ON FACTORIZATION OF COMPLETE GRAPHS INTO ISOMORPHIC CATERPILLARS OF DIAMETER 6

Tomáš Vetrík

The problem of factorization of complete graphs into isomorphic caterpillars of given diameter was opened by Kovářová [4, 5], Kubesa [6, 7] and Fronček [2]. These authors and Kovář presented a complete solution of the problem for the diameter $d \leq 5$ (see [2–7]). In this note we focus on a factorization into isomorphic caterpillars of diameter 6.

A caterpillar R is a tree such that by deleting of all vertices of degree one we obtain a path P called a spine of R. By a $(t_1, t_2, t_3, t_4, t_5)$-caterpillar we mean a caterpillar of diameter 6 with the spine $P = p_1p_2p_3p_4p_5$ where $deg(p_i) = t_i \geq 2$, $i = 1, 2, ..., 5$. A $[t_1, t_2, t_3, t_4, t_5]$-caterpillar is a caterpillar with a spine containing the vertices r_1, r_2, r_3, r_4, r_5 where $deg(r_i) = t_i$, $i = 1, 2, ..., 5$, and $t_j \geq t_{j+1}$, $j = 1, 2, 3, 4$. A factorization of a graph G into n isomorphic caterpillars is decomposition of G into caterpillars $R_1, R_2, ..., R_n$, all isomorphic to R, such that every edge of G belongs to exactly one of the caterpillars. If a caterpillar R on $2n$ vertices of diameter 6 factorizes K_{2n}, then it is easy to show that $t_1 + t_2 + t_3 + t_4 + t_5 = 2n + 3$ (Necessary Condition A) and the degree of every vertex of R is at most n (Necessary Condition B).

We state other conditions on the degrees of the spine of the caterpillar R which factorizes a complete graph.

Theorem 1 *If a $[t_1, t_2, t_3, t_4, t_5]$-caterpillar factorizes K_{2n} into n isomorphic copies, then either $t_1 = n$ or $t_1 + t_5 \leq n + 1$ and $t_2 + t_4 \leq n$.*

Proof: Let $R_1, R_2, ..., R_n$ be a factorization of K_{2n} into n isomorphic caterpillars. By the Necessary Condition B, one has $t_1 \leq n$. We prove that $t_1 \leq n - 1$ implies $t_1 + t_5 \leq n + 1$ and $t_2 + t_4 \leq n$. Suppose that x is a vertex such that $deg_{R_1}(x) = t_1$. Since $\sum_{i=1}^{n} deg_{R_i}(x) = 2n - 1$, it is evident that $deg(x) \geq 2$ in some $R_i, i \in \{2, 3, ..., n\}$, which means that $deg_{R_i}(x) \geq t_5$. Further, $\sum_{i=1}^{n} deg_{R_i}(x) \geq t_1 + t_5 + (n - 2)$ and it is easy to show that $t_1 + t_5 \geq n + 2$ gives a contradiction.

It remains to show that $t_2 + t_4 \leq n$. If $t_2 + t_4 \geq n + 1$, by the Necessary Condition A, $t_1 + t_3 + t_5 \leq n + 2$. On the other hand, $t_5 \geq 2$ and the inequality $t_2 + t_4 \geq n + 1$ implies $t_1 + t_3 \geq n + 1$. Hence $t_1 + t_3 + t_5 \geq n + 3$, a contradiction.

In the proof of our second result we will be using the following definition of blended labelling and Theorem A, established by Fronček [1].

Definition A Let G be a graph with $4n + 1$ edges, $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 =$ and $|V_0| = |V_1| = 2n + 1$. Let λ be an injection, $\lambda : V_i \to \{0_i, 1_i, ..., (2n)_i\}$, $i = 0, 1$. The pure length of an edge (x_i, y_i) with $x_i, y_i \in V_i$, where $i \in \{0, 1\}$, is defined as $l_{ii}(x_i, y_i) = \min\{|\lambda(x_i) - \lambda(y_i)|, 2k + 1 - |\lambda(x_i) - \lambda(y_i)|\}$ for $i = 0, 1$ and the mixed length of an edge (x_0, y_1) as $l_{01}(x_0, y_1) = (\lambda(y_1) - \lambda(x_0)) \bmod 2n + 1$ for $x_0 \in V_0$, $y_1 \in V_1$. Then G has a blended labelling if

(i) $\{l_{ii}(x_i, y_i) | (x_i, y_i) \in E(G)\} = \{1, 2, ..., n\}$ for $i = 0, 1$,

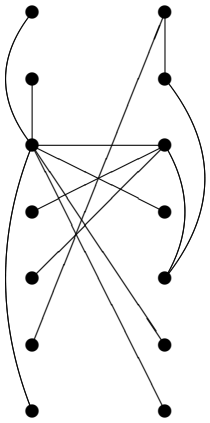
(ii) $\{l_{01}(x_0, y_1) | (x_0, y_1) \in E(G)\} = \{0, 1, 2, ..., 2n\}$.

