One-dimensional loss networks and conditioned $M/G/\infty$ queues

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Abstract

We study one dimensional continuous loss networks with length distribution G and cable capacity C. We prove that the unique stationary distribution η_L of the network for which the restriction on the number of calls to be less than C is imposed only in the segment $[-L, L]$ is the same as the distribution of a stationary $M/G/\infty$ queue conditioned to be less than C in the time interval $[-L, L]$. For distributions G which are of phase-type (= absorbing times of finite state Markov processes) we show that the limit as $L \to \infty$ of η_L exists and is unique. The limiting distribution turns out to be invariant for the infinite loss network. This was conjectured by Kelly (1991).

Key words: Loss networks, Poisson process, projection method, stationary distribution, quasi stationary distributions.

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1 Introduction

Kelly (1991) introduced a continuous unbounded loss network described as follows. Imagine that users are arranged along an infinitely long cable and that a call between two points on the cable s_1, $s_2 \in \mathrm{R}$ involves just that section of the cable between s_1 and s_2. Past any point along its length the cable has the capacity to carry simultaneously up to C calls: a call attempt between s_1, $s_2 \in \mathrm{R}$, $s_1 < s_2$, is lost if past any point of the interval $[s_1, s_2]$ the cable is already carrying C

calls. Suppose that calls are attempted at points in R following a homogeneous Poisson process with rate κ. Assume that the section of the cable needed by a call has distribution G with finite mean μ^{-1} and the duration of a call has exponential distribution with mean one. Assume that the location of a call, the cable section needed and its duration are independent. Let $m(s,t)$ be the number of calls in progress past point s on the cable at time t. Kelly (1991) claims that $((m(s,t), s \in \mathrm{R}), t \geq 0)$ defines a Markov process and he conjectures that this process has a unique invariant measure, constructed as follows: let $X(s)$ be the number of customers at time s in a stationary $M/G/\infty$ queue with arrival rate κ and with service distribution G. Then $(X(s), s \in \mathrm{R})$ is a stochastic process, stationary with respect to the parameter s. Now condition the whole process $(X(s), s \in \mathrm{R})$ on the event $\{X(s) \leq C, s \in [-L, L]\}$ and let $L \to \infty$. The limiting conditioned process would be the unique invariant measure for $((m(s,t), s \in \mathrm{R}), t \geq 0)$.

We first use a projection method introduced by Kurtz (1989) and studied by Garcia (1995) to construct the loss network process and to show that, for fixed L, the stationary distribution for $m_L(s,t)$ —the process for which $m(s,t) \leq C$ is required only for $s \in [-L, L]$— is the same as $\tilde{\eta}_L$ —the distribution of $X(\cdot)$ conditioned to be less than C in $[-L, L]$.

The problem left is to show that, as L goes to infinity, the limit of $\tilde{\eta}_L$ exists and is unique. We show the existence and uniqueness of the limit for a particular class of service time distributions G. To describe this class let $U = \{1, \dots, k\}$, ϕ be a probability distribution on U and Φ be a positive function on U; let $q(i,j)$ be transition rates for a Markov process in U. Assume that $\Phi(i)$ is the rate at which the process jumps to some state outside of U when being at $i \in U$. Define

$$G(s) = G_{U,\phi,\Phi,q}(s) = \sum_{i \in U} \phi(i) \mathrm{P}(Y_s \in U \mid Y_0 = i)$$

In other words, G is the distribution of the absorption time of a transient Markov chain in a finite state space. This class of distributions is called phase-type (Neuts (1981)) and contains in particular mixtures of Gamma distributions and many other distributions. The idea of the proof is somehow present in Kelly (1987) where the exponential case ($G(s) = 1 - \exp(-\lambda s)$, for some $\lambda > 0$) is treated and in Ziedins (1987) who mimics the behavior of the invariant measure for the loss network with a birth and death chain. In both cases they related the desired limit with the quasi stationary distribution of a finite state Markov chain. In the exponential case the process $X(s)$ is Markov itself and theorems for limiting behavior of finite state Markov processes conditioned to stay in a certain set as in Darroch and Seneta (1966) (1967) can be applied. The limit is called a quasi stationary distribution (qsd). For arbitrary G's, $X(s)$ is not Markov, but using the fact that G is phase-type we can introduce a finite state Markov process ξ_t that keeps track of the "current" service rate of each customer in the system. In particular $X(s)$, the number of customers in the system at time s can be expressed as a function of ξ_s. Then applying the quasi stationary limit theorems we obtain the existence and uniqueness of the limit of the distribution of $X(s)$ conditioned on being less than C in the time interval $[-L, L]$.

