well. So the mean flow would be along a principal direction (horizontal) of the anisotropy. However, the Utah data are based on spatially small but 3D flow-fields associated with a gas

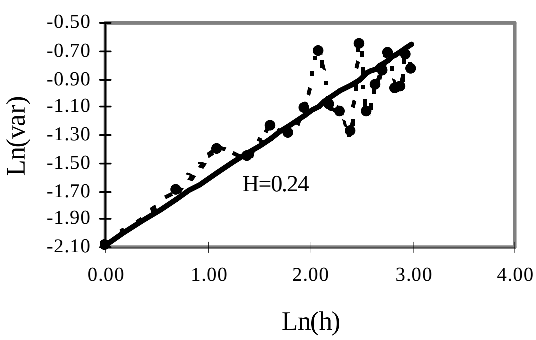

Figure 1. Dispersional analysis of the horizontal Utah data results in a Hurst coefficient $H = 0.24$. A reference line with slope $2H$ is plotted against a log-log plot of the actual variance as a function of lag, computed independent of the dispersional analysis.

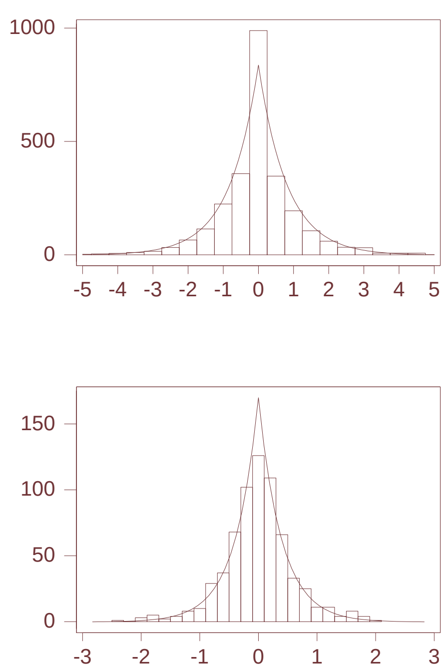

Figure 2. Increments of $\ln(K)$ at the MADE site (top) and the Cape Cod site (bottom) fit a Laplace distribution, with exponential tails. Both measurements based on horizontal flow in a mean sense.

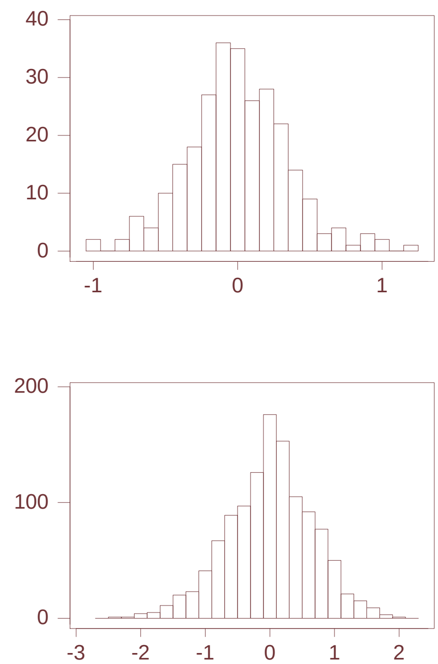

Figure 3. Increments of $\ln(K)$ at the Utah site (top) and the Borden site (bottom) are more bell-shaped, but peaking around the mean is evident in both data sets. Utah data based on 3D flow of unknown detail. Borden data based on highly disturbed samples.

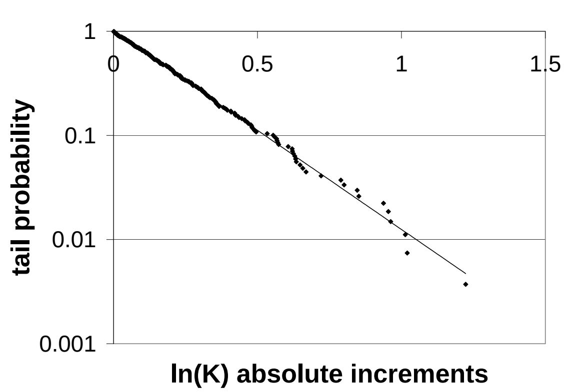

Figure 4. Absolute values of increments in $\ln(K)$ raised to the power $\alpha = 1.167$ at the Utah site show an excellent fit to an exponential distribution.

mini-permeameter, which would involve flow at all angles relative to the principal anisotropy directions. Also, the Borden data were based on remixed samples. It is certainly plausible that the selected measurement procedures at these two sites could have modified a more pure Laplace structure in the principal directions.