Theorem A If G with $4n+1$ edges has a blended labelling, then there exists a bi-cyclic decomposition of K_{4n+2} into $2k+1$ copies of G.

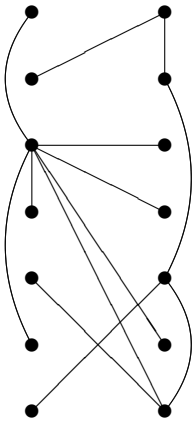
We show 19 different factorizations of K_{14} into isomorphic caterpillars of diameter 6 with a vertex of degree 7.

Theorem 2 *A $(7,2,3,3,2)$-caterpillar, a $(7,s_1,s_2,s_3,s_4)$-caterpillar, a $(u_1,7,u_2,u_3,u_4)$-caterpillar, a $(v_1,v_2,7,v_3,v_4)$-caterpillar of order 14, where $2 \leq s_i,u_j,v_k \leq 4$, $i,j,k=1,2,3,4$, $s_1 \neq 2$, $v_2 \neq 2$, factorizes K_{14}.*

Proof: There are 19 (up to isomorphism) different caterpillars satisfying the conditions: $R^1=(7,4,2,2,2)$, $R^2=(7,3,3,2,2)$, $R^3=(7,3,2,3,2)$, $R^4=(7,3,2,2,3)$, $R^5=(7,2,3,3,2)$, $R^6=(4,7,2,2,2)$, $R^7=(3,7,3,2,2)$, $R^8=(3,7,2,3,2)$, $R^9=(3,7,2,2,3)$, $R^{10}=(2,7,4,2,2)$, $R^{11}=(2,7,3,3,2)$, $R^{12}=(2,7,3,2,3)$, $R^{13}=(2,7,2,4,2)$, $R^{14}=(2,7,2,3,3)$, $R^{15}=(2,7,2,2,4)$, $R^{16}=(3,3,7,2,2)$, $R^{17}=(3,2,7,3,2)$, $R^{18}=(2,4,7,2,2)$, $R^{19}=(2,3,7,3,2)$. Since the caterpillars in the following figures allows a blended labelling, by Theorem A, R^i, $i=1,2,...,19$, factorize K_{14}.



$(7,4,2,2,2)$



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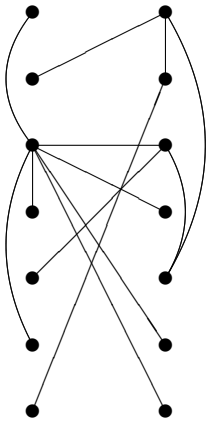

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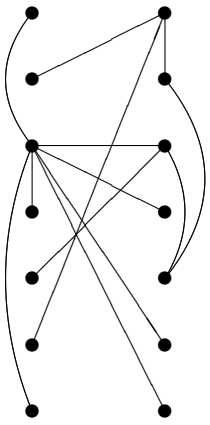

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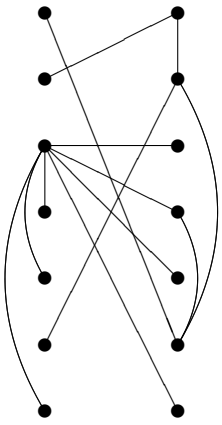

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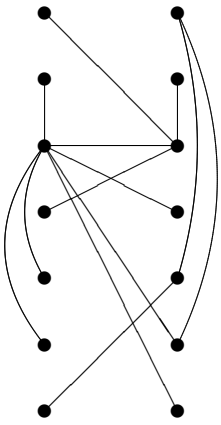
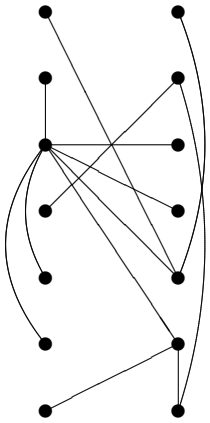
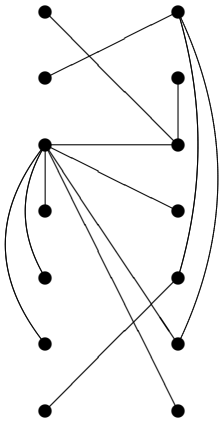


