Some class of distal groups are Tortrat

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Abstract

We prove linear distal groups and groups whose compact subgroups are finite are Tortrat.

Let G be a locally compact group and e denote the identity of G. Let $\mathcal{P}(G)$ be the space of all regular Borel probability measures on G, equipped with the weak* topology with respect to all bounded continuous functions on G. A locally compact group G is called *distal* if e is not a limit point of $\{xgx^{-1} \mid x \in G\}$ for any $g \in G \setminus (e)$. A locally compact group G is called *Tortrat* if a sequence of the form $(g_n \lambda g_n^{-1})$, where $\lambda \in \mathcal{P}(G)$ and (g_n) is a sequence in G, has an idempotent limit point only if λ is an idempotent. Tortrat class was introduced by P. Eisele and he proved some interesting properties of limits of shifted convolution powers of measures on these groups (see [E]). Here we recall the following (see Theorem 2.2 of [E]).

Let G be a Tortrat group and μ be a probability measure on G. Let (x_n) be a sequence in G such that $(\mu^n x_n)$ relatively compact. Then there exists a compact subgroup H of G such that for all y in the support of μ,

$$x \in Hy \iff \mu^n x^{-n} \to \omega_H \iff (\mu^n x^{-n}) \text{ converges}$$

and H is the smallest compact subgroup with $Hy = yH$ containing the support of μ. This result may be applied to prove Krengel-Lin decomposition for probability measures on Tortrat groups; see [BR] for more details on Krengel-Lin decomposition.

Remark 1 Let G be a locally compact group and K be a compact subgroup of G. Then we say that the pair $(G,\ K)$ is *Gelfand* if the semigroup of K-biinvariant probability measures on G is commutative. The above Krengel-Lin decomposition may be proved for all K-biinvariant probability measures on G: one may prove that in the above situation when $(\mu^n x_n)$ is relatively compact, we have $\mu^n * \check{\mu}^n \to \omega_L$ for some compact subgroup L containing K where $\check{\mu}$ is the usual adjoint of μ.

A *local field* $\mathbb{K}$ is a commutative non-discrete locally compact field (see [W]). A locally compact group G is said to be a *linear group* if G is a closed subgroup of $GL(V)$, the general linear group on a finite-dimensional vector space V over a local field $\mathbb{K}$. It is easy to see that a Tortat group is always distal. It is shown in [DR] that an almost connected locally compact group is distal if and only if it is Tortrat. We prove the following:

Theorem 1 *Let G be a linear group over a local field. Then the following are equivalent:*

(1) G is distal;

(2) G is Tortrat.

Theorem 2 *Let G be a locally compact group. Suppose compact subgroups of G are finite. Then the following are equivalent:*

(1) G is distal;

(2) G is Tortrat.

The following lemma is a vital tool in proving the above theorems and it is akin to a Theorem of Azencott.

Lemma 1 *Let T be a group acting on a locally compact first countable group X. Let μ be a probability measure on X and (t_n) be a sequence in T and F be a compact set in X such that for every open set U containing F, $\mu(t_n^{-1}U) \to 1$. Then there exist a subsequence t_{n_k} of t_n and a Borel set B such that $B \subset \{x \in X \mid (t_{n_k}x)$ is relatively compact$\}$ and $\mu(B) = 1$. Moreover, for any $x \in B$, any limit point of the sequence $(t_{n_k}x)$ is in F.*

Proof Let (U_k) be a decreasing sequence of relatively compact open sets which form a system of base at identity and $F_k = FU_k$. Let t_{n_k} be such that $\mu(t_{n_k}^{-1}F_k) \geq 1 - 1/2^k$. Let $B = \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} t_{n_i}^{-1}F_i$. Then it is easy to see that $\mu(B) = 1$ and for $x \in B$, there is a k such that $t_{n_l}x \in F_i$ for all $l > i > k$ and hence $(t_{n_k}x)$ is relatively compact. Let y be a limit point of the sequence $(t_{n_k}x)$. Suppose y is not in F then there is a F_k whose closure does not contain y but $t_{n_l}x \in F_i$ for some $i \geq k$ and all $l \geq i$ which is a contradiction.

