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Counting hyperbolic components

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Abstract

We give formulae for the numbers of type II and type IV hyperbolic components in the space of quadratic rational maps, for all fixed periods of attractive cycles.

1. *Introduction*

The aim of this paper is to solve some basic counting problems which arise in the study of quadratic rational maps as dynamical systems acting on the Riemann Sphere. In particular, given any integers $n, m \geqslant 1$, we compute the number of quadratic rational maps such that both critical points are periodic, one of period n and the other of period m. A computation with a finite output is only possible (and interesting) if our counting takes place in an appropriate moduli space, namely, the moduli space $\mathcal{M}_2^{\mathrm{cm}}$ formed by conjugacy classes of quadratic rational maps with marked critical points (see Subsection 1.1).

Without neglecting the intrinsic interest that enumerative problems in moduli spaces have, our motivation finds its origin in the study of the open and conjecturally dense subset of $\mathcal{M}_2^{\mathrm{cm}}$ formed by hyperbolic maps. In short, we say that a connected component of this subset is a hyperbolic component. Relevant dynamical features such as number and period of attractors remain unchanged within a hyperbolic component. In fact, the dynamics over the Julia set of maps within a hyperbolic component are quasiconformally conjugate. According to [**16**], counting hyperbolic components and computing numbers as the one mentioned above are essentially equivalent problems.

Hyperbolic maps are uniformly expanding on the Julia set and are characterized as the maps for which all critical points lie in the basin of some attracting periodic orbit. The various possible combinatorial arrangements that the orbits of the Fatou components containing critical points might have give a first rough classification of hyperbolic components. More precisely, given a hyperbolic component $\mathcal{H}$ in $\mathcal{M}_2^{\mathrm{cm}}$, then one and exactly one of the following holds for all maps in $\mathcal{H}$ (see [**16**]).

(i) Both critical points lie in the Fatou component of an attracting fixed point.
(ii) There is one periodic orbit of Fatou components and the critical points belong to different components of this orbit.
(iii) There is one periodic orbit of Fatou components and only one critical point belongs to a component of this orbit, and the other eventually maps into this orbit of components.
(iv) There are two periodic orbits of Fatou components, each one containing a critical point.

We say that $\mathcal{H}$ is of type I, II, III or IV, according to which one of the statements above holds.

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There is exactly one type I hyperbolic component in $\mathcal{M}_2^{\mathrm{cm}}$, it is formed by maps having a disconnected Julia set. The other hyperbolic components are formed by maps having connected Julia set [16].

According to [16], each hyperbolic component $\mathcal{H} \subset \mathcal{M}_2^{\mathrm{cm}}$ of quadratic rational maps with connected Julia sets contains a unique critically marked postcritically finite rational map, modulo conjugacy, called the *centre* of the hyperbolic component. Thus, counting centres and counting hyperbolic components are completely equivalent.

A type II component $\mathcal{H}$ such that its periodic orbit of Fatou components has period n is centred at a map with both critical points in the same cycle of period n. Similarly, a type IV component $\mathcal{H}$, with the first critical point in a Fatou component of period n and the second in one of period m, is centred at a map where the first critical point has period n and the second m. In the first case, we say that $\mathcal{H}$ is a type II component of period n, and in the second case we say that $\mathcal{H}$ is a type IV component of period (n, m).

The aim of this paper is to obtain formulae for the following numbers:

$$\begin{aligned}\eta_{\mathrm{II}}(n) &= \#\{\mathcal{H} \subset \mathcal{M}_2^{\mathrm{cm}} \mid \mathcal{H} \text{ is a type II component of period dividing } n\},\\ \eta_{\mathrm{IV}}(n,m) &= \#\{\mathcal{H} \subset \mathcal{M}_2^{\mathrm{cm}} \mid \mathcal{H} \text{ is a type IV component of period } (j,k)\\ &\qquad \text{where } j \mid n \text{ and } k \mid m\}.\end{aligned}$$

The value of $\eta_{\mathrm{IV}}(1, m)$ is well known. In fact, quadratic rational maps that fix a critical point are quadratic polynomials, modulo putting the fixed critical point at ∞. Thus, $\eta_{\mathrm{IV}}(1, m)$ is the number of elements of the quadratic family $Q_c(z) = z^2 + c$ for which the critical point $z = 0$ is periodic of period dividing m. In the early 1980s, it was established that $\eta_{\mathrm{IV}}(1, m) = 2^{m-1}$ (see [4, Exposé XIX] for three different proofs of the fact that all solutions of $Q_c^{m-1}(c) = 0$ are simple).

For higher degree polynomials, it is likely that the available techniques may lead to answers for analogue counting problems. However, similar problems for moduli spaces of rational maps of degree at least 3 are open.

1.1. *Statement of the results*

The elements of the *critically marked moduli space of quadratic rational maps* $\mathcal{M}_2^{\mathrm{cm}}$ are the conjugacy classes of triples (f, ω_1, ω_2), where f is a quadratic rational map with critical points at ω_1 and ω_2. More precisely, (f, ω_1, ω_2) and $(g, \omega_1', \omega_2')$ are conjugate if there exists a Möbius transformation γ such that $\gamma \circ f = g \circ \gamma$ and $\gamma(\omega_j) = \omega_j'$ for $j = 1, 2$. We will denote the conjugacy class of (f, ω_1, ω_2) by $[f, \omega_1, \omega_2]$. The space $\mathcal{M}_2^{\mathrm{cm}}$ is naturally identified to a complex algebraic surface with a unique singular point at the centre of the unique type II component of period 2, that is, $[z^{-2}, 0, \infty]$ (see [12, Section 6]).

To state our results, we will also need to introduce the numbers $\nu_q(n)$ as follows. For $q > 1$ and $n \geqslant 1$, let

$$r \equiv n \bmod q,$$

such that $0 \leqslant r < q$.

If $n < q$, then define $\nu_q(n) = 0$, otherwise let

$$\nu_q(n) = \begin{cases} \dfrac{2^{n-1} - 2^{r-1}}{2^q - 1} & \text{if } q \nmid n,\\[2ex] \dfrac{1}{2} + \dfrac{2^{n-1} - 2^{-1}}{2^q - 1} & \text{if } q \mid n.\end{cases}$$

In Lemma 3.8, we deduce from well-known results about the quadratic polynomial family that $\nu_q(n)$ is the number of hyperbolic components of period dividing n in the p/q-limb of the Mandelbrot set, for any given p (for example, for the definition of limbs see [4] or [13]).

Denote by $\phi(n)$ the Euler phi function of n (that is, the number of integers $1 \leqslant k \leqslant n$ which are relatively prime to n).

We shall prove the following theorems.

THEOREM 1.1. *For all integers $m \geqslant n \geqslant 1$, we have*

$$\eta_{\mathrm{IV}}(n,m) = \frac{1}{3}(5 \cdot 2^{n+m-3} + 2^{n-2} + 2^{m-2}) - \frac{1}{2}\sum_{2 \leqslant q \leqslant n} \phi(q)\nu_q(n)\nu_q(m) - \eta_{\mathrm{II}}(\gcd(n,m)) + \frac{1}{6}((-1)^n + (-1)^m + (-1)^{n+m}). \tag{1.1}$$

In particular, given $n \geqslant 1$,

$$\eta_{\mathrm{IV}}(n,m) = \left(\frac{5}{3} \cdot 2^{n-3} + \frac{1}{12} - \frac{1}{4}\sum_{2 \leqslant q \leqslant n} \frac{\phi(q)\nu_q(n)}{2^q - 1}\right) \cdot 2^m + \varepsilon_n(m),$$

where $\varepsilon_n(m)$ is a bounded function of m. More precisely,

$$|\varepsilon_n(m)| \leqslant 2^n + 2^{2\gcd(n,m)}.$$

For example, the asymptotic behaviour, as $m \to \infty$, of the number of type IV hyperbolic components with one cycle of period exactly n and the other of period dividing m, when $n \leqslant 7$ is:

$$\begin{cases} 2^{m-1} & \text{if } n = 1, \\ \frac{1}{3} \cdot 2^m + O(1) & \text{if } n = 2, \\ \frac{23}{21} \cdot 2^m + O(1) & \text{if } n = 3, \\ \frac{78}{35} \cdot 2^m + O(1) & \text{if } n = 4, \\ \frac{6103}{1085} \cdot 2^m + O(1) & \text{if } n = 5, \\ \frac{202\,371}{19\,530} \cdot 2^m + O(1) & \text{if } n = 6, \\ \frac{29\,316\,701}{1\,240\,155} \cdot 2^m + O(1) & \text{if } n = 7. \end{cases}$$

REMARK 1. The analogue result for type III components is proved when $n = 3$ in [**18**, 3.4], in a very simple-minded way, where it is pointed out that the type IV calculation can be done similarly. The simple-minded calculation for $n = 3$ agrees with the above result.

It is easier to write the formula for η_{II} in terms of the number $\eta'_{\mathrm{II}}(m)$ of type II hyperbolic components of period exactly m, so that

$$\eta_{\mathrm{II}}(m) = \sum_{d \,|\, m} \eta'_{\mathrm{II}}(d).$$

THEOREM 1.2. *For $m \geqslant 3$,*

$$\sum_{d\,|\,m, d\geqslant 3} \frac{m}{d}\eta'_{\mathrm{II}(d)} = \frac{7}{36}m2^m - \frac{37}{108}2^m - \frac{m}{4} - (-1)^m\frac{5}{36}m + \frac{1}{2} + (-1)^m\frac{5}{54} - \frac{1}{2}\sum_{3\leqslant q, q\leqslant j\leqslant m-q} \phi(q)\nu_q(j)\nu_q(m-j). \tag{1.2}$$

Thus, for $m \leqslant 8$, the number $\eta'_{\mathrm{II}}(m)$ of type II components of period m is

$$\begin{array}{ll} 1 & \text{if } m = 2,\\ 2 & \text{if } m = 3,\\ 6 & \text{if } m = 4,\\ 20 & \text{if } m = 5,\\ 46 & \text{if } m = 6,\\ 128 & \text{if } m = 7,\\ 284 & \text{if } m = 8. \end{array}$$

There is a (negative) contribution from $\phi(q)\nu_q(j)\nu_q(m-j)$ only for $m \geqslant 6$.

As usual in enumerative problems of a geometric nature, the proofs of our main results rely on counting a larger set contained in partial compactifications of moduli space and then subtracting off the intersections at infinity. Mostly, we will consider a large portion $\mathcal{R}$ of $\mathcal{M}_2^{\mathrm{cm}}$, and identify it with an open and dense subset of the projective plane $\mathbb{CP}^2$. The centres of the hyperbolic components of types II and IV of given periods will be the intersection locus in $\mathcal{R}$ of two algebraic curves. To apply Bezout's Theorem, we use well-known results which show that intersections are transverse and which are also useful to compute the degrees of these curves. These well-known results are concerned with parametrizations of hyperbolic components of quadratic rational maps [16]. To count the intersections of these curves at infinity (that is, outside $\mathcal{R}$ in $\mathbb{CP}^2$), we rely on results from Stimson's Thesis [21] as well as some new results.

The proof of Theorem 1.1 regarding type IV components is contained in Section 3, and the proof of Theorem 1.2 regarding type II components is contained in Section 4. In Section 2, we discuss some of the necessary background results.

2. *Preliminaries: hyperbolic components and periodic curves*

Recall that we work in the moduli space $\mathcal{M}_2^{\mathrm{cm}}$ of conjugacy classes $[f, \omega_1, \omega_2]$ where f is a quadratic rational map, and ω_1, ω_2 are its critical points. According to [16], each hyperbolic component of type II or IV in $\mathcal{M}_2^{\mathrm{cm}}$ contains a unique postcritically finite quadratic rational map, called the *centre* of the hyperbolic component. We consider the *periodic curves* V_n (respectively, W_m) in $\mathcal{M}_2^{\mathrm{cm}}$, where ω_1 has period exactly n (respectively, ω_2 has period exactly m). Thus, our task is related to understanding and computing the cardinality of $V_n \cap W_m$, since the elements of this intersection are the centres of type II or IV hyperbolic components with a Fatou component of period n containing ω_1 and a Fatou component of period m containing ω_2.

2.1. *Parametrization of type II and IV components*

In order to compute the cardinality of $V_n \cap W_m$, it is convenient to know that V_n and W_m have transversal intersections except at the singular point of $\mathcal{M}_2^{\mathrm{cm}}$. This well-known result is a direct consequence of the parametrizations [16] of type II and IV hyperbolic components, which we proceed to describe.

According to part 2(a) of the Main Theorem in [16], we have that multipliers parametrize type IV components.

THEOREM 2.1. *Denote the open unit disc in* $\mathbb{C}$ *by* $\mathbb{D}$. *Let* $\mathcal{H} \subset \mathcal{M}_2^{\mathrm{cm}}$ *be a type* IV *hyperbolic component. Given* $\mathbf{f} = [f, \omega_1, \omega_2] \in \mathcal{H}$, *for* $i = 1, 2$, *denote by* $\lambda_i(\mathbf{f})$ *the multiplier of the attracting periodic orbit which contains* ω_i *in its basin. Then the map* $\lambda : \mathcal{H} \to \mathbb{D} \times \mathbb{D}$ *given by* $\lambda(\mathbf{f}) = (\lambda_1(\mathbf{f}), \lambda_2(\mathbf{f}))$ *is biholomorphic.*

Observe that a non-empty intersection of a periodic curve V_n (respectively, W_m) with a type IV hyperbolic component $\mathcal{H}$ corresponds to the disc $\lambda_1 = 0$ (respectively, $\lambda_2 = 0$).

In order to parametrize type II components, we will model the first return map to critical Fatou components by a pair of Blaschke products of the form

$$\beta_a(z) = z\frac{1-\bar{a}}{1-a}\frac{z-a}{1-\bar{a}z},$$

where $a \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Note that $\beta_a(\mathbb{D}) = \mathbb{D}$, $\beta_a(0) = 0$ and $\beta_a(1) = 1$.

THEOREM 2.2. *Consider a period* $m \geqslant 3$ *type* II *hyperbolic component* $\mathcal{H} \subset \mathcal{M}_2^{\mathrm{cm}}$ *such that the Fatou component containing the first critical point maps in* $j < m$ *iterates onto the one containing the second critical point. Then there exists a homeomorphism* $h : \mathcal{H} \to \mathbb{D} \times \mathbb{D}$ *such that the following holds.*

If $h(\mathbf{f}) = (a, b)$ *where* $\mathbf{f} = [f, \omega_1, \omega_2] \in \mathcal{H}$ *and* U_1 (*respectively*, U_2) *is the Fatou component of* f *containing* ω_1 (*respectively*, ω_2), *then for an appropriate choice of conformal maps* $h_i : \mathbb{D} \to U_i$ *where* $i = 1$ *and* 2 *we have that* $f^j : U_1 \to U_2$ *coincides with* $h_2 \circ \beta_a \circ h_1^{-1}$ *and* $f^{m-j} : U_2 \to U_1$ *coincides with* $h_1 \circ \beta_b \circ h_2^{-1}$.

Although this theorem is a consequence of the main result in [16], it is easier to deduce the above statement from the literature with the aid of [14, Theorem 9.3].