3. Fractional Laplace motion

In this section we propose an alternative fractal model for $\ln(K)$ based on the Laplace PDF rather than the normal or Lévy stable PDF. The stable PDF family (which includes normal as a special case) appears in the extended central limit theorem [Feller, 1971]. Sums of IID random variables are asymptotically stable, normal if their variance is finite. Sums of IID stable random variables maintain the same PDF (up to a change of scale), leading to elegant self-affine stochastic models.

The Laplace PDF emerges from a different and less well-known central limit theorem. The sum of a random number of IID variates with finite variance is asymptotically Laplace if the random count is geometrically distributed. For example, daily price changes in a stock are the sum of a random number of price jumps, one for each trade. The probability that the number of trades exceeds n falls off geometrically, like p^n for some $0 < p < 1$, and then the geometric central limit theorem implies that daily price changes are approximately Laplace, as often seen in financial data. A similar explanation for the appearance of the Laplace PDF in sedimentation may be due to a geometric number of depositional events. Although a detailed analysis is beyond the scope of the present letter, it would seem likely that the geometric central limit theorem would play an important role in a physical explanation for the appearance of the Laplace PDF in both turbulence and sedimentation. Such considerations are topics for future research in a variety of disciplines. We mention the idea here merely to give some initial rough hypothesis for the Laplace PDFs that we and others have observed.

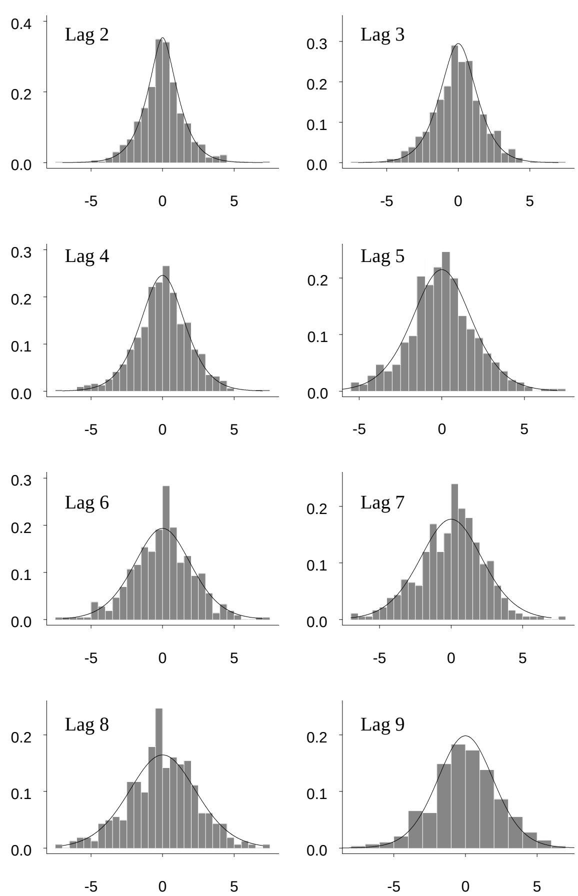

Figure 5. The distribution of $\ln(K)$ increments at the MADE site converges to a normal as the scale increases. Increments are computed as the difference between every nth value of $\ln(K)$ for $n = 2, 3, \ldots, 9$. Fractional Laplace motion PDFs give a good fit.

The Laplace PDF, as the limit of a random number of observations, can also be considered as a normal PDF with a random variance or spread. A similar idea was used by Painter [2001] to develop realistic simulations of K fields. The simplest way to implement this model is by subordination. In subordination, a stochastic process $X(t)$ is replaced by another process $X(T(t))$ where $T(t)$ is a randomized version of the independent variable t. In groundwater hydrology, $T(t)$ has been used to represent the "operational time" a particle experiences, a time that passes more rapidly in high velocity zones [Baeumer, et al., 2001]. When modeling characteristics of a porous medium such as hydraulic conductivity, the subordinator may represent the number of depositional features encountered over a distance t. Since the depositional process is probably chaotic, the same Laplace and stretched exponential probability models that describe turbulent flow are also relevant here.