$(4,7,2,2,2)$



$(3, 7, 3, 2, 2)$



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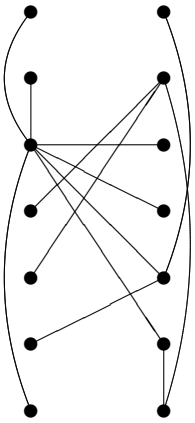

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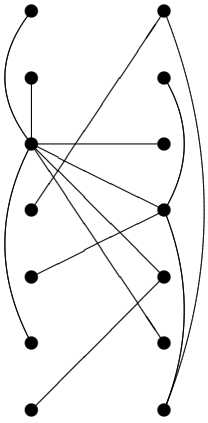
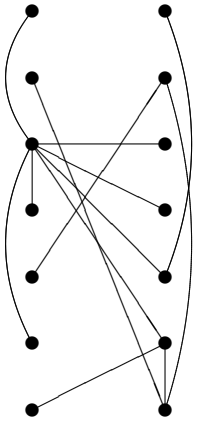


$(2, 7, 4, 2, 2)$



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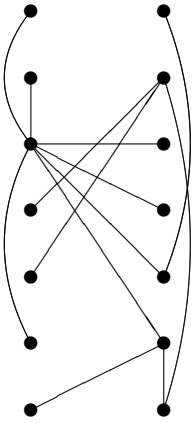
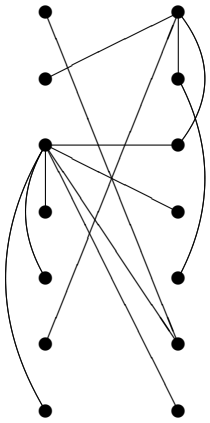
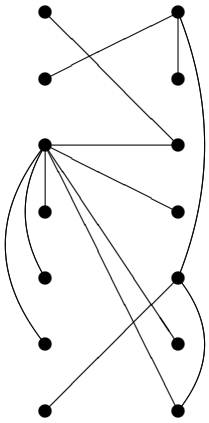


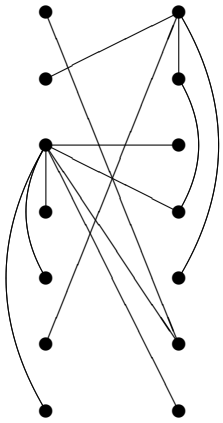
$(2, 7, 3, 2, 3)$



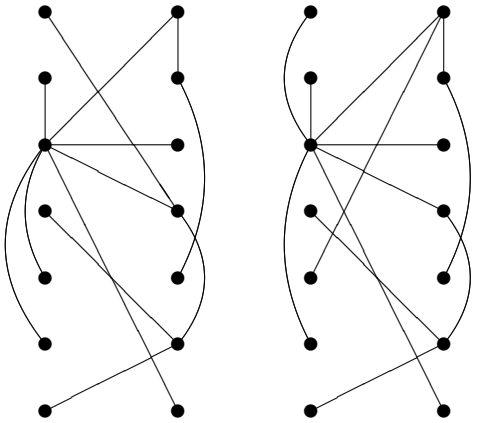
$(2, 7, 2, 4, 2)$



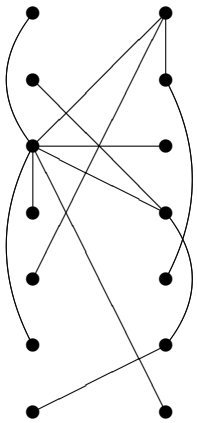
$(2, 7, 2, 3, 3)$



$(2, 7, 2, 2, 4)$



$(3, 3, 7, 2, 2)$ $(3, 2, 7, 3, 2)$ $(2, 4, 7, 2, 2)$



$(2, 3, 7, 3, 2)$

Finally, we would like to formulate the following conjecture.

Conjecture 1 Except for the caterpillars presented in Theorem 2, there exist no other caterpillars of diameter 6 with the maximum degree 7 allowing an isomorphic factorization of K_{14}.

Acknowledgement The research was supported by the VEGA Grant No. 1/2004/05 and the APVT Grant No. 20–023302.

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Author:

Tomáš Vetrík,
Department of Mathematics,
Faculty of Civil Engineering,
Slovak University of Technology,
Radlinského 11,
813 68 Bratislava, Slovakia.
e-mail: vetrik@math.sk