Proof. In [14], Milnor works in the moduli space $\mathcal{M}_2^{\mathrm{fm}}$ of quadratic rational maps with marked fixed points. This moduli space $\mathcal{M}_2^{\mathrm{fm}}$ is formed by conjugacy classes of (f, x_1, x_2, x_3), where f is a quadratic rational map with fixed points at x_1, x_2, x_3 listed with repetitions according to multiplicity. The conjugacy class of (f, x_1, x_2, x_3) is formed by the quadruples $(\gamma \circ f \circ \gamma^{-1}, \gamma(x_1), \gamma(x_2), \gamma(x_3))$, where γ is a Möbius transformation. It follows that $\mathcal{M}_2^{\mathrm{fm}}$ is a complex (affine) algebraic surface with a singularity at the class of the map possessing a triple fixed point [12, Lemma 6.6]. We will also employ the totally marked moduli space $\mathcal{M}_2^{\mathrm{tm}}$ defined as the conjugacy classes of $(f, \omega_1, \omega_2, x_1, x_2, x_3)$, where ω_1, ω_2 are the critical points of f and the x_i are the fixed points as above. This latter moduli space is a smooth complex algebraic surface [12, Lemma 6.6]. According to Milnor [12, p. 51], the forgetful map $\mathcal{M}_2^{\mathrm{tm}} \to \mathcal{M}_2^{\mathrm{cm}}$ is a degree 2 covering ramified only over $[z^{-2}, 0, \infty]$ and $\mathcal{M}_2^{\mathrm{tm}} \to \mathcal{M}_2^{\mathrm{fm}}$ is a degree 6 covering ramified only over the unique singular point of $\mathcal{M}_2^{\mathrm{fm}}$.

Given a type II hyperbolic component $\mathcal{H}'$ in $\mathcal{M}_2^{\mathrm{fm}}$, [14, Theorem 9.3] produces a homeomorphism $h' : \mathcal{H}' \to \mathbb{D} \times \mathbb{D}$ that assigns to each element of $\mathcal{H}'$ a pair of Blaschke products (β_a, β_b) which model the first return map to critical Fatou components (as in the statement of the theorem). The conformal maps h_1 and h_2, which uniquely determine β_a and β_b, are themselves determined by the images in U_1 and U_2 of 0 and 1, with 0 mapped to the unique fixed point of f^m in U_1 or U_2, and 1 mapped by h_1 to one of the three fixed points of f^m on ∂U_1, and by h_2 to the image of this fixed point under f^j. (It turns out not to matter which fixed point on ∂U_1 is chosen, provided it is done continuously in f.) Let us now translate this result to

a period $m \geqslant 3$ type II hyperbolic component $\mathcal{H} \subset \mathcal{M}_2^{\rm cm}$. From [16], we know that $\mathcal{H}$ is simply connected and from [14, Theorem 9.3] the same holds for hyperbolic components in $\mathcal{M}_2^{\rm fm}$. Thus, via lifting $\mathcal{H}$ to $\mathcal{M}_2^{\rm tm}$, and then projecting into $\mathcal{M}_2^{\rm fm}$, we obtain a natural biholomorphic map from $\mathcal{H}$ onto a hyperbolic component in $\mathcal{M}_2^{\rm fm}$. The postcomposition of this homeomorphism with Milnor's parametrization gives us the desired parametrization $h : \mathcal{H} \to \mathbb{D} \times \mathbb{D}$. □

COROLLARY 2.3. *For all $n \geqslant 1$ and $m \geqslant 1$, any intersection of V_n with W_m at a non-singular point of $\mathcal{M}_2^{\rm cm}$ is transverse.*

Proof. Recall that two curves in $\mathbb{CP}^2$ are said to intersect transversally at a point p if the curves are smooth at p and their tangent spaces span the ambient tangent space. Equivalently, the intersection number of two curves at a point p is 1, that is, small perturbations of the curves have an intersection point nearby.

Any point $\mathbf{f}$ in $V_n \cap W_m$ lies in a type II or IV hyperbolic component. In view of Theorems 2.1 and 2.2, there is a homeomorphism which maps a neighbourhood of the origin in $\mathbb{C}^2$ onto a neighbourhood of $\mathbf{f}$ in $\mathcal{M}_2^{\rm cm}$ such that the germ of $\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$ at the origin maps onto the germ of $V_n \cup W_m$ at $\mathbf{f}$. Transversality of curves is a topological invariant. In fact, one might argue as follows. From [19, Theorem 1], we know that the intersection of $V_n \cup W_m$ with a small 3-sphere around $\mathbf{f}$ consists of a link of two unknotted curves with linking number 1. This linking number agrees with the intersection number of the curves (for example, see [24, Lemma 5.2.3]). Thus, the intersection at $\mathbf{f}$ is transverse. □

We will also need a result regarding transversal intersections of other critical orbit relations. More precisely, given $m \geqslant 2$ and $1 \leqslant j < m$ we consider the curve $\mathcal{P}'_j$ consisting of all elements $[f, \omega_1, \omega_2] \in \mathcal{M}_2^{\rm cm}$ such that $f^j(\omega_1) = \omega_2$. Also, we let $\mathcal{Q}'_{m-j}$ be the curve formed by all $[f, \omega_1, \omega_2] \in \mathcal{M}_2^{\rm cm}$ such that $f^{m-j}(\omega_2) = \omega_1$. Clearly, any intersection point of these curves is the centre of a type II hyperbolic component of period n which divides m.

COROLLARY 2.4. *For all $m \geqslant 3$ and $1 \leqslant j < m$, any intersection of $\mathcal{P}'_j$ with $\mathcal{Q}'_{m-j}$ at a non-singular point of $\mathcal{M}_2^{\rm cm}$ is transverse.*

Proof. To prove this corollary, it is more convenient to reparametrize any type II hyperbolic component by pairs of Blaschke products having a critical point at the origin. With this purpose, for each $a \in \mathbb{D}$ let us consider

$$\gamma_a(z) = \frac{1 - \bar{a}}{1 - a} \frac{z^2 - a}{1 - \bar{a}z^2}.$$

Note that $\gamma_a(\mathbb{D}) = \mathbb{D}$, $\gamma'(0) = 0$ and $\gamma(1) = 1$. Since γ_a has only one critical point $z = 0$ in $\mathbb{D}$, if h_1 and h_2 are the Möbius transformations of $\mathbb{D}$, which fix 1 and map 0 to the critical points of β_a and β_b, respectively, then $\varphi : \mathbb{D} \times \mathbb{D} \to \mathbb{D} \times \mathbb{D}$ given by $\varphi(a, b) = (c, d)$ is a diffeomorphism, where $\gamma_c = h_2^{-1} \circ \beta_a \circ h_1$ and $\gamma_d = h_2^{-1} \circ \beta_b \circ h_1$.

It follows that given a period $n \geqslant 3$ type II hyperbolic component $\mathcal{H}$, a non-empty intersection $\mathcal{P}'_j \cap \mathcal{H}$ is mapped under $\varphi \circ h : \mathcal{H} \to \mathbb{D} \times \mathbb{D}$ onto $\{0\} \times \mathbb{D}$, where h is as in Theorem 2.2 and φ is as in the previous paragraph. Similarly, $\mathcal{Q}'_{m-j} \cap \mathcal{H}$ is mapped onto $\mathbb{D} \times \{0\}$. The corollary now follows from [19, Theorem 1] as in the previous proof. □

2.2. *Periodic curves: smoothness*

Although periodic curves are well known to be smooth, we are unable to provide a published reference for this fact. A sketch of its proof is included below only for the sake of completeness, since we make no essential use of it.

THEOREM 2.5. *For all $n \geqslant 1$ and $m \geqslant 1$, the curves V_n and W_m are smooth at non-singular points of $\mathcal{M}_2^{\mathrm{cm}}$.*

Proof (*Sketch*). For $n, m \leqslant 2$, this can be checked by a direct calculation, so we assume that n and m are at least 3. To fix ideas, we prove smoothness for V_n, smoothness of W_m follows along the same lines. We consider $\mathbf{g} = [g, \omega_1(g), \omega_2(g)] \in V_n$ and proceed to prove that V_n is smooth in a neighbourhood U of $\mathbf{g}$.

First assume that $\omega_2(g)$ is not in the immediate basin of the period n cycle containing $\omega_1(g)$. Normalizing the critical points to 0 and ∞, and one of the critical values to 1, it is not difficult to conclude that there exists a holomorphic section $(f_u, \omega_1(u), \omega_2(u))$ of quadratic rational maps defined for all u in a neighbourhood U of $\mathbf{g}$ (that is, $u = [f_u, \omega_1(u), \omega_2(u)]$ for all $u \in U$). Taking U sufficiently small, we may assume that the periodic orbit of $\omega_1(u)$ for $u = \mathbf{g}$ has a well-defined analytic continuation to a periodic orbit $\mathcal{O}_u$ of f_u for all $u \in U$. Let $\lambda : U \to \mathbb{C}$ be the map that assigns to each $u \in U$ the multiplier $\lambda(u)$ of $\mathcal{O}_u$. Note that $\lambda(u) = 0$ if and only if $u \in V_n$. We claim that the gradient of λ does not vanish at $\mathbf{g}$. In fact, note that each component of the immediate basin of $\mathcal{O}_{\mathbf{g}}$ is simply connected. Thus, we may apply quasiconformal surgery, as in the quadratic polynomial case (see [**3**, Théoréme 4]) to obtain a smooth family $u(\mu) \in U$ such that $\lambda(u(\mu)) = \mu$, defined for a parameter μ varying in a neighbourhood of $0 \in \mathbb{D}$ and such that $u(0) = \mathbf{g}$. Hence, the gradient of λ does not vanish at $\mathbf{g}$, which guarantees smoothness of V_n near $\mathbf{g}$.

REMARK 2. The original surgery described in [**3**, Théoréme 4] yields a continuous family $u(\mu)$. However, following the proof of [**14**, Theorem 5.8] the surgery may be upgraded to one producing a real analytic family $u(\mu)$.

It remains to check that V_n is smooth at elements where the second critical point is in the basin of the first periodic critical point. This again follows from Theorem 2.2, since such elements of V_n are contained in type II hyperbolic components and, after applying [**19**, Theorem 1], we conclude that V_n intersects a small 3-sphere around these points in an unknotted simple closed curve. Thus, V_n is also smooth at these points. □

REMARK 3. Other approaches to these results are possible, and indeed, have been announced by Buff, Epstein and Hubbard, based on the Epstein preprint on transversality [**7**]. Also, a very recent preprint by Favre and Gauthier [**8**] includes a detailed generalization of Douady's surgery for higher degree polynomials with simple critical points.

2.3. *A convenient subset $\mathcal{R}$ of $\mathcal{M}_2^{\mathrm{cm}}$ and projective curves*

For reasons that will be apparent later, for $n, m \geqslant 1$, we consider the curve $X_n \subset \mathcal{M}_2^{\mathrm{cm}}$ consisting on all maps such that the first critical point is periodic of period dividing n, and similarly the curve Y_m formed by maps with the second critical point periodic of period dividing

m. The curves X_n and Y_m are the union of periodic curves. That is,

$$X_n = \bigcup_{p\,|\,n} V_p,$$
$$Y_n = \bigcup_{p\,|\,n} W_p.$$

For our purpose, it is convenient to work with the set

$$\mathcal{R} = \mathcal{M}_2^{\mathrm{cm}} \setminus X_2,$$

which may be parametrized as follows. For $(c,d) \in \mathbb{C}\times\mathbb{C}^*$, consider the quadratic rational map

$$f_{c,d}(z) = 1 + \frac{c}{z} + \frac{d}{z^2}.$$

Then $(c,d) \mapsto [f_{c,d}, 0, -2d/c]$ parametrizes $\mathcal{R} = \mathcal{M}_2^{\mathrm{cm}} \setminus X_2$. That is, we identify $\mathcal{R}$ with $\mathbb{C}\times\mathbb{C}^*$.

For us, it is also convenient to regard $\mathcal{R}$ as a subset of $\mathbb{CP}^2$. By adding the line $d=0$ and a projective line at infinity to $\mathcal{R}$, we obtain $\mathbb{CP}^2$, with a preferred affine plane $\mathbb{C}^2$ parametrized by (c,d). Thus, we will regard $\mathcal{R}$ both as a subset of $\mathcal{M}_2^{\mathrm{cm}}$ and of $\mathbb{CP}^2$ according to convenience.

In order to be precise, we let $\mathcal{X}_n = X_n \cap \mathcal{R} \subset \mathbb{CP}^2$ and $\mathcal{Y}_m = Y_m \cap \mathcal{R} \subset \mathbb{CP}^2$. Similarly, let $\mathcal{V}_n = V_n \cap \mathcal{R}$ and $\mathcal{W}_n = W_n \cap \mathcal{R}$. Denote by $\bar{\mathcal{X}}_n$, $\bar{\mathcal{Y}}_m$, $\bar{\mathcal{V}}_n$ and, $\bar{\mathcal{W}}_n$ their closure in $\mathbb{CP}^2$. It follows that $\bar{\mathcal{X}}_n$, $\bar{\mathcal{Y}}_m$, $\bar{\mathcal{V}}_n$ and, $\bar{\mathcal{W}}_n$ are projective algebraic varieties.

3. *Type* IV *components*

The general strategy to compute $\eta_{\mathrm{IV}}(n,m)$ involves two main steps. The first one consists of computing the degrees of the curves $\bar{\mathcal{X}}_n$ and $\bar{\mathcal{Y}}_m$ introduced above, in Subsection 2.3. Then we apply Bezout's Theorem to obtain the total number of intersections of these curves. Since we are only interested in intersections which are relevant to our count (that is, those in $\mathcal{R}$), the second main step is to compute the number of intersections at 'infinity' and subtract them from the total number of intersections. Both steps involve translating dynamical information into algebraic information about intersections. While in the first step, the main ingredients come from results summarized in Section 2 about transversal intersections, in the second step, Stimson's results (Theorem 3.6) about how certain dynamical systems are organized close to infinity in parameter space, will play a key role.

Complementary to the main ideas described in the previous paragraph, the proof of Theorem 1.1 involves rewriting the formula as follows:

$$\begin{aligned}\eta_{\mathrm{IV}}(n,m) = {} & \frac{1}{36}(2^n - 3 - (-1)^n)(7\cdot 2^m + 3 - (-1)^m) - \frac{1}{2}\sum_{3\leqslant q\leqslant n}\phi(q)\nu_q(n)\nu_q(m)\\ & - \frac{1}{2}\left(\frac{2^n}{6} + \frac{(-1)^n}{3}\right)\left(\frac{2^m}{6} + \frac{(-1)^m}{3}\right) + \frac{1}{2}\left(\frac{2^n}{6} + \frac{(-1)^n}{3}\right)\left(\frac{2^m}{6} + \frac{(-1)^m}{3}\right)\\ & + \frac{(4+(-1)^n)}{6}2^m - \frac{(1+(-1)^n)(-1)^m}{6} + \frac{(1+(-1)^n)(1+(-1)^m)}{4}\\ & - \eta_{\mathrm{II}}(\gcd(n,m)).\end{aligned} \tag{3.1}$$

The first line is the size of $\mathcal{X}_n \cap \mathcal{Y}_m$, for $n \geqslant 3, m \geqslant 1$, that is, the total number of type II and IV components with an attracting cycle of period $\geqslant 3$ dividing n and an attracting cycle of period dividing m (maybe the same cycle). The first term is the product of the degrees of the curves $\overline{\mathcal{X}_n}$ and $\overline{\mathcal{Y}_m}$ (see Subsection 3.1) and the second term is their intersection number outside $\mathcal{R}$ (see Subsections 3.2 and 3.3).

The second line adds to zero. However, the first term in this line is $-(1/2)\phi(2)\nu_2(n)\nu_2(m)$ so that we may insert this number in the sum at the end of the first line of (3.1).

The third line corresponds to $\eta_{\mathrm{IV}}(1,m)$ if n is odd, to $\eta_{\mathrm{IV}}(2,m)$ if n is even and m is odd, and to $\eta_{\mathrm{IV}}(2,m)+\eta_{\mathrm{II}}(2)$ if n and m are even.