A flexible model for $\ln(K)$, consistent with all of the statistical analysis shown in the previous section, can be obtained by subordinating a (fractional) Brownian motion $X(t)$ with Hurst coefficient $0 < H < 1$ to a Gamma process $T(t)$, a stationary independent increment process with an exponential PDF at $t = 1$, and more generally a Gamma PDF (the sum of IID exponentials has a Gamma PDF). For $H = 1/2$ the subordinated process $Y(t) = X(T(t))$ is called a Laplace motion [Kotz, et al., 2001]. For other values of $0 < H < 1$ we call $Y(t)$ a *fractional Laplace motion*. A simple conditioning argument [Kozubowski, et al., 2004] shows that the fractional Laplace motion variable $Y(t)$ has stretched exponential tails that fall off like $|x|^\alpha \exp(-\lambda|x|^\beta)$ for large x where $\lambda > 0$, $\alpha = 2t/(1+2H)-1$ and $\beta = 2/(1+2H)$, so that $\beta = 1$ (exponential tails) when $H = 1/2$. As t increases, the subordinator $T(t) \sim ct$ so the variables $Y(t)$ converge to a normal, which is consistent with the data. If $\ln(K)$ has exponential probability tails, then K has power law tails, consistent with the findings of Benson, et al. [2001].

Laplace motion has many of the familiar properties of Brownian motion, except that now increments follow a Laplace rather than a normal PDF. Fractional Brownian motion $X(t)$ has stationary increments and covariance function $<X(t)\,X(s)> = (1/2)\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right]$. Fractional Laplace motion $Y(t) = X(T(t))$ also has stationary increments, and its covariance function $<Y(t)\,Y(s)> =$

$$\frac{1}{2}\left[\frac{\Gamma(|t|+2H)}{\Gamma(|t|)} + \frac{\Gamma(|s|+2H)}{\Gamma(|s|)} - \frac{\Gamma(|t-s|+2H)}{\Gamma(|t-s|)}\right]$$

is asymptotically equivalent to that of fBm, using the well-known fact that $\Gamma(x+c) \sim x^c\,\Gamma(x)$ as $x \to \infty$. The increments of fractional Brownian motion $X(t)$ at lag h, defined by $N(t) = X(t) - X(t-h)$, are called fractional Gaussian noise. This stationary process with stationary increments has covariance function $<N(t)\,N(t+r)> = (1/2)[|r+h|^{2H} + |r-h|^{2H} - 2|r|^{2H}]$. Increments of fractional Laplace motion defined by $L(t) = Y(t) - Y(t-h)$, called fractional Laplace noise, form a stationary process with stationary increments and covariance function $<L(t)\, L(t+r)> =$

$$\frac{1}{2}\left[\frac{\Gamma(|r+h|+2H)}{\Gamma(|r+h|)} + \frac{\Gamma(|r-h|+2H)}{\Gamma(|r-h|)} - 2\frac{\Gamma(|r|+2H)}{\Gamma(|r|)}\right],$$

asymptotically equivalent to the covariance function of fractional Gaussian noise. Details may be found in a forthcoming paper [Kozubowski, et al., 2004]. Fractional Laplace motion can be simulated by the method of Painter [2001] using an exponential subordinator, or directly from a Laplace noise field as described in Benson, et al. [2003].

4. Summary

Based on an examination of K data from 4 different sites, a new stochastic fractal model, fractional Laplace motion (fLam), is proposed. This model is based on the assumption of spatially stationary $\ln(K)$ increments governed by the Laplace PDF, which has been observed at a number of sites. Similar behavior has been reported for other increment processes (often called fluctuations) in the fields of finance and turbulence. A possible stochastic connection between turbulence and sedimentation as reflected by $\ln(K)$ distributions is intriguing and deserves more study. Sometimes the PDF observed is stretched Laplace, and the reason for this deviation from pure Laplace behavior is not clear, but may relate to imperfect measurement procedures.

The Laplace PDF serves as the basis for a stochastic fractal as a result of the so-called geometric central limit theorem, which states that a Laplace PDF will appear from random sums of IID variables with finite variance if the random count is geometrically distributed. FLam and fractional Laplace noise (fLan) are named in analogy to the corresponding Gaussian processes (fBm and fGn), and the covariance function for each stochastic process is presented. A generation procedure for fLam based on subordination of fBm to a Gamma PDF is suggested. All Laplace processes reduce to their Gaussian analogs for sufficiently large lags, consistent with the K field data, and this property also potentially removes an apparent contradiction between the large-scale fBm model of Neuman (1990) and the consistent observation of non-Gaussian PDFs on smaller scales.

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