At the end of the paper a continuous percolation argument is used to show the result under a different set of hypotheses: that G has a finite third moment and that the arrival rate κ is sufficiently small. The percolation argument can be also used to prove that the process $m(s,t)$ is ergodic, that is, starting from no matter which distribution, it will converge to a unique invariant distribution. Ycart (1993) proved ergodicity for the discrete one dimensional case $s \in \mathbb{Z}$ with $G(s) = \mathbf{1}\{s \leq 2\}$, that is, when the calls have deterministic length 2. These are the only results we know about ergodicity of this process. The difficulty here is that loss networks are not attractive processes and hence the usual coupling techniques do not work.

The paper is organized as follows. In Section 2 we construct the process using the projection model and show that for fixed L, there is a unique invariant distribution for loss network for which the restriction on the number of calls to be less than C is imposed only in the cable interval $[-L, L]$. In that section we also show that this invariant distribution coincides with the distribution of the stationary process $X(\cdot)$ conditioned to be less than C in the time interval $[-L, L]$.

In Section 3 we recall some results on quasi stationary distributions for finite state Markov processes of Darroch and Seneta (1967) in a convenient form.

In Section 4 we introduce a Markov process ξ_s with the property that in the invariant (for t) non restricted case $m(s,t)$ can be expressed as a function of ξ_s and show the existence and uniqueness of the limit as $L \to \infty$ of $\xi_\cdot$ conditioned to $m(s,t) \leq C$ for $s \in [-L, L]$.

In Section 5 we discuss briefly the percolation argument to show ergodicity of the loss network for κ sufficiently small.

2 Representation through the projection method

The loss network process described above can be represented through the projection method introduced by Kurtz (1989) and studied by Garcia (1995). The basic idea is the construction of the loss network process on $\mathrm{R} \times [0, \infty)$ from a higher dimensional Poisson process on $\mathrm{R} \times [0, \infty)^3$ by constructing a sequence of random sets of the higher dimensional space and projecting the points of the Poisson process lying in that set onto the lower dimensional subspace. Let N be a Poisson process on $\mathrm{R} \times [0, \infty)^3$ with mean measure $\nu = \kappa m \times \mu_G \times m \times \mu_e$, where m is Lebesgue measure, μ_G is the measure on $[0, \infty)$ corresponding to G and μ_e is the exponential distribution on $[0, \infty)$. We need to construct a family of random sets $\Delta_{m(t)} \subset \mathrm{R} \times [0, \infty)^3$ and a family of corresponding point processes $\widetilde{M}_{m(t)}$ by projecting the points of N that lie in $\Delta_{m(t)}$ onto $\mathrm{R} \times [0, \infty)$. Consider the following system:

$$\begin{cases} \Delta_{m(t)} = \{(x, y, z, s) \in \mathrm{R} \times [0, \infty)^3; m(x, z) \leq C, y \geq 0, z \leq t, s > t - z\} \\ \widetilde{M}_{m(t)}(B) = N(\Delta_{m(t)} \cap (B \times [0, \infty)^2)), \quad B \in \mathcal{B}(\mathrm{R} \times [0, \infty)) \\ m(x, t) = \int_{-\infty}^{x} \int_{x-\eta}^{\infty} d\widetilde{M}_{m(t)}(\eta, w). \end{cases} \tag{2.1}$$

Then, $\widetilde{M}_{m(t)}$ is the loss network process of completed calls and $m(x, t)$ is the number of calls in progress past point x on the cable at time t. It is easy to see that m satisfies the assumptions given in Section 1.