Thus, the sum up to the third line gives the number of hyperbolic components with one critical Fatou component of period dividing n and another of period dividing m. This count includes some type II components, namely, the ones with period dividing both n and m. The last line is the necessary correction to not count these $\eta_{\mathrm{II}}(\gcd(n,m))$ components and only consider type IV components.

3.1. *The degrees of the curves*

LEMMA 3.1. *The following statements hold.*

For all $n \geqslant 3$, the degree of $\bar{\mathcal{X}}_n$ is

$$\tfrac{1}{6}2^n - \tfrac{1}{2} - \tfrac{1}{6}(-1)^n.$$

For all $m \geqslant 1$, the degree of $\bar{\mathcal{Y}}_m$ is

$$\tfrac{7}{6}2^m + \tfrac{1}{2} - \tfrac{1}{6}(-1)^m.$$

None of the points $[1:0:0], [0:1:0], [0:0:1]$ belongs to $\bar{\mathcal{X}}_n$, for all $n \geqslant 3$.

Before proving the lemma, it is convenient to introduce notation, and state and prove some intermediate results contained in Lemma 3.2. These results will also be useful when counting type II components.

To compute the degree of the curves $\bar{\mathcal{X}}_n$, we recursively define, for $n \geqslant 3$, polynomials $P_n(c,d), Q_n(c,d) \in \mathbb{C}[c,d]$ as follows:

$$\begin{aligned} P_3(c,d) &= 1+c+d, & Q_3(c,d) &= 1,\\ P_{n+1} &= P_n^2 + cP_nQ_n + dQ_n^2, & Q_{n+1} &= P_n^2. \end{aligned}$$

With these definitions, we have that, for all $(c,d) \in \mathcal{R}$,

$$f_{c,d}^n(0) = \frac{P_n(c,d)}{Q_n(c,d)}.$$

Thus, $\mathcal{X}_n$ is the set of all $(c,d) \in \mathcal{R}$ such that $P_n(c,d) = 0$.

Similarly, to compute the degrees of the curves $\bar{\mathcal{Y}}_m$ we define, for $m \geqslant 1$,

$$\begin{aligned} R_1(c,d) &= 4d - c^2, & S_1(c,d) &= 4d,\\ R_{m+1} &= R_m^2 + cR_mS_m + dS_m^2, & S_{m+1} &= R_m^2. \end{aligned}$$

Note that

$$f_{c,d}^m(-2d/c) = \frac{R_m(c,d)}{S_m(c,d)}.$$

Thus, $\mathcal{Y}_m$ is the set of all $(c,d) \in \mathcal{R}$ such that $cR_m(c,d) + 2dS_m(c,d) = 0$.

LEMMA 3.2. *For all $n \geqslant 3$ and $m \geqslant 1$, the following statements hold.*

(i)

$$\begin{aligned} \deg(P_n) &= \tfrac{1}{6}2^n - \tfrac{1}{2} - \tfrac{1}{6}(-1)^n,\\ \deg(Q_n) &= \tfrac{1}{6}2^n - 1 + \tfrac{1}{3}(-1)^n. \end{aligned}$$

(ii) *For all even n, $\deg(P_n) = \deg(Q_n)$. For all odd n, $\deg(P_n) = \deg(Q_n) + 1$.*

(iii) *The constant term of P_n and of Q_n is 1.*

(iv) *Among the terms of* P_n (*respectively*, Q_n) *with maximal degree* $\delta = \deg(P_n)$ (*respectively*, $\delta = \deg(Q_n)$), *both the monomial* c^δ *and the monomial* d^δ *always appear multiplied by a positive integer coefficient.*

(v) *The minimum degree of the monomials of* R_m *and of* S_m *is* 2^{m-1} *and are uniquely realized by monomials in* d.

(vi) *Let* $G_m(c,d) = cR_m(c,d) + 2dS_m(c,d)$. *Then*

$$\deg(G_m) = \tfrac{7}{6}2^m + \tfrac{1}{2} - \tfrac{1}{6}(-1)^m.$$

Proof. Since (i) through (iv) hold for $n = 3$, we proceed by induction assuming that these assertions are true for n. From the definitions, $P_{n+1}(0,0) = P_n^2(0,0) = Q_{n+1}(0,0)$, thus (iii) holds for $n+1$, since $P_n(0,0) = 1$. When n is even, from (iii), (iv) and the formulae $P_{n+1} = P_n^2 + cP_nQ_n + dQ_n^2$, $Q_{n+1} = P_n^2$, it follows that (i), (ii) and (iv) hold for $n+1$ since $\deg(P_{n+1}) = 1 + \deg(P_n) + \deg(Q_n) = 1 + 2\deg(P_n) = 1 + \deg(Q_{n+1})$. Similarly, when n is odd, (i), (ii) and (iv) also hold for $n+1$ since $\deg(P_{n+1}) = 1 + \deg(P_n) + \deg(Q_n) = 2\deg(P_n) = \deg(Q_{n+1})$.

Assertion (v) clearly holds for $m = 1$. Assuming that (v) holds for m, from the definitions it follows that the monomials of minimum degree of R_{m+1} and of S_{m+1} coincide with the square of the monomial of minimum degree of R_m. Thus, (v) holds for $m+1$ and the assertion holds, by induction.

To prove (vi), we first establish by induction the assertions that $\deg(R_m) \geqslant \deg(S_m)$, and for all $m \geqslant 2$, among the terms of R_m (respectively, S_m) with maximal degree $\delta = \deg(R_m)$ (respectively, $\delta = \deg(S_m)$) the monomial c^δ appears multiplied by a positive integer coefficient. In fact, for $m = 2$, directly from the definitions it follows that c^4 is a term of maximal degree of R_2 and S_2. Assuming the assertions are true for m, it follows that $S_{m+1} = R_m^2$ has degree $2\deg(R_m)$ and a term $c^{\deg(S_{m+1})}$ multiplied by a positive integer. Moreover, $\deg(R_{m+1}) = \max\{2\deg(R_m), 1 + \deg(R_m) + \deg(S_m)\}$ and in both cases R_{m+1} has a term of maximal degree of the form c^δ multiplied by a positive integer coefficient. We also conclude that $\deg(R_{m+1}) \geqslant \deg(S_{m+1})$.

From the previous paragraph, it is not difficult to conclude by induction that for all $m \geqslant 1$ odd, $\deg(R_{m+1}) = \deg(S_{m+1}) = 2\deg(R_m)$ and, for all $m \geqslant 1$ even, $\deg(R_{m+1}) = \deg(S_{m+1}) + 1 = 2\deg(R_m) + 1$. Also, $\deg(G_m) = \deg(R_m) + 1$, since for $\delta = \deg(R_m)$ the monomial c^δ appears multiplied by a non-zero coefficient as a term of R_m. Now an induction readily checks that $\deg(G_m)$ is given by the formula of assertion (vi). □

Proof of Lemma 3.1. Denote by $P_n^h(c,d,e)$ (respectively, $G_m^h(c,d,e)$) the homogeneous version of $P_n(c,d)$ (respectively, $G_m(c,d)$). Then $\bar{\mathcal{X}}_n$ (respectively, $\bar{\mathcal{Y}}_m$) is the projective curve where $P_n^h(c,d,e)$ (respectively, $G_m^h(c,d,e)$) vanishes.

From Lemma 3.2(iii), we conclude that $[0:0:1] \notin \bar{\mathcal{X}}_n$ since $P_n^h(0,0,1) = 1$. Lemma 3.2(iv) implies that $P_n^h(1,0,0) \neq 0 \neq P_n^h(0,1,0)$. Thus, none of the points $[1:0:0], [0:1:0], [0:0:1]$ belongs to $\bar{\mathcal{X}}_n$.

Since $\deg(P_n^h) = \deg(P_n)$ and $\deg(G_m^h) = \deg(G_m)$, in view of Lemma 3.2, it is sufficient to establish that P_n^h generates the ideal of $\bar{\mathcal{X}}_n$ and, similarly that G_m^h generates the ideal of $\bar{\mathcal{Y}}_m$.

The birational automorphism of $\mathbb{CP}^2$ induced by interchanging the role of the critical points will allow us to only check the above for P_n^h. More precisely, for all $(c,d) \in \mathcal{R} \setminus \mathcal{Y}_2$, let $M_{c,d}$ be the Möbius transformation such that:

$$M_{c,d}(f_{c,d}^j(-2d/c)) = \begin{cases} 0 & \text{if } j = 0, \\ \infty & \text{if } j = 1, \\ 1 & \text{if } j = 2. \end{cases}$$

Then there exists $(c', d') \in \mathcal{R}$ such that

$$f_{c',d'} = M_{c,d}^{-1} \circ f_{c,d} \circ M_{c,d}.$$

It follows that c' and d' are rational functions of (c, d). Moreover, $M_{c',d'} = M_{c,d}^{-1}$. Therefore, the map $(c, d) \mapsto (c', d')$ is a birational map $\varphi : \mathbb{CP}^2 \dashrightarrow \mathbb{CP}^2$. Furthermore,

$$\frac{G_m^h}{G} = \varphi^*(P_m^h) = P_m^h \circ \varphi,$$

where $G = G_1^h$ if m is odd, and $G = G_2^h$ if m is even. After checking that G_1^h and G_2^h are free of perfect square factors in $\mathbb{C}[c, d, e]$, it follows that G_m^h generates the ideal of $\bar{\mathcal{Y}}_m$ if and only if P_m^h generates the ideal of $\bar{\mathcal{X}}_m$ for all $m \geqslant 3$.

To show that P_n^h is square factor free, it is sufficient to show that the degree of $\bar{\mathcal{X}}_n$ coincides with that of P_n. For this, we count the intersections of $\bar{\mathcal{X}}_n$ with the degree 3 curve $\bar{\mathcal{Y}}_1$. The curve $\bar{\mathcal{Y}}_1$ is defined by the equation $c^3 - 4dce - 8d^2e = 0$. Thus, $\bar{\mathcal{Y}}_1$ intersects the line $d = 0$ at $[0 : 0 : 1]$ and the line $e = 0$ at $[0 : 1 : 0]$. Therefore, $\bar{\mathcal{Y}}_1 \cap \bar{\mathcal{X}}_n$ is contained in $\mathcal{R}$. Hence, $\bar{\mathcal{Y}}_1 \cap \bar{\mathcal{X}}_n$ consists of points of transverse intersection between $\mathcal{Y}_1$ and $\mathcal{X}_n$. It follows that $3\deg(\bar{\mathcal{X}}_n)$ coincides with the cardinality of $\mathcal{Y}_1 \cap \mathcal{X}_n$.

Recall that $\mathcal{Y}_1$ consists of all elements of critically marked moduli space outside of X_2 with the second critical point fixed. Conjugacy classes in $\mathcal{Y}_1$ are exactly the conjugacy classes of maps of the form $z \mapsto z^2 + v$ with $v \in \mathbb{C}$ and $v \neq 0, -1$. The correspondence follows by normalizing the fixed critical point at infinity and requiring the resulting polynomial to be monic. Therefore, the cardinality of $\mathcal{Y}_1 \cap \mathcal{X}_n = 2^{n-1} - 1 - \delta$, where $\delta = 1$ if n is odd, and $\delta = 2$ if n is even. That is,

$$3\deg(\bar{\mathcal{X}}_n) = 2^{n-1} - 1 - \frac{1 + (-1)^n}{2} = 3\deg(P_n^h),$$

and the lemma follows. □

To prove Theorem 1.1, we need to compute the cardinality of the intersection

$$\mathcal{X}_n \cap \mathcal{Y}_m.$$

But by Bezout's theorem, the number of intersections (with multiplicities) of $\bar{\mathcal{X}}_n$ and $\bar{\mathcal{Y}}_m$ in $\mathbb{CP}^2$ is simply the product of their degrees. So now we need to compute the number of intersections, with multiplicities, in $\mathbb{CP}^2 \setminus \mathcal{R}$. Counting the intersections in $\mathbb{CP}^2 \setminus \mathcal{R}$ is divided into two parts. First we show that there are no intersections in the line $[c : d : 0]$, and then we count the intersections in the line $[c : 0 : 1]$.

3.2. *No intersections at* ∞

In order to understand the behaviour at infinity of the curves $\mathcal{X}_n$ and $\mathcal{Y}_m$, we consider limit maps for f, say in $\mathcal{X}_n$, approaching infinity, or more generally limits of f^q for a given q. Quadratic rational maps with a parabolic fixed point were obtained as such limits by Stimson in his Thesis. This phenomenon was generalized by Epstein [6] to study compactness properties of certain hyperbolic components in quadratic moduli space. Also compare with subsequent developments by DeMarco [2] and Kiwi [10]. However, since we will only consider limits along parameters with a periodic critical point, our approach is very close to the original one by Stimson.

LEMMA 3.3. *For all* $n \geqslant 3$ *and* $m \geqslant 1$,

$$\bar{\mathcal{X}}_n \cap \bar{\mathcal{Y}}_m \cap \{x \in \mathbb{CP}^2 \mid x = [c : d : 0]\} = \emptyset.$$

Proof. Since $[1:0:0]$ and $[0:1:0] \notin \bar{\mathcal{X}}_n$, it is sufficient to consider $x = [1:s:0]$ with $s \neq 0$ and show that $x \notin \bar{\mathcal{X}}_n \cap \bar{\mathcal{Y}}_m$. First note that if $f_{c,d} \in \mathcal{R}$, then

$$f_{c,d}^2(z) = 1 + \frac{z^2}{z^2/c + z + d/c} + \frac{d}{c^2}\frac{z^4}{(z^2/c + z + d/c)^2}.$$

Thus, as $[c:d:1]$ converges to x,

$$f_{c,d}^2(z) \longrightarrow 1 + \frac{z^2}{z+s} = g_s(z),$$

uniformly in compact subsets of $\mathbb{C} \setminus \{-s\}$.

Note that g_s has a parabolic fixed point at ∞, and that $g_s(-s) = \infty$. Moreover, since one of the forward orbits of the two critical points $0, -2s$ of g_s must be infinite and converge to ∞, one of the two critical points of g_s, call it ω, has an infinite forward orbit entirely contained in $\mathbb{C} \setminus \{-s\}$.

If $\omega = 0$ and $[c:d:1]$ is sufficiently close to x, then the first $n+1$ iterates of $\omega_1 = 0$ are all distinct and we deduce that $x \notin \bar{\mathcal{X}}_n$. Similarly, if $\omega = -2s$, then $x \notin \bar{\mathcal{Y}}_m$. So $x \notin \bar{\mathcal{X}}_n \cap \bar{\mathcal{Y}}_m$. □

3.3. *Counting intersections at $d = 0$*

Now our aim is to count the intersections at $d = 0$. We start by establishing where the intersections occur.