Proposition 1 *Let G be a metrizable distal group. Let (x_n) be a sequence in G and $\lambda \in \mathcal{P}(G)$. Suppose $x_n\lambda x_n^{-1} \to \omega_F$ where F is a finite group. Then $\lambda = \omega_E$ and $|E| = |F|$ where $|\cdot|$ is the usual counting measure on G.*

Proof By Theorem 2.1 of [B] and Lemma 1, by passing to a subsequence we may assume that there exists a Borel set $B \subset H$ such that $\lambda(B) = 1$, $(x_n b x_n^{-1})$ is relatively compact and limit points of $(x_n b x_n^{-1})$ are in F for all $b \in B$. Let H be the set of all g in G such that the sequence $(x_n g x_n^{-1})$ is relatively compact and any limit point of the sequence $(x_n g x_n^{-1})$ is in F. Then H is a group and contains B. Let $h \in H$ and $l(h)$ be a limit point of $(x_n h x_n^{-1})$. Then $l(h)^k = e$ where $k = |F|$. Since G is distal, $l(h_1) \neq l(h_2)$ for $h_1 \neq h_2$ in H. Since F is finite, this implies that H is finite. Thus, by passing to a subsequence, we may assume that $x_n h x_n^{-1} \to h_\alpha \in F$ for all $h \in H$. Since $B \subset H$ and $\lambda(B) = 1$, we have $x_n\lambda x_n^{-1} \to \omega_F$. Let $\alpha: H \to F$ be the map defined by $\alpha(h) = h_\alpha$. Then α is an isomorphism. This implies that $\alpha(\lambda) = \omega_F$ and hence $\lambda = \omega_{\alpha^{-1}(F)}$. This proves the proposition.

Proof (of Theorem 1) Let G be a linear group over a local field. Suppose G is distal. Let $\lambda \in \mathcal{P}(G)$ and (x_n) be a sequence in G such that $x_n \lambda x_n^{-1} \to \omega_K$, the normalized Haar measure on some compact subgroup K of G. Then by Theorem 2.1 of [B] and Lemma 1, by passing to a subsequence there exists a Borel subset B of G such that $\lambda(B) = 1$ and $(x_n g x_n^{-1})$ is relatively compact in G for all $g \in B$. Let W be the algebra of all linear transformations on V. Now, for any $g \in G$, let $\phi_g : W \to W$ be the map defined by $\phi_g(w) = gwg^{-1}$ for all $w \in W$ and let $\phi_n = \phi_{x_n}$. Let W_1 be the subalgebra of elements $w \in W$ such that $(\phi_n(w))$ relatively compact. Then $B \subset W_1$. Since W_1 is a finite-dimensional vector space, by passing to a subsequence, we may assume that $(\phi_n(w))$ converges for all $w \in W_1$. Let $\phi(w) = \lim \phi_n(w)$ for all $w \in W_1$. Then ϕ is a linear map from W_1 into W. Since $\phi_n \lambda \to \omega_K$, we have that $\phi(\lambda) = \omega_K$. Let H be the closed sugroup of G such that $H = \phi^{-1}(K) \cap G$. Then $B \subset H$ and hence $\lambda(H) = 1$. Let $\alpha : H \to K$ be the restriction of ϕ to H. Then α is a continuous map and $\alpha(\lambda) = \omega_K$. We now claim that $\alpha(H)$ is closed. Let C be a compact subset of H such that $\lambda(C) > 0$. Then $\omega_K(\alpha(C)) > 0$ and hence $\alpha(H) \supset \alpha(C)\alpha(C)^{-1} = \alpha(CC^{-1})$ which contains an open subset of K. This implies that $\alpha(H)$ is an open subgroup of K and hence $\alpha(H)$ is closed. Since $\lambda(H) = 1$ and $\alpha(\lambda) = \omega_K$, we have $\alpha(H) = K$. Thus, α is onto. Now by open mapping theorem α is an open map. Since G is distal, α is one-one. This implies that α is an isomorphism and $\lambda = \omega_{\alpha^{-1}(K)}$. This proves the theorem.

Proof (of Theorem 2) Let G be a distal group. Let (x_n) be a sequence in G and $\lambda \in \mathcal{P}(G)$. Suppose $(x_n \lambda x_n^{-1})$ has an idempotent limit point ρ. Then ρ is the normalized Haar measure on a finite subgroup of G. Suppose G is metrizable. Then by Lemma 1, λ is a normalized Haar measure on a compact subgroup of G. Let G be any arbitrary locally compact group. Then arguing along the lines of Theorem 4.1 of [E] one may prove that G is a Tortrat group.

The following is immediate from Theorem 2.

Corollary 3 *Let G be a locally compact aperiodic group. Then G is distal if and only if G is Tortrat.*

Remark 2 On one hand a finitely generated group has polynomial growth if and only if it has a nilpotent subgroup of finite index (see [P]) and on the other hand discrete groups are both distal and Tortrat (see [R] and [E]). Thus, neither of the conditions in Theorem 2 imply polynomial growth.

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