Theorem 2.1 *There is a unique solution for the system (2.1).*

Proof: Let $\mathcal{C}$ be the space of functions $f : \mathrm{R} \times [0, \infty) \to \{0, 1, 2, \ldots, C\}$, and $\mathcal{H}$ be the space of $\mathcal{C}$-valued random variables. For $m \in \mathcal{H}$, define its norm as

$$\| m \|_t = \int_0^t \sup_x \mathrm{E}[|m(x, u)|] \, du.$$

Note that (2.1) is equivalent to $m = \phi(m)$ where $\phi : \mathcal{C} \to \mathcal{C}$ is given by

$$\phi(m)(x, t) = N(\{(\eta, y, z, s); m(\eta, z) \leq C, \eta \leq x, y \geq x - \eta, z \leq t, s > t - z\}). \tag{2.2}$$

For any $m \in \mathcal{C}, t \geq 0$, we have $\Delta_{m(t)} = B_{m(t)} \setminus A_{m(t)}$, where

$$A_{m(t)} = \{(x, y, z, s); m(x, z) \leq C, y \geq 0, z \leq t, s \leq t - z\} \tag{2.3}$$

$$B_{m(t)} = \{(x, y, z, s); m(x, z) \leq C, y \geq 0, z \leq t\} \tag{2.4}$$

and the sets $A_{m(t)}$ and $B_{m(t)}$ given by (2.3) and (2.4) are predictable as defined by Kurtz (1983). Define $\mathcal{H}_u = \sigma(N(A) : A \in \mathcal{B}, A \subset \mathrm{R} \times [0,u])$. We define the σ-algebra of *predictable* sets as $\mathcal{P} \subset \mathcal{F} \times \mathcal{B}(\mathrm{R} \times [0,\infty))$ the σ-algebra generated by $\mathcal{P}_0 = \{H \times A \times (u,\infty) : H \in \mathcal{H}_u, A \in \mathcal{B}(\mathrm{R}), u \geq 0\} \cup \{\Omega \times A \times [0,\infty) : A \in \mathcal{B}(\mathrm{R})\}$. Letting

$$
\begin{aligned}
B^l_{m(t)} &= \{(x,y,z,s); x \in \mathrm{R}, y \geq 0, z \leq l^{-1}\} \\
&\quad \cup \cup_{k=1}^{[tl]} \{(x,y,z,s); m(x,\frac{k}{l}) < C, y \geq 0, \frac{k}{l} < z \leq \frac{k+1}{l}\}
\end{aligned}
$$

and

$$
\begin{aligned}
A^l_{m(t)} &= \{(x,y,z,s); x \in \mathrm{R}, y \geq 0, z \leq l^{-1}, s \leq t\} \\
&\quad \cup \cup_{k=1}^{[tl]} \{(x,y,z,s); m(x,\frac{k}{l}) < C, y \geq 0, \frac{k}{l} < z \leq \frac{k+1}{l}, s \leq t - \frac{k}{l}\}
\end{aligned}
$$

we have $B^l_{m(t)} \to B_{m(t)}$ and $A^l_{m(t)} \to A_{m(t)}$ a.s. as $l \to \infty$ and $m(x,u-)$ is $\{\mathcal{H}_u\}$-measurable.

Uniqueness: Let m_1 and m_2 be two solutions of (2.1), then

$$
\begin{aligned}
\| \phi(m_1) - \phi(m_2) \|_t &= \int_0^t \sup_x \mathrm{E}[|m_1(x,u) - m_2(x,u)|]\, du \\
&= \int_0^t \sup_x \mathrm{E}[\int_{-\infty}^x \int_{x-\eta}^\infty T(\widetilde{M}_{m_1(u)} - \widetilde{M}_{m_2(u)}, (d\eta, dw))]\, du \qquad (2.5)
\end{aligned}
$$

where $T(\xi, B)$ denotes the total variation of the measure ξ on the set B. Moreover, by the predictability of the sets $A_{m_1(t)}$, $B_{m_1(t)}$, $A_{m_2(t)}$ and $B_{m_2(t)}$ we have