LEMMA 3.4. *For all $n \geqslant 3$, if $[c:0:1] \in \bar{\mathcal{X}}_n$, then $c^{-1} = -4\cos^2 \pi p/q$ for some $1 \leqslant p < q \leqslant n$ with $\gcd(p,q) = 1$ and $q \neq 2$.*

Proof. For all $(c,d) \in \mathcal{R}$, we have that the cross ratio

$$[\omega_1, \omega_2, f_{c,d}(\omega_1), f_{c,d}(\omega_2)] = \frac{c^3}{8d^2} - \frac{c}{2d},$$

where $\omega_1 = 0, \omega_2 = -2d/c$ and

$$[0, z_2, \infty, z_4] = \frac{z_4}{z_2}.$$

It follows that, given $c_0 \neq 0$ and a sequence $(c_k, d_k) \in \mathcal{X}_n$ which converges to $(c_0, 0)$, the conjugacy class of f_{c_k,d_k} diverges to infinity in moduli space. According to [**6**, Proposition 1] (and also [**12**, Lemma 4.1], but the case $q = 1$ is not excluded there), f_{c_k,d_k} has three fixed points, one with multiplier diverging to ∞ and the other two multipliers converge to reciprocal roots of unity of order q where $2 \leqslant q \leqslant n$. Uniformly on compact subsets of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the maps f_{c_k,d_k} converge to

$$M_{c_0}(z) = 1 + \frac{c_0}{z}.$$

Since $\mathbb{C}^*$ contains the fixed points of M_{c_0}, the multipliers of the fixed points of M_{c_0} are of the form $\exp(\pm 2\pi i p/q)$, where $\gcd(p,q) = 1$. This means that $r\,e^{\pm \pi i (p/q)}$ are the zeros of the characteristic polynomial $\lambda^2 - \lambda - c_0$ of the matrix associated to M_{c_0}, for some $r \neq 0$, in fact, for $r^2 = -c_0$. It follows that

$$c_0^{-1} = -4\cos^2(\pi p/q).$$

In particular, $q \neq 2$ and the lemma follows. □

The rest of this section is devoted to proving the formula below that computes the intersection numbers at the relevant points of $d = 0$.

PROPOSITION 3.5. *Consider $n \geqslant 3$ and $m \geqslant n$. Let $1 \leqslant p < q \leqslant n$ with $\gcd(p,q) = 1$ and $q \neq 2$. If $c_{p,q}^{-1} = -4\cos^2(\pi p/q)$, then*

$$\bar{\mathcal{X}}_n \bullet_{\mathbf{c}_{p,q}} \bar{\mathcal{Y}}_m = \nu_q(n)\nu_q(m),$$

where $\mathbf{c}_{p,q} = [c_{p,q} : 0 : 1]$.

We have seen that the points at infinity on X_n and Y_m are described by the multipliers of their fixed points. So the multiplier of a fixed point is a natural parameter to use near infinity in X_n (or Y_n). To prove the above proposition, it is therefore convenient to work with the *totally marked moduli space* $\mathcal{M}_2^{\mathrm{tm}}$, which is a smooth complex manifold of dimension 2 (see [12, Lemma 6.6]), namely, the space of quadratic rational maps with marked critical points and marked fixed points. The elements of $\mathcal{M}_2^{\mathrm{tm}}$ are conjugacy classes of sextuples $(f, \omega_1, \omega_2, x_1, x_2, x_3)$, where f is a quadratic rational map, ω_j are critical points of f and x_1, x_2, x_3 is a complete list of the fixed points of f (with repetitions when f has a multiple fixed point). As before, two such sextuples are conjugate if there is a Möbius transformation conjugating the rational maps and respecting markings. The conjugacy class of $(f, \omega_1, \omega_2, x_1, x_2, x_3)$ will be denoted by $[f, \omega_1, \omega_2, x_1, x_2, x_3]$.

In $\mathcal{M}_2^{\mathrm{tm}}$, we let A_n (respectively, B_m) be the curves where the first (respectively, the second) critical point is periodic of period n (respectively, m). Following [17, 7.4], to study the ends of A_n and B_m it is convenient to consider the family of quadratic rational maps defined for $(\zeta, \rho) \in \mathbb{C}^* \times \mathbb{C} \setminus \{-1, -2\}$:

$$\begin{aligned} h_{\zeta,\rho}(z) &= \zeta z \left(1 - \frac{2+\rho}{2(1+\rho)} z\right) \left(1 - \frac{2}{2+\rho} z\right)^{-1} \\ &= \zeta z \left(1 - \frac{\rho^2 z}{4(1+\rho)(1+\frac{1}{2}\rho - z)}\right). \end{aligned}$$

A similar parametrization was used by Stimson in his thesis [21]. The great advantage of this type of parametrization is that $h_{\zeta,0}^k(z)$ is just $\zeta^k z$, and estimates on $h_{\zeta,\rho}^k(z) - \zeta^k z$ for small ρ, and for a suitable set of z, are easily obtained.

The critical points of $h_{\zeta,\rho}$ are $\omega_1 = 1$ and $\omega_2 = 1 + \rho$. There are two distinguished fixed points, one at $x_1 = 0$ and the other at $x_2 = \infty$. The third fixed point x_3 is given by the formula

$$x_3 = \frac{(1+\frac{1}{2}\rho)(1+\rho)(\zeta - 1)}{\zeta\rho^2 + (1+\rho)(\zeta - 1)}.$$

Therefore, we may parametrize a subset $\mathcal{S}$ of $\mathcal{M}_2^{\mathrm{tm}}$ by $\mathbb{C}^* \times \mathbb{C} \setminus \{0, -1, -2\}$, using $[h_{\zeta,\rho}, 1, 1+\rho, 0, \infty, x_3]$.

A partial compactification of $\mathcal{S}$ is achieved by adding the line $\mathbb{V}(\rho) = \{\rho = 0\}$. That is, we identify $\mathcal{S}$ with a subset of $\mathbb{CP}^2$, by identifying (ζ, ρ) with $[\zeta : \rho : 1]$. Now let $\mathcal{A}_n = A_n \cap \mathcal{S}$ and $\mathcal{B}_m = B_m \cap \mathcal{S}$. Consider the closures $\bar{\mathcal{A}}_n$ and $\bar{\mathcal{B}}_m$ in $\mathbb{CP}^2$. Here, we say that a *branch* of a curve $\mathcal{C}$ near a point z is a representative of an irreducible germ of $\mathcal{C}$ at z (See [9, Sections 6.12 and 6.13]). The following was proved in Stimson's Thesis.

THEOREM 3.6 (Stimson [21]). *The following statements hold.*

(i) *The intersection of $\bar{\mathcal{A}}_n$ (respectively, $\bar{\mathcal{B}}_n$) with the line $\mathbb{V}(\rho)$ is contained in*

$$\{[1:0:1]\} \cup \{[\exp(2\pi i p/q) : 0 : 1] \mid 1 \leqslant p < q \leqslant n, \gcd(p,q) = 1\}.$$

(ii) *For any* $\kappa \in \bar{\mathcal{A}}_n \cap \mathbb{V}(\rho)$ (*respectively*, $\kappa \in \bar{\mathcal{B}}_n \cap \mathbb{V}(\rho)$), *let* E (*respectively*, F) *be a branch of the curve* $\bar{\mathcal{A}}_n$ (*respectively*, $\bar{\mathcal{B}}_n$) *at* $[\kappa : 0 : 1]$. *Then,*

$$\frac{\zeta - \kappa}{\kappa\rho} \longrightarrow a_1 \in \{a \in \mathbb{C} \mid \Re a > 0\},$$

as $E \ni (\zeta, \rho) \to (\kappa, 0)$ *and,*

$$\frac{\zeta - \kappa}{\kappa\rho} \longrightarrow a_2 \in \{a \in \mathbb{C} \mid \Re a < 0\},$$

as $F \ni (\zeta, \rho) \to (\kappa, 0)$.

(iii) *Let* $\kappa = \exp(2\pi ip/q)$ *where* $1 \leqslant p < q \leqslant n, \gcd(p, q) = 1$. *The intersection number of* $\bar{\mathcal{A}}_n$ (*respectively*, $\bar{\mathcal{B}}_n$) *with the line* $\mathbb{V}(\rho)$ *at* $[\kappa : 0 : 1]$ *is equal to the number of hyperbolic components of period* n *in the* p/q*-limb of the Mandelbrot set.*

Proof. From the formula, we see that

$$h_{\zeta,\rho}(1) = \zeta\left(1 - \frac{\rho}{2(1+\rho)}\right). \tag{3.2}$$

For ρ near 0, and for $z \neq 1 + \frac{1}{2}\rho + O(\rho^2)$,

$$h_{\zeta,\rho}(z) = \zeta z(1 + o(1)). \tag{3.3}$$

Therefore, for ρ near 0, unless ζ^q is close to 1 for some $1 \leqslant q \leqslant k$, we have $h^\ell_{\zeta,\rho}(1) = \zeta^\ell(1 + o(1))$, which is bounded from 1 for $0 < \ell \leqslant k$. Putting $k = n$, we see that we can only have $(\zeta, \rho) \in \mathcal{A}_n$ for ρ close to 0 if ζ^q is close to 1 for some $1 \leqslant q \leqslant n$. So (i) follows.

For part (ii), let q be the smallest positive integer such that $\kappa^q = 1$. Write:

$$h_{\zeta,\rho}(1 + z\rho) = \zeta(1 + z\rho)\left(1 - \frac{\rho}{4(1+\rho)(\frac{1}{2} - z)}\right). \tag{3.4}$$

It follows from (3.2) and (3.3), and using a local Puiseux series in ρ for $\zeta - \kappa$, that $\zeta - \kappa = O(\rho)$ in a neighbourhood of $[\kappa : 0 : 1]$ in $\bar{\mathcal{A}}_n$, and hence, if $\rho \to 0$, then $(\zeta - \kappa)/(\kappa\rho) \to a$ for some $a \in \mathbb{C}$. So, for sufficiently small ρ, given z in a compact subset of $\mathbb{C} \setminus \{\frac{1}{2}\}$, by the definition of $h_{\zeta,\rho}$ and (3.3) and (3.4), applying $h_{\zeta,\rho}$ to $h^{j-1}_{\zeta,\rho}(1 + z\rho)$ for $1 \leqslant j \leqslant q$,

$$\begin{aligned} h^q_{\zeta,\rho}(1 + z\rho) &= \zeta^q(1 + z\rho)\left(1 + \frac{\rho}{4(z - \frac{1}{2})} + o(\rho)\right) \\ &= 1 + \rho\left(qa + z + \frac{1}{4(z - \frac{1}{2})}\right) + o(\rho). \end{aligned}$$

So, uniformly in compact subsets of $\mathbb{C} \setminus \{\frac{1}{2}\}$,

$$\frac{h^q(1 + z\rho) - 1}{\rho} \longrightarrow qa + z + \frac{1}{4(z - \frac{1}{2})} = g_a(z).$$

The map g_a has a parabolic fixed point at ∞ and critical points at $z = 0$ and $z = 1$. If $\Re a \leqslant 0$, then it is not difficult (by a direct calculation) to check that $\Re g^k_a(0)$ is a strictly decreasing sequence and $g^k_a(0)$ diverges to ∞. Similarly, if $\Re a \geqslant 0$, then $\Re g^k_a(1)$ is a strictly increasing sequence and $g^k_a(1)$ diverges to ∞. Part (ii) of the theorem follows for $\mathcal{A}_n$. The result for $\mathcal{B}_n$ follows immediately because, from the conjugacy by $z \mapsto (1 + \rho)z^{-1}$, we see that $(\zeta, \rho) \in \mathcal{A}_n$ if and only if $(\zeta_1, \rho) \in \mathcal{B}_n$, where

$$\zeta_1 = \zeta^{-1}\frac{4(1+\rho)}{(2+\rho)^2} = \zeta^{-1}(1 + O(\rho^2)).$$

We now consider (iii), so let $\kappa = 1$ or $\kappa = \exp(2\pi ip/q)$ where $q > 1$ and $1 \leqslant p < q$, and p and q are relatively prime. The multipliers at the fixed points 0 and ∞ are ζ and ζ_1, respectively.

By (ii), either of these multipliers is a local coordinate on $\mathcal{A}_n$ in a deleted neighbourhood of $\mathbb{V}(\rho) \cap \bar{\mathcal{A}}_n$. So, considering the multiplier ζ_1, the intersection number of $\bar{\mathcal{A}}_n$ with $\mathbb{V}(\rho)$ at $[\kappa : 0 : 1]$ is the number of points $[\zeta : \rho : 1] \in \mathcal{A}_n$, for any fixed ζ_1 sufficiently near κ^{-1}, but not equal to it.

If $\kappa = 1$, then this number is at least 2^{n-1}. We can see this, because after blowing up the point $[\zeta = 1 : \rho = 0 : 1]$ we may observe that in the coordinates given by $(\rho, a = (\zeta - 1)/\rho)$ the intersection of the proper transform of $\bar{\mathcal{A}}_n$ with the line $\rho = 0$ is the set of parameters a such that $g_a^n(0) = 0$, which has degree 2^{n-1} in a.

If $\kappa = \exp(2\pi ip/q)$, then the number is at least the number of hyperbolic components in the $-p/q$ limb of period n. In fact, consider a parameter v in the $-p/q$ limb of the Mandelbrot set such that the critical point of $Q_v(z) = z^2 + v$ is periodic of period dividing n. Then, for any $0 < r < 1$ and $\zeta_1(r) = r \exp(2\pi ip/q)$ there exist $\rho(r)$ and $\zeta(r)$ such that the map $h_{\rho(r),\zeta(r)} \in \mathcal{A}_n$ has an attracting fixed point of multiplier $\zeta_1(r)$ at ∞ and in the complement of the basin of ∞ is hybrid equivalent to Q_v. According to Petersen [**15**, Corollary 2], as $r \nearrow 1$, we have that $\zeta_1(r)\zeta(r) \to 1$, thus $\rho(r) \to 0$. Hence, the intersection number of $\bar{\mathcal{A}}_n$ with $\mathbb{V}(\rho)$ at $[\kappa = \exp(2\pi ip/q) : 0 : 1]$ is at least the number of hyperbolic components of period n in the $-p/q$ limb of the Mandelbrot set, which via $z \mapsto \bar{z}$ is easily seen to coincide with the corresponding number in the p/q limb.

REMARK 4. There is no contradiction with known results in the case $\kappa = 1$ which discard intersections in the corresponding point at infinity in moduli space. In fact, let U be a neighbourhood of $[1 : 0 : 1]$ and consider the forgetful map from $U \setminus \mathbb{V}(\rho)$ to critically marked moduli space. This forgetful map is not proper. As is suggested above, after blowing up the point $[1 : 0 : 1]$, the forgetful map extends to one where the exceptional divisor maps to the locus of quadratic rational maps with one parabolic fixed point.

To prove equality, we need to compute the degree in ζ of the curve $\mathcal{A}_n$ in $\mathbb{CP}^2$. We can then obtain the result for $\mathcal{B}_n$ by the usual birational equivalence. In fact, it suffices to compute the degree in ζ of $H_{n,1}^h - H_{n,2}^h$ where

$$\frac{H_{n,1}}{H_{n,2}} = h_{\zeta,\rho}^n(1),$$

and $H_{n,j}^h$ is the homogenized version of $H_{n,j}$. The zero set of $H_{n,1}^h - H_{n,2}^h$ is the union of sets $\mathcal{A}_d$ for d dividing n. We claim that the degree is $2^n - 1$. Since the total number of hyperbolic components of periods dividing n is 2^{n-1} by Douady and Hubbard [**4**], and all but one of these are in limbs of the Mandelbrot set, the exception being the main cardioid, this suffices to show that for each d dividing n, $\mathcal{A}_d$ occurs with multiplicity 1 in the zero set of $H_{n,1}^h - H_{n,2}^h$, and that we have the equality claimed in (iii).