$$
\begin{aligned}
&\mathrm{E}[T(\widetilde{M}_{m_1(t)} - \widetilde{M}_{m_2(t)}, B)] \\
&= \mathrm{E}[N((\Delta_{m_1(t)} \cup \Delta_{m_2(t)}) \cap (B \times [0,\infty)^2))] - \mathrm{E}[N(\Delta_{m_1(t)} \cap (B \times [0,\infty)^2))] \\
&\quad + \mathrm{E}[N((\Delta_{m_1(t)} \cup \Delta_{m_2(t)}) \cap (B \times [0,\infty)^2))] - \mathrm{E}[N(\Delta_{m_2(t)} \cap (B \times [0,\infty)^2))] \\
&= \mathrm{E}[\nu((\Delta_{m_1(t)} \cup \Delta_{m_2(t)}) \cap (B \times [0,\infty)^2))] - \mathrm{E}[\nu(\Delta_{m_1(t)} \cap (B \times [0,\infty)^2))] \\
&\quad + \mathrm{E}[\nu((\Delta_{m_{1(t)}} \cup \Delta_{m_2(t)}) \cap (B \times [0,\infty)^2))] - \mathrm{E}[\nu(\Delta_{m_2(t)} \cap (B \times [0,\infty)^2))] \\
&= \kappa \int_B \int_{[0,\infty)} \int_0^t \int_{t-z}^\infty |\mathrm{E}[\mathbf{I}_{[m_1(x,z) \leq C]} - \mathbf{I}_{[m_2(x,z) \leq C]}|] dx\, d\mu_G(y)\, dz\, d\mu_e(s) \\
&= \kappa \int_B \int_0^\infty \int_0^t \mathrm{E}[|\mathbf{I}_{[m_1(x,z) \leq C]} - \mathbf{I}_{[m_2(x,z) \leq C]}|] e^{-(t-z)}\, dz\, d\mu_G(y)\, dx \qquad (2.6)
\end{aligned}
$$

By substituting (2.6) into (2.5) we get

$$
\| \phi(m_1) - \phi(m_2) \|_t = \|m_1 - m_2\|_t
$$

$$
\begin{aligned}
&= \kappa \int_0^t \{\sup_x \int_{-\infty}^x \int_{x-\eta}^{\infty} \int_0^u \mathrm{E}[|\mathbf{I}_{[m_1(\eta,z)\le C]} - \mathbf{I}_{[m_2(\eta,z)\le C]}|]e^{-(u-z)}\, dz\, d\mu_G(y)\, d\eta\}du \\
&\le \kappa \int_0^t \{\sup_x \int_{-\infty}^x \int_{x-\eta}^{\infty} \int_0^u \mathrm{E}[|m_1(\eta,z) - m_2(\eta,z)|]e^{-(u-z)}\, dz\, d\mu_G(y) d\eta\}du \\
&\le \kappa \int_0^t \{\int_0^u \sup_\eta \mathrm{E}[|m_1(\eta,z) - m_2(\eta,z)|]dz\}\{\sup_x \int_{-\infty}^x \int_{x-\eta}^{\infty} d\mu_G(y) d\eta\}du \\
&\le \kappa\mu^{-1} \int_0^t \|m_1 - m_2\|_u du \\
&= 0
\end{aligned}
$$

by Gronwall's inequality.

Existence: Consider $m_0(t) \equiv 0$, and $m_{i+1} = \phi(m_i)$ for $i \ge 0$. Then, by the same argument as in the uniqueness proof, we have

$$
\begin{aligned}
\|m_{i+1} - m_i\|_t &= \|\phi(m_i - \phi(m_{i-1})\|_t \\
&\le \kappa\mu^{-1} \int_0^t \|m_i - m_{i-1}\|_u du \qquad (2.7)
\end{aligned}
$$