We write $\deg_\zeta(H)$ for the degree of a polynomial H in ζ. It is easily checked that $H_{k,1}$ has ζ as a factor for all $k \geqslant 1$, and hence has no constant term, and that $H_{k,2}$ has a constant term for all $k \geqslant 1$. The recursive equations are

$$\begin{aligned} H_{k+1,1} &= \zeta H_{k,1}(2(1+\rho)(2+\rho)H_{k,2} - (2+\rho)^2 H_{k,1}), \\ H_{k+1,2} &= 2(1+\rho)H_{k,2}((2+\rho)H_{k,2} - 2H_{k,1}). \end{aligned}$$

It follows that $H_{k,1}$ has ζ^{2^k-1} as a factor, has a term $(-4)^{2^{k-1}}\zeta^{2^k-1}$ and has no constant term, and that $H_{k,2}$ has constant term $2 \cdot 4^{k-1}$. It also follows that $\deg_\zeta(H_{k,2}) \leqslant 2^k - k - 1$. Thus, the degree of $H_{n,1}^h - H_{n,2}^h$ is $2^n - 1$ and the theorem follows. □

REMARK 5. Stimson's result is in fact far more precise than the theorem we have stated and proved above. Via the Thurston class of obstructed postcritically finite branched coverings, he gives a complete characterization of hyperbolic components in the $-p/q$ limb that lead to the same irreducible germ of $\bar{\mathcal{A}}_n$ at a point $[\exp(2\pi ip/q) : 0 : 1]$. However, his proof that there is only one such hyperbolic component per irreducible germ is not correct.

LEMMA 3.7. *Let* $1 \leqslant p < q \leqslant \min\{n, m\}$ *with* $\gcd(p, q) = 1$ *and* $q \neq 2$. *Let* $c_{p,q}^{-1} = -4\cos^2(\pi p/q)$ *and* $\kappa_{p,q} = \exp(2\pi ip/q)$. *Write* $\mathbf{c}_{p,q} = [c_{p,q} : 0 : 1]$ *and* $\mathbf{k}_{p,q} = [\kappa_{p,q} : 0 : 1]$. *Then*

$$\bar{\mathcal{V}}_n \bullet_{\mathbf{c}_{p,q}} \bar{\mathcal{W}}_m = \bar{\mathcal{A}}_n \bullet_{\mathbf{k}_{p,q}} \bar{\mathcal{B}}_m.$$

Proof. It is sufficient to show that there is a biholomorphic map between a neighbourhood U of $\mathbf{c}_{p,q}$ in the (c, d)-plane, and a neighbourhood U' of $\mathbf{k}_{p,q}$ in the (ζ, ρ)-plane, that maps $\mathcal{V}_n$ to $\mathcal{A}_n$ and $\mathcal{W}_m$ to $\mathcal{B}_m$. If U is small, then for any $(c, d) \in U$, the multipliers of two of the fixed points of $f_{c,d}$ are close to $\exp(\pm 2\pi ip/q)$, and if $d \neq 0$, then the multiplier of the third fixed point is large. By taking U sufficiently small, we can ensure that the multipliers of any map in U are all different from the multipliers of the fixed points of any other map in U, and that none of the multipliers is 1. Therefore, we know that none of the maps $f_{c,d}$, for $(c, d) \in U$, is Möbius conjugate to any other, and the fixed points vary holomorphically in U. We write $x_1(c, d)$ and $x_2(c, d)$ for the fixed points which have multipliers close to $\exp(2\pi ip/q)$ and $\exp(-2\pi ip/q)$, respectively. Since $x_1(c_{p,q}, 0)$ and $x_2(c_{p,q}, 0)$ are the solutions of $z^2 - z + c_{p,q} = 0$, the points $x_1(c, d)$, $x_2(c, d)$ and 0 are all distinct for all $(c, d) \in U$, if U is sufficiently small. Let $\tau_{c,d}$ denote the unique Möbius transformation which maps $x_1(c, d)$, $x_2(c, d)$ and 0 to 0, ∞ and 1. Then $\tau_{c,d} \circ f_{c,d} \circ \tau_{c,d}^{-1}$ is a quadratic rational map with fixed points 0 and ∞ and critical point at 1. It is therefore of the form $h_{\zeta,\rho}$ with $\zeta = \zeta(c, d)$ and $\rho = \rho(c, d)$. In fact, we have

$$\zeta(c, d) = f'_{c,d}(x_1(c, d))$$

and

$$\rho(c, d) = \tau_{c,d}(-2d/c) - 1.$$

The map (ζ, ρ) is clearly a holomorphic injection on U which maps $\mathbf{c}_{p,q}$ to $\mathbf{k}_{p,q}$, because all the maps in U are distinct up to Möbius conjugacy. Hence, it must be a holomorphic bijection onto a neighbourhood U' of $\mathbf{k}_{p,q}$. In any case, the inverse map is easily obtained by using conjugation by a Möbius transformation which maps 1, $h_{\zeta,\rho}(1)$ and $h_{\zeta,\rho}^2(1)$ to 0, ∞ and 1. Note that $q = 2$ has to be omitted, because we need the points 1, $h_{\zeta,\rho}(1)$ and $h_{\zeta,\rho}^2(1)$ to be bounded apart. □

The proof of Proposition 3.5 follows, since (ii) of Theorem 3.6 implies that at an intersection point of $\bar{\mathcal{A}}_n$ and $\bar{\mathcal{B}}_m$ in $\rho = 0$, the tangents to $\bar{\mathcal{A}}_n$ and the tangents to $\bar{\mathcal{B}}_m$ are distinct.

3.4. *Low periods and the proof of Theorem* 1.1

The last ingredients needed to prove Theorem 1.1 are contained in the lemma below.

LEMMA 3.8.

$$\begin{aligned} \eta_{\mathrm{IV}}(1, m) &= 2^{m-1}, \\ \eta_{\mathrm{IV}}(1, m, p/q) &= \nu_q(m), \\ \eta_{\mathrm{II}}(2) &= 1, \\ \eta_{\mathrm{IV}}(2, m) - \eta_{\mathrm{IV}}(1, m) &= \frac{1}{3}2^m - \frac{(-1)^m}{3}. \end{aligned}$$

Here, $\eta_{\mathrm{IV}}(1, m, p/q)$ denotes the number of type IV *hyperbolic components in the p/q-limb of the Mandelbrot set with an attractive cycle of period dividing m.*

Proof. As already mentioned, the formula for $\eta_{\mathrm{IV}}(1, m)$ is a consequence of the Douady–Hubbard classification [4] of hyperbolic components in the Mandelbrot set for the parameter family of quadratic polynomials $\{z^2 + v \mid v \in \mathbb{C}\}$, which is naturally identified with $\mathbb{C}$. In fact, the complement of the Mandelbrot set in $\mathbb{C}$ is the intersection with $\mathbb{C}$ of the type I hyperbolic component of rational maps which is mentioned in Section 1. The type I hyperbolic component contains no critically finite maps. In fact, it coincides with the set of v for which the forward orbit of 0 under $z \mapsto z^2 + v$ diverges to ∞. The main cardioid of the Mandelbrot set (as it is known) is the hyperbolic component of $z \mapsto z^2$. The set of all the other hyperbolic components in the Mandelbrot set, for which the attractive cycle is of period dividing m, is in two-to-one correspondence with the set of odd-denominator rationals

$$\left\{ \frac{k}{2^m - 1} : 0 < k < 2^m - 1 \right\}.$$

Therefore, we obtain

$$\eta_{\mathrm{IV}}(1, m) = 1 + \frac{2^m - 2}{2} = 2^{m-1}.$$

Furthermore, the set of all hyperbolic components in the p/q limb of the Mandelbrot set, for which the attractive cycle is of period dividing m, is in two-to-one correspondence with the set of odd-denominator rationals $k/(2^m - 1)$ in a closed interval $[1/(2^q - 1), 2/(2^q - 1)]$, since the arguments of the root are of the form $r/(2^q - 1)$ and $(r + 1)/(2^q - 1)$ for some r coprime to q. This is the number $\nu_q(m)$ (see [4], [13] and [20]).

The fact that $\eta_{\mathrm{II}}(2) = 1$ follows immediately from fixing critical points of a quadratic rational map to be 0 and ∞. If these are in a period two cycle, then the map must be of the form $z \mapsto \lambda z^{-2}$ for some $\lambda \neq 0$, and any such maps are conjugate to $z \mapsto z^{-2}$, by a conjugacy of the form $z \mapsto \mu z$.

We now give a proof of the formula for $\eta_{\mathrm{IV}}(2, m) - \eta_{\mathrm{IV}}(1, m)$ which is close to the methods of Subsection 3.3. Since a great deal is known about V_2, other proofs are possible. For example, one can use the results of a recent paper of Aspenberg–Yampolsky, [1], some of which occur also in work of Timorin [23], and both of which consolidate the structure that has been known in outline for some time, as evidenced, for example, by the thesis of Luo [11] in the 1990s. One can also make use of the theory of matings, which involves using Thurston's criterion [5] and Tan Lei's theorem to decide which matings are realizable [22].

Instead of using $\mathcal{R}$ as in the case of V_n for $n \geqslant 3$, we use a parametrization which extends that used in [1, 23]:

$$g_{a,b}(z) = \frac{a}{z^2 + 2z + b}.$$

If $a \neq 0$, then critical points of $g_{a,b}$ are ∞ and -1, and $g_{a,b}(\infty) = 0$. Every Möbius conjugacy class in $\mathcal{M}_2^{\mathrm{cm}} \setminus X_1$, apart from that of $z \mapsto z^{-2}$, is represented by exactly one map $g_{a,b}$ for $a \neq 0$, and therefore $\mathcal{M}_2^{\mathrm{cm}} \setminus (X_1 \cup \{z^{-2}\})$ can be identified with

$$\{[a : b : 1] \mid a, b \in \mathbb{C}, a \neq 0\} \subset \mathbb{CP}^2.$$

We write $\mathcal{Z}_m$ for the set of $[a : b : 1]$ such that $g_{a,b}$ represents an element of Y_m. Then

$$\eta_{\mathrm{IV}}(2, m) - \eta_{\mathrm{IV}}(1, m) = \#(\mathcal{Z}_m \cap \{[a : 0 : 1] \mid a \neq 0\}).$$

As in the earlier cases, all intersections are transverse. We write $\bar{\mathcal{Z}}_m$ for the closure of $\mathcal{Z}_m$ in $\mathbb{CP}^2$. The plane $b = 0$ in $\mathbb{CP}^2$ is the set of all $[a : 0 : 1]$ together with $[1 : 0 : 0]$. It will be denoted by $\mathbb{V}(b)$, the vanishing set of the linear polynomial b. Provided that all zeros are simple, by

Bezout's Theorem,

$$\#(\bar{\mathcal{Z}}_m \cap \mathbb{V}(b)) = \deg(\bar{\mathcal{Z}}_m).$$

Therefore,

$$\eta_{\mathrm{IV}}(2,m) - \eta_{\mathrm{IV}}(1,m) = \deg(\bar{\mathcal{Z}}_m) - \bar{\mathcal{Z}}_m \bullet_{[0:0:1]} \mathbb{V}(b) - \bar{\mathcal{Z}}_m \bullet_{[1:0:0]} \mathbb{V}(b),$$

where $\bar{\mathcal{Z}}_m \bullet_{[0:0:1]} \mathbb{V}(b)$ denotes the intersection number of $\bar{\mathcal{Z}}_m$ with $\mathbb{V}(b)$ at $[0:0:1]$. To compute $\deg(\bar{\mathcal{Z}}_m)$, we write

$$g^m_{a,b}(-1) = \frac{T_m(a,b)}{U_m(a,b)}.$$

Then $\mathcal{Z}_m$ is the zero set of $T^h_m + U^h_m$, where, following the notation of Subsection 3.1, T^h_m and U^h_m are the homogenized versions of the polynomials T_m and U_m. Induction shows that the degrees of T_m, U_m and $T_m + U_m$ are $2^m - 1$ for all $m \geqslant 1$. In fact, the highest degree terms of T_{m+1} and U_{m+1} come from the highest degree terms of aU^2_m and bU^2_m, respectively. As in Subsection 3.1, we see that, for all $m \geqslant 3$, there is a birational map $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that

$$P^h_m \circ \psi = \begin{cases} T^h_m + U^h_m & \text{if } m \text{ is odd} \\ \dfrac{T^h_m + U^h_m}{T^h_2 + U^h_2} & \text{if } m \text{ is even.} \end{cases}$$

Hence, as in Subsection 3.1, since P^h_m is square-free for all $m \geqslant 3$, and since $T^h_m + U^h_m$ is square-free for $m = 1$ or $m = 2$, this is also true for $T^h_m + U^h_m$ for all $m \geqslant 3$. So

$$\deg(\mathcal{Z}_m) = \deg(T_m + U_m) = 2^m - 1.$$

So now we need to compute $\bar{\mathcal{Z}}_m \bullet_{[0:0:1]} \mathbb{V}(b)$ and $\bar{\mathcal{Z}}_m \bullet_{[1:0:0]} \mathbb{V}(b)$. We use homogenized coordinates $[a:b:t]$. We write

$$\begin{aligned} T_m(a,b) &= T^1_m(a) + bT^2_m(a,b), \quad T^h_m(a,b,t) = T^{h,1}_m(a,t) + bT^{h,2}_m(a,b,t), \\ U_m(a,b) &= U^1_m(a) + bU^2_m(a,b), \quad U^h_m(a,b,t) = U^{h,1}_m(a,t) + bU^{h,2}_m(a,b,t). \end{aligned}$$

Then $\bar{\mathcal{Z}}_m \bullet_{[0:0:1]} \mathbb{V}(b)$ and $\bar{\mathcal{Z}}_m \bullet_{[1:0:0]} \mathbb{V}(b)$ are the maximum powers of a and t, respectively, which divide $T^{h,1}_m + U^{h,1}_m$. The first of these is the maximum power of a dividing $T^1_m + U^1_m$. The second is $\deg(T_m + U_m) - \deg(T^1_m + U^1_m)$. So if we write

$$g^m_{a,0}(-1) = \frac{T^0_m(a)}{U^0_m(a)},$$

where T^0_m and U^0_m have no non-zero power of a as a common factor, and, provided that $T^0_m + U^0_m$ has a non-zero constant term, we see that

$$\eta_{\mathrm{IV}}(2,m) - \eta_{\mathrm{IV}}(1,m) = \deg(T^0_m + U^0_m).$$

A straightforward induction gives

$$\begin{aligned} T^0_1 &= a, \quad U^0_1 = -1, \\ T^0_2 &= 1, \quad U^0_2 = a - 2, \\ T^0_{2k+1} &= a(U^0_{2k})^2, \\ U^0_{2k+1} &= T^0_{2k}(T^0_{2k} + 2U^0_{2k}), \\ T^0_{2k+2} &= (U^0_{2k+1})^2 = (T^0_{2k})^2(T^0_{2k} + 2U^0_{2k})^2, \\ U^0_{2k+2} &= T^0_{2k+1}(T^0_{2k+1} + 2U^0_{2k+1})/a = (U^0_{2k})^2(a(U^0_{2k})^2 + 2(T^0_{2k})^2 + 4T^0_{2k}U^0_{2k}). \end{aligned}$$

Induction also gives $\deg(U^0_{2k}) > \deg(T^0_{2k})$ for all $k \geqslant 1$ and $\deg(T^0_{2k+1}) > \deg(U^0_{2k+1})$ for all $k \geqslant 0$. Therefore, the degree of $T^0_m + U^0_m$ is the maximum of the degrees of T^0_m and U^0_m. More precisely, we have

$$\begin{aligned}
\deg(T^0_{2k+1}) &= 2\deg(U^0_{2k}) + 1,\\
\deg(U^0_{2k+1}) &= \deg(T^0_{2k}) + \deg(U^0_{2k}),\\
\deg(T^0_{2k+2}) &= 2\deg(U^0_{2k+1}) = 2(\deg(T^0_{2k}) + \deg(U^0_{2k})),\\
\deg(U^0_{2k+2}) &= 4\deg(U^0_{2k}) + 1.
\end{aligned}$$

This gives

$$\begin{aligned}
\deg(U^0_{2k}) &= \sum_{i=0}^{k-1} 4^i = \frac{4^k - 1}{3} = \frac{2^{2k} - 1}{3},\\
\deg(T^0_{2k+1}) &= \frac{2^{2k+1} + 1}{3}.
\end{aligned}$$

So we obtain

$$\deg(T^0_m + U^0_m) = \frac{2^m - (-1)^m}{3}.$$

It remains to show that the constant term of $T^0_m + U^0_m$ is non-zero. For this, it suffices to show that if a_{2k} and b_{2k} are the constant terms of T^0_{2k} and U^0_{2k}, then

$$b_{2k} < 0 < a_{2k} < -b_{2k},$$

for all $k \geqslant 1$. This is true for $k = 1$. It then follows for all k by induction, from

$$a_{2k+2} = a^2_{2k}(a_{2k} + 2b_{2k})^2, \quad b_{2k+2} = a_{2k}b_{2k}(2b_{2k})(a_{2k} + 2b_{2k}).$$ □

4. *Type* II *components*

As before, we let $\eta_{\mathrm{II}}(m)$ denote the number of type II components of period dividing m, in the space $\mathcal{M}^{\mathrm{cm}}_2$, and $\eta'_{\mathrm{II}}(m)$ is the number of type II components of period exactly m. For $1 \leqslant j < m$, let $\eta_{\mathrm{II}}(m, j)$ denote the number of type II components of period at least 3 dividing m such that the second marked critical point ω_2 is in the jth iterate of the immediate attractive basin of ω_1. Note that

$$\sum_{d \,|\, m, d \geqslant 3} \frac{m}{d}\eta'_{\mathrm{II}}(d) = \sum_{j=1}^{m-1} \eta_{\mathrm{II}}(m, j).$$

We have already seen that $\eta_{\mathrm{II}}(2) = \eta_{\mathrm{II}}(2, 1) = 1$. The aim of this section is to prove Theorem 1.2. In fact, we obtain a sharper result by giving a formula for $\eta_{\mathrm{II}}(m, j)$.

THEOREM 4.1. *For all $m \geqslant 3$, and all $1 \leqslant j < m$,*

$$\begin{aligned}
\eta_{\mathrm{II}}(m, j) = {} & \frac{7}{36}2^m - \frac{1}{12}(2^j + 2^{m-j}) - \frac{(-1)^m}{36}((-2)^j + (-2)^{m-j})\\
& - \frac{1}{4} - \frac{5}{36}(-1)^m + \frac{1}{12}((-1)^j + (-1)^{m-j})\\
& - \frac{1}{2}\sum_{3 \leqslant q \leqslant j} \phi(q)\nu_q(j)\nu_q(m - j).
\end{aligned} \tag{4.1}$$

Hence,

$$\sum_{d\,|\,m,d\geqslant 3}\frac{m}{d}\eta'_{\mathrm{II}}(d)=\frac{7}{36}m2^m-\frac{37}{108}2^m-\frac{m}{4}-(-1)^m\frac{5}{36}m+\frac{1}{2}+(-1)^m\frac{5}{54}$$
$$-\frac{1}{2}\sum_{3\leqslant q\leqslant j\leqslant m-q}\phi(q)\nu_q(j)\nu_q(m-j). \tag{4.2}$$

REMARK 6. (1) The formula is symmetric in j and $m-j$, as expected, because we can interchange ω_1 and ω_2.

(2) In particular,

$$\begin{aligned}
&\eta_{\mathrm{II}}(3,1)=\eta_{\mathrm{II}}(3,2)=1,\\
&\eta_{\mathrm{II}}(4,1)=\eta_{\mathrm{II}}(4,2)=\eta_{\mathrm{II}}(4,3)=2,\\
&\eta_{\mathrm{II}}(5,j)=5\quad\text{for } 1\leqslant j\leqslant 4,\\
&\eta_{\mathrm{II}}(6,j)=10\quad\text{for } 1\leqslant j\leqslant 5,\\
&\eta_{\mathrm{II}}(7,j)=\begin{cases}21 & \text{for } j=1,2,5,6,\\ 22 & \text{for } j=3,4,\end{cases}\\
&\eta_{\mathrm{II}}(8,j)=\begin{cases}42 & \text{for } j=1,2,3,5,6,7,\\ 44 & \text{for } j=4.\end{cases}
\end{aligned}$$

The only non-zero contribution to $\eta_{\mathrm{II}}(m,j)$ from the last row of (4.1), in this list, is when $m=6$, $j=q=3$, or $m=7$, $j=3$ or 4 and $q=3$, or $m=8$ and $j=3$ or 5 and $q=3$, or $j=4$ and $q=3$ or 4.

4.1. *Outline proof*

We use the representation of the elements of $\mathcal{R}=\mathcal{M}_2^{\mathrm{cm}}\setminus X_2$ by maps $f_{c,d}$, where $(c,d)\in\mathbb{C}\times\mathbb{C}^*$ as in Subsection 2.2. Given $m\geqslant 3$, for $j\geqslant 1$ we consider the curves in $\mathcal{R}$:

$$\mathcal{P}_j=\left\{(c,d)\in\mathcal{R}\;\middle|\;f_{c,d}^j(0)=-\frac{2d}{c}\right\}$$

and

$$\mathcal{Q}_{m-j}=\left\{(c,d)\in\mathcal{R}\;\middle|\;f_{c,d}^{m-j}\left(-\frac{2d}{c}\right)=0\right\}.$$

The number $\eta_{\mathrm{II}}(m,j)$ is then given by the cardinality of $\mathcal{P}_j\cap\mathcal{Q}_{m-j}$. All intersections between $\mathcal{P}_j$ and $\mathcal{Q}_{m-j}$ are transverse, by Corollary 2.4.

As in the case of type IV components, we consider the closure of $\mathcal{P}_j$ and $\mathcal{Q}_{m-j}$ in $\mathbb{CP}^2$, and apply Bezout's Theorem.

We will start by computing the degree of $\bar{\mathcal{P}}_j$ and $\bar{\mathcal{Q}}_{m-j}$ and then continue to subtract the intersections at 'infinity' from the product of the degrees. The extra difficulty for counting intersections at infinity arises from the fact that we will have to establish an analogue of Stimson's Theorem 3.6. Both computing the degrees, and computing the intersections at infinity, will rely heavily on the parametrization of the unique type I component introduced in [**16**].

4.2. *The v coordinate*

We proceed to summarize the relevant results contained in [**16**] related to the parametrization of the type I hyperbolic component.

For each $\zeta \in \mathbb{D}^* = \{\zeta \mid 0 < |\zeta| < 1\}$, we consider the quadratic rational map

$$\tau_\zeta(z) = z\frac{z+\zeta}{1+\bar{\zeta}z}.$$

In terms of the notation of Subsection 2.1,

$$\beta_{-\zeta} = \frac{1+\bar{\zeta}}{1+\zeta}\tau_\zeta.$$

Note that both $z = 0$ and $z = \infty$ are attracting fixed points of τ_ζ. The basin of $z = 0$ is $\mathbb{D}$ and contains a unique critical point which we denote by $c(\zeta)$. The dynamics in $\mathbb{D}$ is semiconjugate, via the Königs coordinate, to multiplication by ζ. The Königs coordinate is the unique holomorphic map $\phi_\zeta : \mathbb{D} \to \mathbb{C}$ such that $\zeta\phi_\zeta(z) = \phi_\zeta \circ \tau_\zeta(z)$ and $\phi_\zeta(0) = 0, \phi_\zeta(c(\zeta)) = 1$. Such a map is an isomorphism between a neighbourhood of the origin and the unit disc. More precisely, there exists a conformal map $\psi_\zeta : \mathbb{D} \to \psi_\zeta(\mathbb{D})$ such that $\psi_\zeta(0) = 0$ which is an inverse of ϕ_ζ (that is, $\phi_\zeta \circ \psi_\zeta(z) = z$, and for all $w \in \psi_\zeta(\mathbb{D})$, we also have that $\psi_\zeta \circ \phi_\zeta(w) = w$).

Now consider an element $[f, \omega_1, \omega_2] \in \mathcal{M}_2^{\mathrm{cm}}$ which lies in the type I component. Then f has an attracting fixed point z_0, which we assume has multiplier $\zeta \neq 0$. Similarly to above, there exists a conformal isomorphism $\psi_f : \mathbb{D} \to \psi_f(\mathbb{D})$ with image contained in the basin of z_0 such that $\psi_f(0) = z_0$, $\psi_f(\zeta z) = f(\psi_f(z))$ and at least one critical point of f lies in $\partial\psi_f(\mathbb{D})$. (This map ψ_f always extends to a homeomorphism from $\bar{\mathbb{D}}$ onto its image.)

Let $\mathcal{U}$ be the subset of the type I component formed by all $[f, \omega_1, \omega_2]$ with an attracting fixed point of non-vanishing multiplier such that $\omega_1 \notin \partial\psi_f(\mathbb{D}) \ni \omega_2$. Note that $\mathcal{U}$ is well defined, since the required properties are invariant under conjugacies that respect critical markings.

Given $[f, \omega_1, \omega_2] \in \mathcal{U}$, with attracting fixed point z_0 with multiplier $\zeta = \zeta(f)$, and basin of attraction U_f, there exists a unique Königs coordinate $\phi_f : U_f \to \mathbb{C}$ that semiconjugates f with multiplication by $\zeta(f)$ such that $\phi_f(\omega_2) = 1$, and which is equal to ψ_f^{-1} near z_0. In the sequel, we will always assume that ϕ_f is this unique semiconjugacy. In a neighbourhood of z_0, the map $\psi_\zeta \circ \phi_f$ is a conjugacy between f and τ_ζ. Taking iterated preimages, the conjugacy $\psi_\zeta \circ \phi_f$ uniquely extends to a simply connected domain contained in U_f containing the critical value $f(\omega_1)$. The v-coordinate $v(f)$ of $f \in \mathcal{U}$ is, by definition, the image of $f(\omega_1)$ under the conjugacy extending $\psi_\zeta \circ \phi_f$.

According to [16],

$$\begin{aligned} \mathcal{U} &\longrightarrow \mathbb{D}^* \times \mathbb{D}, \\ [f, \omega_1, \omega_2] &\longmapsto (\zeta(f), v(f)) \end{aligned}$$

is a real-analytic homeomorphism.

4.3. *The degrees of the curves*

LEMMA 4.2. *Let $m \geqslant 3$. The following statements hold.*

For all $j \geqslant 1$, the degree of $\bar{\mathcal{P}}_j$ is

$$\tfrac{1}{6}(2^j - (-1)^j) + \tfrac{1}{2}.$$

For all $j \geqslant 1$, the degree of $\bar{\mathcal{Q}}_{m-j}$ is

$$\tfrac{1}{6}(7 \cdot 2^{m-j} - (-1)^{m-j}) - \tfrac{1}{2}.$$

Proof. We write $(cP_j + 2dQ_j)^h$ for the homogenized version of $cP_j + 2dQ_j$. Similarly, R^h_{m-j} is the homogenized version of R_{m-j}, where P_j, Q_j, R_{m-j} are as in Subsection 3.1. Observe that $\bar{\mathcal{P}}_j$ and $\bar{\mathcal{Q}}_{m-j}$ are the varieties of $(cP_j + 2dQ_j)^h$ and R^h_{m-j}, respectively. Using parts (i) and (iv) of Lemma 3.2 in Subsection 3.1, we obtain the first line below, and part (vi) of the same

lemma implies the second:

$$\deg((cP_j + 2dQ_j)^h) = \tfrac{1}{6}(2^j - (-1)^j) + \tfrac{1}{2},$$
$$\deg(R^h_{m-j}) = \tfrac{7}{6}2^{m-j} - \tfrac{1}{6}(-1)^{m-j} - \tfrac{1}{2}.$$

Hence, to establish the lemma we need to show that these polynomials are square-free. From the birational equivalence argument, we just need to check that $cP_j + 2dQ_j$ is square-free.

Given $0 < r < 1$, let L_r be the set formed by the maps $f_{c,d}$ in $\mathcal{R}$ that have an attracting fixed point with multiplier r. From [16], the intersection of $\mathcal{P}_j$ with L_r is contained in $\mathcal{U}$ and, as summarized above, it is in bijection with the points $z \in \mathbb{D}$ which map onto $c(r)$ under $j-1$ iterates of the map τ_r. It follows that the cardinality of the intersection $\mathcal{P}_j \cap L_r$ is 2^{j-1}, all of them transverse.

It is sufficient to show that the number of intersections of $\bar{\mathcal{P}}_j$ and $\bar{L}_r$ coincides with the product of the degrees of $cP_j + 2dQ_j$ and the polynomial equation of L_r, which we proceed to compute.

The curve L_r is given by the equations

$$1 + \frac{c}{z} + \frac{d}{z^2} = z, \quad -\frac{c}{z^2} - 2\frac{d}{z^3} = r.$$

Adding twice the first equation to z times the second gives a quadratic equation in z:

$$z^2 - \frac{2}{2+r}z - \frac{c}{2+r} = 0 = \left(z - \frac{1}{2+r}\right)^2 - \frac{c}{2+r} - \frac{1}{(2+r)^2}.$$

We then obtain

$$z^3 - z^2 - cz = -\frac{1}{(2+r)^2}\left((c(2+r)(1+r) + 2r)\left(z - \frac{1}{2+r}\right) + c(1+2r) + \frac{2r}{2+r}\right).$$

Substituting this in the first equation for L_r, rearranging, squaring and using the quadratic equation in z again, we obtain

$$(c(2+r)+1)(c(2+r)(1+r)+2r)^2 = (d(2+r)^3 + c(1+2r)(2+r) + 2r)^2.$$

This simplifies to

$$(1+r)^2c^3 + rc^2 - (2+r)^3d^2 - 2(2+r)(1+2r)cd - 4rd = 0.$$

The only term of degree 3 is c^3, and there is no constant term. So there are no elements of $\bar{L}_r$ at $[c:0:1]$ for $c \neq 0$. Since both the above equations of L_r and $cP_j + 2dQ_j$ have linear terms but no constant terms (Lemma 3.2(iii)), $[0:0:1] \in \bar{\mathcal{P}}_j \cap \bar{L}_r$ and the intersection number is 1.

Finally, $[0:1:0]$ lies in the intersection only when j is odd. By Lemma 3.2(ii) and (iv), when j is even, both P_j and Q_j have the same degree and both have powers of d of maximal degree, thus $[0:1:0] \notin \bar{\mathcal{P}}_j$ if j is even. When j is odd, $\deg(P_j) = \deg(Q_j) + 1$, so $[0:1:0] \in \bar{\mathcal{P}}_j$ and the intersection number at $[0:1:0]$ is also 1.

Thus, the total intersection number between $\bar{\mathcal{P}}_j$ and $\bar{L}_r$ is $2^{j-1} + 1 + (1-(-1)^j)/2$ which is equal to $3\deg(cP_j + 2dQ_j)$. Hence, $cP_j + 2dQ_j$ is square-free. □

4.4. *No intersections at ∞ and intersections at $c = 0$*

LEMMA 4.3. *We have the following*:

$[1:s:0] \notin \bar{\mathcal{P}}_j \cap \bar{\mathcal{Q}}_{m-j}$ *for all* $s \in \mathbb{C}$.
$\bar{\mathcal{P}}_j \bullet_{[0:0:1]} \mathbb{V}(d) = 1$, *where* $\mathbb{V}(d)$ *denotes the plane* $d = 0$, *and* $\bar{\mathcal{P}}_j \bullet_{[0:0:1]} \bar{\mathcal{Q}}_{m-j} = 2^{m-j-1}$ *for all* $1 \leqslant j < m$,
$\bar{\mathcal{P}}_j \bullet_{[0:1:0]} \bar{\mathcal{Q}}_{m-j} = \frac{1}{6}(1-(-1)^j)(2^{m-j} - (-1)^{m-j})$.