for $i \ge 1$ and

$$
\begin{aligned}
\|m_1 - m_0\|_t &= \|\phi(m_0)\|_t \\
&= \int_0^t \sup_x \mathrm{E}[\int_{-\infty}^x \int_{x-\eta}^{\infty} \int_0^u \int_{u-z}^{\infty} N(d\eta, dy, dz, ds)]du \\
&= \kappa \int_0^t \sup_x [\int_{-\infty}^x \int_{x-\eta}^{\infty} \int_0^u e^{-(u-z)} dz\, d\mu_G(dy)\, d\eta]du \\
&= \kappa \int_0^t [\int_0^u e^{-(u-z)}\, dz \quad \sup_x \int_{-\infty}^x (1 - G(x-\eta))d\eta]\, du \\
&= \kappa\mu^{-1} \int_0^t \int_0^u (1 - F(u-z))dz\, du \\
&< \kappa\mu^{-1} t. \qquad (2.8)
\end{aligned}
$$

By induction, we can prove that $\|m_{n+1} - m_n\|_t \le (\kappa\mu^{-1})^{n+1}[t^{n+1}/(n+1)!]$. Therefore, $\{m_n\}$ converges and

$$
m(t) = \sum_{n=1}^{\infty} (m_n(t) - m_{n-1}(t))
$$

is a solution of (2.1). □

Note that N is invariant under translation $S_{(0,0,-T,0)}$, consequently constructing the process N as a Poisson process on $\mathrm{R} \times [0,\infty) \times \mathrm{R} \times [0,\infty)$ with mean measure $\kappa m \times \mu_G \times m \times \mu_e$, where m is Lebesgue measure on R, μ_G is the measure on $[0,\infty)$ corresponding to G and μ_e is the exponential measure on $[0,\infty)$ and considering the backward simulation of the loss network process

$$\begin{cases} \Delta_{T,t} = \{(x,y,z,s); m_T(x,z) \le C, y \ge 0, -T \le z \le t-T, s > t-T-z\} \\ \widetilde{M}_{T,t}(B) = N(\Delta_{T,t} \cap (B \times [0,\infty)^2)), \quad B \in \mathcal{B}(\mathrm{R} \times [0,\infty)) \\ m_T(x,t) = \int_{-\infty}^{x} \int_{x-\eta}^{\infty} d\widetilde{M}_{T,t}(\eta, w) \end{cases} \tag{2.9}$$

that is,

$$m_T(x,t) = \int_{-\infty}^{x} \int_{x-\eta}^{\infty} \int_{-T}^{t-T} \int_{t-T-z}^{\infty} \mathbf{I}_{\{m_T(x,z) \le C\}} N(d\eta, dw, dz, ds). \tag{2.10}$$

we shall have $m(T) \stackrel{\mathcal{D}}{=} m_T(T)$.

The $M/G/\infty$ queue. Let $X(x)$ be the number of customers at time $x \in \mathrm{R}$ in a stationary $M/G/\infty$ queue with arrival rate κ and service distribution G. We can construct the process $\{X(x), x \in \mathrm{R}\}$ by the projection method as

$$X(x) = \int_{-\infty}^{x} \int_{x-\eta}^{\infty} \int_{-\infty}^{0} \int_{-z}^{\infty} N(d\eta, dy, dz, ds). \tag{2.11}$$

Comparing (2.10) as $L \to \infty$ and (2.11) it is intuitive to conjecture that the invariant measure for the process m is the limit as $L \to \infty$ of the process X conditioned on the event $\{X(x) \le C, x \in [-L, L]\}$.

First we prove that the unique stationary distribution of the network for which the restriction of the number of calls to be less than C is imposed only in the segment $[-L, L]$ is the same as the distribution of a stationary $M/G/\infty$ queue conditioned to be less than C in the time interval $[-L, L]$.