Proof. *Intersections at* $[1:s:0]$. For $s \neq 0$, this follows directly from Lemma 3.3, since $\bar{\mathcal{P}}_j \cap \bar{\mathcal{Q}}_{m-j} \subset \bar{\mathcal{X}}_m \cap \bar{\mathcal{Y}}_m$. There are no intersections at $[1:0:0]$ either, because $cP_j + 2dQ_j$ always has a term c^k of maximal degree by Lemma 3.2(iv).

Intersections at $[0:0:1]$. By (v) of Lemma 3.2, the minimum degree $\mathrm{mindeg}(R_{m-j})$ of the monomials in $R_{m-j}(c,d)$ is 2^{m-j-1} and this is uniquely realized by a monomial in d for all $1 \leqslant j < m$. By (iii) of Lemma 3.2, the constant term of both P_j and Q_j is 1 for all $j \geqslant 3$. Hence, $cP_j + 2dQ_j$ has non-zero linear terms in c and d, and the claim follows for all $j \geqslant 3$. Since $\mathcal{P}_1 = \mathbb{V}(c)$ and $\mathcal{P}_2 = \mathbb{V}(c+d)$, the claim also follows for $j = 1, 2$.

Intersections at $[0:1:0]$. There are no intersections if j is even, because, by (ii) of Lemma 3.2, both P_j and Q_j have the same degree, and, by (iv) of Lemma 3.2 both have powers of d of maximal degree. So the degree of $cP_j + 2dQ_j$ is realized by a power of d in $2dQ_j$. But $\deg(P_j) = \deg(Q_j) + 1$ for all odd $j \geqslant 3$ by (ii) of Lemma 3.2, and hence the degree of intersection of $(cP_j + 2dQ_j)^h = 0$ with $c = 0$ is 1 for all odd $j \geqslant 3$ (using, again, that linear terms in c and d are non-zero). The degree of the intersection of $(cP_j + 2dQ_j)^h$ with R^h_{m-j} is then given by $\deg(R_{m-j}) - \deg(R^1_{m-j})$, where we write

$$R_k(c,d) = R^1_k(d) + cR^2_k(c,d), \quad S_k(c,d) = S^1_k(d) + cS^2_k(c,d).$$

Inductively, we see that

$$R^1_1 = S^1_1 = 4d,$$
$$R^1_{k+1} = (R^1_k)^2 + d(S^1_k)^2, \quad S^1_{k+1} = (R^1_k)^2,$$

and hence for all $k \geqslant 1$,

$$\deg(R^1_{2k+1}) = 2\deg(R^1_{2k}) = 1 + 4\deg(R^1_{2k-1}).$$

It follows that

$$\deg(R^1_{m-j}) = \tfrac{5}{6}2^{m-j} - \tfrac{1}{2} + \tfrac{1}{6}(-1)^{m-j},$$

and hence

$$\deg(R_{m-j}) - \deg(R^1_{m-j}) = \tfrac{1}{3}(2^{m-j} - (-1)^{m-j}). \qquad \square$$

4.5. *Preliminaries on intersections at* $[c:0:1]$

LEMMA 4.4. *If* $[c:0:1] \in \bar{\mathcal{P}}_j \cup \bar{\mathcal{Q}}_j$, *then* $c = 0$, *or* $c = c_{p,q}$ *for some* q *with* $3 \leqslant q \leqslant j$ *and* p *with* $\gcd(p,q) = 1$, *where* $c_{p,q}$ *is as in Proposition* 3.5, *that is,* $c^{-1}_{p,q} = -4\cos^2(\pi p/q)$.

Proof. If $f_{c,0} \in \bar{\mathcal{P}}_j$ for $c \neq 0$, then it is the limit of maps f_{c_n,d_n} in $\mathcal{P}_j$ for a sequence (c_n, d_n). The critical points of f_{c_n,d_n} are 0 and $-2d_n/c_n$, where $-2d_n/c_n \to 0$. The critical values are ∞ and $1 - c_n/4d_n^2$, which converges to ∞, and hence $f^2_{c_n,d_n}(0)$ converges to 1. Since $f^j_{c_n,d_n}(0) = -2d_n/c_n$ and since the maps f_{c_n,d_n} converge uniformly to $f_{c,0}$ outside any neighbourhood of 0, restricting to a subsequence there must be some least integer $3 \leqslant k \leqslant j$ such that $f^k_{c_n,d_n}(0) \to 0$, and hence $f^k_{c,0}(0) = 0$ and $f_{c,0}$ has order k. So $c = c_{p,q}$ as claimed. The proof for $\bar{\mathcal{Q}}_j$ is similar. $\square$

In order to compute intersection numbers, we use the family of maps $h_{\zeta,\rho}$ introduced in Proposition 3.5, and varieties $\mathcal{C}_j$ and $\mathcal{D}_{m-j}$ corresponding to Möbius conjugates in the $h_{\zeta,\rho}$ family of the maps $f_{c,d}$ in $\mathcal{P}_j$ and $\mathcal{Q}_{m-j}$. That is, $\mathcal{C}_j$ consists of those parameters (ζ,ρ) such that the critical point $\omega_1 = 1$ of $h_{\zeta,\rho}$ maps in j iterates onto the critical point $\omega_2 = 1 + \rho$. Similarly, $\mathcal{D}_{m-j}$ consists of those parameters (ζ,ρ) such that the critical point $\omega_2 = 1 + \rho$ of $h_{\zeta,\rho}$ maps in j iterates onto the critical point $\omega_1 = 1$. As in Theorem 3.6, we may restrict our attention to study the intersections of $\bar{\mathcal{C}}_j$ and $\bar{\mathcal{D}}_{m-j}$ with the line $\rho = 0$. In particular, we will only be interested on parameters (ζ,ρ) where $|\rho|$ is close to 0.

LEMMA 4.5. *Let $\bar{\mathcal{C}}_j$ and $\bar{\mathcal{D}}_{m-j}$ be the varieties in the (ζ,ρ) coordinates that correspond to $\bar{\mathcal{P}}_j$ and $\bar{\mathcal{Q}}_{m-j}$. Then, for $c_{p,q}^{-1} = -4\cos^2(\pi p/q)$ and $\kappa_{p,q} = e^{2\pi i p/q}$, we have the following.*

(i) *The following numbers coincide, where $\mathbf{c}_{p,q} = [c_{p,q} : 0 : 1]$ and $\mathbf{k}_{p,q} = [\kappa_{p,q} : 0 : 1]$:*
$\bar{\mathcal{P}}_j \bullet_{\mathbf{c}_{p,q}} \mathbb{V}(d)$,
$\bar{\mathcal{Q}}_j \bullet_{\mathbf{c}_{p,q}} \mathbb{V}(d)$,
$\bar{\mathcal{C}}_j \bullet_{\mathbf{k}_{p,q}} \mathbb{V}(\rho)$,
$\bar{\mathcal{D}}_j \bullet_{\mathbf{k}_{p,q}} \mathbb{V}(\rho)$.

(ii) $\bar{\mathcal{P}}_j \bullet_{\mathbf{c}_{p,q}} \bar{\mathcal{Q}}_{m-j} = (\bar{\mathcal{P}}_j \bullet_{\mathbf{c}_{p,q}} \mathbb{V}(d)) \cdot (\bar{\mathcal{Q}}_{m-j} \bullet_{\mathbf{c}_{p,q}} \mathbb{V}(d))$.

Proof. For (i), note that the biholomorphism of the proof of Lemma 3.7 establishes that the first and third numbers coincide, as well as the second and fourth. Now $(\zeta,\rho) \mapsto (\zeta, -\rho/(1+\rho))$ interchanges $\mathcal{C}_j$ and $\mathcal{D}_j$. Hence, their intersection numbers at $\mathbf{k}_{p,q}$ coincide.

(ii) Once we have established (i), this is proved very similarly to the absence of common tangent lines at $\mathbb{V}(\rho)$ of $\mathcal{A}_m$ and $\mathcal{B}_n$ in Theorem 3.6. As there, we have

$$\frac{h^q(1+z\rho)-1}{\rho} \longrightarrow a + z + \frac{1}{2(2z-1)} = g_a(z).$$

If $\Re a \leqslant 0$, then $\Re g_a^k(0)$ is a strictly decreasing sequence and if $\Re a \geqslant 0$, then $\Re g_a^k(1)$ is a strictly increasing sequence. Therefore, we must have $\zeta = \kappa_{p,q}(1 + a_1\rho + o(\rho))$ with $\Re(a_1) > 0$ on any branch of $\mathcal{C}_j$ near $(\kappa_{p,q}, 0)$ and $\zeta = \kappa_{p,q}(1 + a_2\rho + o(\rho))$ with $\Re(a_2) < 0$ on any branch of $\mathcal{D}_{m-j}$ near $(\kappa_{p,q}, 0)$. The absence of common tangent lines follows, and (ii) is a consequence of this and the biholomorphism of the proof of Lemma 3.7. □

4.6. *Intersection number at $\mathbf{k}_{p,q}$*

THEOREM 4.6. *Let $\kappa_{p,q} = \exp(2\pi i p/q)$ and $\mathbf{k}_{p,q} = [\kappa_{p,q} : 0 : 1]$ (as before). Then*

$$\bar{\mathcal{C}}_j \bullet_{\mathbf{k}_{p,q}} \mathbb{V}(\rho) = \nu_q(j).$$

To compute $\bar{\mathcal{C}}_j \bullet_{\mathbf{k}_{p,q}} \mathbb{V}(\rho)$, we observe that, since local intersection number is constant under sufficiently small perturbation, there exists $\delta > 0$, such that this number coincides with the cardinality of

$$\mathcal{C}_j \cap (\{r\kappa_{p,q}\} \times \{\rho \mid |\rho| \leqslant \delta\}),$$

for all $0 < r < 1$ sufficiently close to 1. According to [16], the parameters in the above intersection correspond to maps in the subset $\mathcal{U}$ of the type I component introduced in Subsection 4.2. To prove the theorem, we need to further understand the image of the above intersection under the parametrization described in Subsection 4.2. The description of this image is contained in the next lemma and proposition.

Recall that $\phi_\zeta : \mathbb{D} \to \mathbb{C}$ denotes the (normalized) Königs coordinate for τ_ζ.

LEMMA 4.7. *For $\kappa = \kappa_{p,q}$, consider the graph*

$$S_\kappa = \{t\kappa^i \mid 1 \leqslant i \leqslant q, t \geqslant 0\} \subset \mathbb{C}.$$

Given $0 < r < 1$, let $\zeta = r\kappa$ and consider

$$\Gamma = \Gamma_\zeta = \phi_\zeta^{-1}(S_\kappa) \subset \mathbb{D}.$$

Then the following statements hold:

(i) *Γ is connected, simply connected and locally homeomorphic to a finite tree*;

(ii) *$\Gamma \setminus \{0\}$ has exactly q connected components*;

(iii) *label by* γ_1^ζ *the connected component of* $\Gamma \setminus \{0\}$ *containing* $\tau_\zeta(c(\zeta))$. *Then*

$$\#\{z \in \gamma_1^\zeta \mid \tau_\zeta^{n-1}(z) = c(\zeta)\} = \nu_q(n).$$

Proof. We can write Γ as an increasing union of sets Γ_n for $n \geqslant 0$, where

$$\Gamma_n = \tau_\zeta^{-n}(\psi_\zeta(S_\kappa \cap \mathbb{D})).$$

For each n, $\tau_\zeta : \Gamma_{n+1} \to \Gamma_n$ is a degree 2 branched cover with a single critical point. So, by induction on n, each Γ_n is a finite tree. The only intersections between $\overline{\Gamma_{n+1} \setminus \Gamma_n}$ and Γ_n are at extreme points of Γ_n, again by induction. It follows that Γ is a connected and locally finite tree. Also by induction, $\Gamma_n \setminus \{0\}$ has q components, and each one of the q components of $\Gamma_{n+1} \setminus \{0\}$ contains one of the q components of $\Gamma_n \setminus \{0\}$, which, in turn, contains one of the points $\tau_\zeta^j(c(\zeta))$ for $1 \leqslant j \leqslant q$. So taking the union of all of these, $\Gamma \setminus \{0\}$ also has q components γ_j^ζ, for $1 \leqslant j \leqslant q$, where γ_j^ζ contains $\tau_\zeta^j(c(\zeta))$, and γ_q^ζ also contains $c(\zeta)$. Also, τ_ζ maps γ_j^ζ homeomorphically onto γ_{j+1}^ζ if $1 \leqslant j \leqslant q-1$, and maps γ_q^ζ onto Γ, mapping one-to-one onto $(\Gamma \setminus \gamma_1^\zeta) \cup \{\tau_\zeta(c(\zeta))\}$ and two-to-one onto $\gamma_1^\zeta \setminus \{\tau_\zeta(c(\zeta))\}$. It follows that the number $a_j(n) = \#(\tau_\zeta^{-n}(c(\zeta)) \cap \gamma_j^\zeta)$ satisfies

$$a_j(0) = \begin{cases} 0 & \text{if } j < q, \\ 1 & \text{if } j = q, \end{cases}$$

$$a_j(n+1) = \begin{cases} a_{j+1}(n) & \text{if } j < q, \\ 2a_1(n) + \displaystyle\sum_{1<i\leqslant q} a_i(n) & \text{if } j = q. \end{cases}$$

Let $s(n) = a_1(n) + \cdots + a_q(n)$. It follows that $s(n+1) = 2s(n)$. Thus, $s(n) = 2^n$. Moreover, given $k \geqslant 0$ and $0 \leqslant r < q$, since $a_1(kq+r) = a_q((k-1)q + r + 1) = a_1((k-1)q + r) + s((k-1)q + r)$, we obtain that

$$\begin{aligned} a_1(kq+r) &= a_1(r) + s(r) + \cdots + s((k-1)q + r) \\ &= a_{1+r}(0) + s(r) + \cdots + s((k-1)q + r) \\ &= \begin{cases} \dfrac{2^{kq+r} - 2^r}{2^q - 1} & \text{if } r < q-1, \\ 1 + \dfrac{2^{kq+r} - 2^r}{2^q - 1} & \text{if } r = q-1. \end{cases} \end{aligned}$$

Setting $n = kq + r + 1$, we obtain that $a_1(n-1) = \nu_q(n)$. □

PROPOSITION 4.8. *Let* X *be a branch of* $\bar{\mathcal{C}}_j$ *at* $(\kappa, 0)$, *for* $\kappa = \kappa_{p,q}$. *There exist* $0 < r_0 < 1$ *and* $\delta > 0$ *such that if* $h = h_{r\kappa,\rho} \in X$ *for some* $r_0 < r < 1$ *and* $|\rho| < \delta$, *then* $v(h) \in \gamma_1^{r\kappa}$.

We assume this proposition and defer, for a moment, its proof.

Proof of Theorem 4.6. From the previous proposition and lemma, we obtain that

$$\bar{\mathcal{C}}_j \bullet_{\mathbf{k}_{p,q}} \mathbb{V}(\rho) \leqslant \nu_q(j).$$

Applying Bezout's Theorem, we obtain the first line below. From Lemma 4.4, we obtain the second identity. The third line is obtained putting together the inequality above with Lemmas 4.5 and 4.3. The fourth line follows since the number of hyperbolic components of period dividing j in the p/q limb is $\nu_q(j)$ (Lemma 3.8). Then the fifth line is deduced using the fact that $\eta_{\mathrm{IV}}(1, j) = 2^{j-1}$ of Lemma 3.8. After rearranging the terms in the sixth line, the

last identity is obtained from Lemma 4.2.

$$\begin{aligned}
\deg \bar{\mathcal{P}}_j &= \bar{\mathcal{P}}_j \bullet \mathbb{V}(d) \\
&= \bar{\mathcal{P}}_j \bullet_{[0:0:1]} \mathbb{V}(d) + \sum_{\mathbf{c}_{p,q}, 3\leqslant q\leqslant j} \bar{\mathcal{P}}_j \bullet_{\mathbf{c}_{p,q}} \mathbb{V}(d) \\
&\leqslant 1 + \frac{1}{2}\sum_{3\leqslant q\leqslant j} \phi(q)\nu_q(j) = 1 + \frac{1}{2}\sum_{2\leqslant q\leqslant j} \phi(q)\nu_q(j) - \frac{\nu_2(j)}{2} \\
&= 1 + \frac{1}{2}(\eta_{\mathrm{IV}}(1,j) - 1) - \frac{1}{2}\left(\frac{2^{j-1} + (-1)^j}{3}\right) \\
&= \frac{1}{2}\left(2^{j-1} - \frac{2^{j-1} + (-1)^j}{3}\right) + \frac{1}{2} \\
&= \frac{1}{6}(2^j - (-1)^j) + \frac{1}{2} \\
&= \deg \bar{\mathcal{P}}_j.
\end{aligned}$$

Thus, equality holds throughout. In particular, by Lemma 4.5,

$$\nu_q(j) = \bar{\mathcal{P}}_j \bullet_{\mathbf{c}_{p,q}} \mathbb{V}(d) = \bar{\mathcal{C}}_j \bullet_{\mathbf{k}_{p,q}} \mathbb{V}(\rho). \qquad \square$$

Proof of Proposition 4.8. We write γ_i' for the image of γ_i^ζ under the conjugacy between τ_ζ and h extending $\phi_h^{-1} \circ \psi_\zeta$. We also write Γ_h' for the union of the γ_i'. It is sufficient to show that $h(\omega_1) \in \gamma_1'$ for all r sufficiently close to 1.

Let $\varepsilon > 0$ be sufficiently small so that the sectors

$$S_i = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z - \arg \kappa^i| < 3\varepsilon\},$$

for $i = 0, \dots, q-1$ (subscripts mod q) are pairwise disjoint. We will show that $\gamma_i' \subset S_i$. Since $h(\omega_1) \in S_1$ the proposition will follow. Now for any integer $M \geqslant j$,

$$h^M(\Gamma_h') \subset \bigcup_{i\geqslant 0} \psi_h(]0,1]\zeta^{M+i-j}) = T_0.$$

We shall see in Lemma 4.10 that $|\psi_h(w) - w| < 3\varepsilon|w|$ for $w \in]0,1]\zeta^\ell$ for all sufficiently large ℓ and r sufficiently close to 1, given ε, and hence $\psi_h(w) \in S_\ell$, since $\zeta = r\kappa$. The idea of Lemma 4.10 is simply to prove ψ_h is sufficiently close to the identity on a sufficiently large set, but we assume this for the moment. So for a fixed M given ε, for all r sufficiently close to 1, we have

$$h^M(\Gamma_h') \subset S = \bigcup_{i=0}^{q-1} S_i.$$

We want to extend this, and show that $\Gamma_h' \subset S$, from which it follows immediately that $\gamma_i' \subset S_i$ for all $0 \leqslant i \leqslant q-1$. In order to prove this statement, given $\delta > 0$, let us denote by V_δ a δ-open neighbourhood of the roots of unity $1, \kappa, \dots, \kappa^{q-1}$. Since h^M converges uniformly (spherical metric) to multiplication by κ^M in $\bar{\mathbb{C}} \setminus V_\varepsilon$ as $r \to 1$, and T_0 contains ω_2, which is in V_ε for r close to 1, and $h^M(T_0) \subset T_0$, we have that for r sufficiently close to 1, if $z \in \bar{\mathbb{C}} \setminus V_\varepsilon$ and $h^M(z) \in T_0 \setminus V_{2\varepsilon}$, then $z \in T_0$. We may also assume that if $z \in \bar{\mathbb{C}} \setminus V_\varepsilon$ and $h^M(z) \in V_{2\varepsilon}$, then $z \in V_{3\varepsilon}$. Thus, if $z \in \Gamma_h'$, then $h^M(z) \in T_0$ and therefore, $z \in V_{3\varepsilon} \cup T_0 \subset S$, as required. $\square$

4.6.1. *Lemmas* 4.9 *and* 4.10. The proof of the assumption made above is, loosely speaking, that the Königs coordinate ϕ_h and its inverse ψ_h are close to the identity in $\mathbb{D}$, as $r \to 1$, since h converges to the identity in $\mathbb{D}$. We consider ϕ_h first, in Lemma 4.9, and then ψ_h, in Lemma 4.10.

LEMMA 4.9. *Let X be a branch of $\bar{\mathcal{C}}_j$ at $(\kappa, 0)$, for $\kappa = \kappa_{p,q}$. Given $\varepsilon > 0$, there exists $0 < r_0 < 1$, $C > 0$ and $N \in \mathbb{N}$ such that if $h = h_{r\kappa,\rho} \in X$ and $r_0 < r < 1$, then*

$$|\phi_h(z) - z| < \varepsilon|z|,$$

for all $|z| \leqslant 1 - C(1-r)$ and,

$$|h^N(1+\rho)| \leqslant 1 - C(1-r).$$

Proof. Note that, there exists $a \in \mathbb{C}$ with $\Re a > 0$ such that, if $h_{\zeta,\rho} \in X$, then

$$\zeta = \kappa(1 + a\rho + o(\rho)),$$

as $\rho \to 0$. Assume that $h_{r\kappa,\rho} \in X$ and $r < 1$. Let $\Delta = 1 - r$. It follows that

$$\rho = -\frac{\Delta}{a}(1 + o(1)),$$

as $\Delta \to 0$.

Given $\varepsilon > 0$, take

$$C \geqslant \frac{1}{2\varepsilon|a|^2} + \frac{1}{|a|}.$$

Assume that

$$|z| \leqslant 1 - C\Delta.$$

Then, for Δ sufficiently small,

$$\left|1 + \frac{\rho}{2} - z\right| \geqslant -\frac{\Delta}{2|a|}(1 + o(1)) + C\Delta \geqslant \frac{\Delta}{2\varepsilon|a|^2}.$$

Also recall that $h_{\zeta,\rho}(z) = \zeta z P_\rho(z)$ where

$$P_\rho(z) = 1 - \frac{\rho^2 z}{4(1+\rho)(1+\rho/2-z)}.$$

Thus, for Δ sufficiently small,

$$|P_\rho(z) - 1| \leqslant \frac{(2\Delta/|a|)^2|z|}{2\Delta/(2\varepsilon|a|^2)} \leqslant 4\varepsilon\Delta|z|.$$

Hence,

$$|h(z)| \leqslant (1-\Delta)|z|(1 + |P_\rho(z) - 1|) \leqslant (1-\Delta)|z|(1 + 4\varepsilon\Delta) \leqslant (1 - (1-4\varepsilon)\Delta)|z|.$$

In particular, $|h(z)| \leqslant 1 - C\Delta$.

Let $z_n = h^n(z)$ and recall that $\phi_h(z) = \lim \zeta^{-n} z_n$.

$$\begin{aligned}\left|\frac{z_{n+1}}{\zeta^{n+1}} - \frac{z_n}{\zeta^n}\right| &= (1-\Delta)^{-n}|z_n| \cdot |P_\rho(z_n) - 1| \\ &\leqslant (1-\Delta)^{-n}|z_n|^2 4\varepsilon\Delta \\ &\leqslant (1-\Delta)^{-n}(1 - (1-4\varepsilon)\Delta)^{2n}|z_0|4\varepsilon\Delta.\end{aligned}$$

Hence,

$$\begin{aligned}|\zeta^{-(n+1)} z_{n+1} - z_0| &\leqslant 4\varepsilon\Delta \sum_{n \geqslant 0} \left(\frac{(1 - (1-4\varepsilon)\Delta)^2}{1-\Delta}\right)^n |z_0| \\ &= \frac{4\varepsilon(1-\Delta)}{1 - 8\varepsilon - (1-4\varepsilon)^2\Delta}|z_0| \leqslant 8\varepsilon|z_0|,\end{aligned}$$

for Δ sufficiently small and ε small, say $\varepsilon < \frac{1}{20}$. This proves the first assertion.

For the second assertion, recall that as $\rho \to 0$,

$$\frac{h^q(1+z\rho)-1}{\rho} \longrightarrow g(z) = qa + z + \frac{1}{2(2z-1)},$$

uniformly in compact subsets of $\mathbb{C} \setminus \{\frac{1}{2}\}$. Since ∞ is a parabolic fixed point of g with attracting direction $(0,+\infty)\cdot a$, it follows that, given $\delta > 0$, for N large, $|g^N(1)|$ is large and

$$|\arg g^N(1) - \arg a| < \delta.$$

Since $\rho = -\Delta(1+o(1))/a$, we have that

$$|1 + g^N(1)\rho| \leqslant 1 - (\cos\delta)|g^N(1)|\Delta.$$

Thus, taking $\delta > 0$ small, N sufficiently large and r sufficiently close to 1, we have that $(h^{qN}(1+\rho)-1)/\rho$ is sufficiently close to a sufficiently large $g^N(1)$ so that

$$|h^{qN}(1+\rho)| \leqslant 1 - C\Delta. \qquad \square$$

LEMMA 4.10. *Let X be a branch of $\bar{\mathcal{C}}_j$ at $(\kappa,0)$, for $\kappa = \kappa_{p,q}$. Given $\varepsilon > 0$, there exists $0 < r_0 < 1$ and $N' \in \mathbb{N}$ such that the following holds. For all $\zeta = r\kappa$ with $r_0 < r < 1$, and all $h = h_{r\kappa,\rho} \in X$ if*

$$w \in \{t\zeta^n \mid t \in]0,1]\},$$

then

$$|\psi_h(w) - w| < 3\varepsilon|w|,$$

for all $n \geqslant N'$.

Proof. Let $\Delta = 1 - r$. Consider N and C as in the previous lemma. Denote by $\mathrm{dist}_{\mathbb{D}}$ the hyperbolic distance (with constant curvature -1) in the open unit disc. Observe that

$$\mathrm{dist}_{\mathbb{D}}(\zeta^n, \zeta^{n+q}) = \log\frac{1+r^n}{1-r^n}\frac{1-r^{n+q}}{1+r^{n+q}} \leqslant q\Delta + \frac{q}{n}.$$

Hence, we may choose $N' > N$ such that

$$\mathrm{dist}_{\mathbb{D}}(\zeta^n, \zeta^{n+q}) < \log\frac{3}{2},$$

for all $n \geqslant N'$ and all Δ sufficiently small.

We may assume that Δ is sufficiently small so that the $\varepsilon/2$ ball centred at κ^n, denoted by $B_{\varepsilon/2}(\kappa^n)$ is not contained in $\psi_h(\mathbb{D})$. In fact, since h converges uniformly in compact subsets of $\mathbb{C} \setminus \{1\}$ to multiplication by κ, any such ball contains an iterated preimage of the critical point $\omega_2 = 1$.

We now claim that for all $n \geqslant N'$, if $\psi_h(\zeta^n)$ and $\psi_h(\zeta^{n+q})$ lie in $B_{\varepsilon/2}(\kappa^n)$, then the hyperbolic geodesic of $\psi_h(\mathbb{D})$ joining these points is contained in $B_\varepsilon(\kappa^n)$. In fact, recall that the (infinitesimal) hyperbolic arc length in $\psi_h(\mathbb{D})$ is bounded below by $1/(2\delta(z))$, where $\delta(z)$ is the distance from z to the boundary of $\psi_h(\mathbb{D})$. Thus, the geodesic from $\psi_h(\zeta^n)$ to $\psi_h(\zeta^{n+q})$ may not exit $B_\varepsilon(\kappa^n)$, for otherwise, it would have length at least $\log(3/2)$.

By the previous lemma, we may take Δ sufficiently small so that the following conditions are satisfied.

(i) For all w such that $|w| \leqslant 1 - \varepsilon/5$,

$$|\psi_h(w) - w| < \epsilon|w|.$$

(ii) For all z such that $|z| < 1 - C\Delta$, we have

$$|\phi_h(z) - z| < \varepsilon/8.$$

(iii) For all n such that $1 \leqslant n < N' + q$, we may also assume that $|\zeta^n - \kappa^n| < \varepsilon/8$.

Recall that $\psi_h(\zeta^k) = h^k(\omega_2)$, for all $k \geqslant 1$. Equivalently, $\phi_h(h^k(\omega_2)) = \zeta^k$. We may assume that n is such that $N' \leqslant n < N' + q$. It follows that

$$|h^n(\omega_2) - \kappa^n| \leqslant |h^n(\omega_2) - \zeta^n| + |\zeta^n - \kappa^n| < \varepsilon/2.$$

Let $\ell \geqslant 0$ be the smallest integer such that

$$|h^{n+q\ell}(\omega_2) - \kappa^n| \leqslant \varepsilon/2$$

and

$$|h^{n+q(\ell+1)}(\omega_2) - \kappa^n| > \varepsilon/2.$$

Note that this number ℓ depends on Δ (that is on ζ).

It follows that

$$|\zeta^{n+q(\ell+1)} - \kappa^n| > \varepsilon/2 - \varepsilon/8.$$

Then, since $\kappa^q = 1$:

$$|\zeta^{n+q(\ell+1)} - \zeta^{n+q\ell}| \leqslant |\zeta^q - 1| < \varepsilon/8.$$

Hence,

$$|\zeta^{n+q\ell} - \kappa^n| > \varepsilon/2 - \varepsilon/8 - \varepsilon/8 > \varepsilon/5.$$

If $w = s\zeta^n$, for some $s < r^{q\ell}$, then $|w| < 1 - \varepsilon/5$ and therefore $|\psi_h(w) - w| < \varepsilon|w|$.

Finally, if $w = s\zeta^n$ where $r^{q(i+1)} \leqslant s \leqslant r^{qi}$ for some $0 \leqslant i < \ell$, then $\psi_h(w)$ lies in a geodesic joining the points $h^{n+qi}(\omega_2)$ and $h^{n+q(i+1)}(\omega_2)$, which are $\varepsilon/2$-close to κ^n. Thus, $\psi_h(w)$ is ε-close to κ^n. For Δ sufficiently small, it follows that $|\psi_h(w) - w| < 3\varepsilon|w|$. □

4.7. *Proof of Theorem* 4.1

By Bezout's Theorem, and our study of intersections at infinity, we have that $\eta_{\mathrm{II}}(m, j)$ is obtained by subtracting from the product of the degrees of $\bar{\mathcal{P}}_j$ and $\bar{\mathcal{Q}}_{m-j}$ (first line below) the intersection numbers at $[0 : 0 : 1]$ (second line), at $[0 : 1 : 0]$ (third line) and at all $\mathbf{c}_{p,q}$ (fourth line).

$$\begin{aligned}\eta_{\mathrm{II}}(m, j) = &\left(\frac{1}{6}(2^j - (-1)^j) + \frac{1}{2}\right)\left(\frac{1}{6}(7 \cdot 2^{m-j} - (-1)^{m-j}) - \frac{1}{2}\right)\\ &- 2^{m-j-1}\\ &- \frac{1}{6}(1 - (-1)^j)(2^{m-j} - (-1)^{m-j})\\ &- \frac{1}{2}\sum_{3 \leqslant q \leqslant j} \phi(q)\nu_q(j)\nu_q(m - j).\end{aligned}$$

Now a calculation shows that the formula above is equivalent to that stated in the theorem.

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