Let $\tilde{\eta}$ be the distribution of the Poisson process with mean measure $\kappa m \times \mu_G$ on $\mathrm{R} \times [0,\infty)$ and

$$\widetilde{C}_C^L = \{n \in \mathcal{N}(\mathrm{R} \times [0,\infty)); n(\{(y,z); y \le x \le y+z\}) \le C, \forall x \in [-L, L]\}.$$

Define a measure on $\widetilde{C}_C^L$ by

$$\tilde{\eta}_L = \frac{\tilde{\eta}}{\tilde{\eta}(\widetilde{C}_C^L)}. \tag{2.12}$$

Define

$$\begin{cases} \Delta_t^L = \{(\eta, y, z, s); m(\eta, z) \le C \text{ if } |\eta| \le L, y \ge 0, z \le t, s > t - z\} \\ \widetilde{M}_t^L(B) = N(\Delta_t \cap ((B \times [0,\infty)^2) \\ m^L(x,t) = \int_{-\infty}^x \int_{x-\eta}^\infty \widetilde{M}_t^L(d\eta, dy) \end{cases} \tag{2.13}$$

we have

$$\begin{aligned} m^L(x,t) &= \int_{-\infty}^{x\wedge(-L)} \int_{x-\eta}^\infty \int_0^t \int_{t-z}^\infty N(d\eta, dy, dz, ds) \\ &+ \int_{x\wedge(-L)}^{x\wedge L} \int_{x-\eta}^\infty \int_0^t \int_{t-z}^\infty \mathbf{I}_{\{m^L(\eta,z)\le C\}} N(d\eta, dy, dz, ds) \\ &+ \int_{x\wedge -L}^{x} \int_{x-\eta}^\infty \int_0^t \int_{t-z}^\infty N(d\eta, dy, dz, ds) \end{aligned} \tag{2.14}$$

which restricts only points along $[-L, L]$ to carry at most C calls. The generator of the restricted loss process defined by (2.14) takes the form

$$\begin{aligned} A_L f(n) &= \int_{\mathrm{R}\times[0,\infty)} (f(n+\delta_{(w_1,w_2)}) - f(n)) \mathbf{I}_{[\int_{-\infty}^x \int_{x-\eta}^\infty (n+\delta_{(w_1,w_2)})(d\eta,dy)\le C, -L\le x\le L]} \kappa\, dw_1 d\mu_G(w_2) \\ &+ \int_{\mathrm{R}\times[0,\infty)} (f(n-\delta_{(w_1,w_2)}) - f(n)) n((dw_1, dw_2)). \end{aligned} \tag{2.15}$$

Lemma 2.2 *The distribution $\widetilde{\eta}_L$ given by (2.12) is the unique stationary distribution for the restricted loss network given by (2.14).*

Proof: It follows by Echeverria's theorem (see Ethier and Kurtz (1986), Theorem 4.9.17) that $\widetilde{\eta}_L$ is a stationary distribution for m^L if

$$\int_{\widetilde{C}_C^L} A_L f(n) \widetilde{\eta}_L(dn) = 0. \tag{2.16}$$

In fact, it is enough to prove (2.16) for $f(n) = \int g(z) n(dz)$. In this case,

$$\begin{aligned} &\int_{\widetilde{C}_L^C} A_L f(n) \widetilde{\eta}_L(dn) \\ &= \frac{1}{\widetilde{\eta}(\widetilde{C}_L^C)} \int_{\widetilde{C}_L^C} [\int_{\mathrm{R}\times[0,\infty)} g(w_1,w_2) \mathbf{I}_{[\int_{-\infty}^x \int_{x-\eta}^\infty (n+\delta_{(w_1,w_2)})(d\eta,dy)\le C, -L\le x\le L]} \kappa\, dw_1 d\mu_G(w_2) \\ &\quad - \int_{\mathrm{R}\times[0,\infty)} g(w_1,w_2) n((dw_1,dw_2))] \widetilde{\eta}(dn). \end{aligned}$$

Let $\widetilde{N}$ be a Poisson process on $\mathrm{R} \times [0,\infty)$ with mean measure $\kappa m \times \mu_G$, that is, $\widetilde{N}$ has distribution $\widetilde{\eta}$, then

$$
\begin{aligned}
&\int_{\widetilde{C}_L^C} A_L f(n)\widetilde{\eta}(dn) \\
&= \frac{1}{\widetilde{\eta}(\widetilde{C}_L^C)}\mathrm{E}\Big[\int_{\mathrm{R}\times[0,\infty)} g(w_1,w_2)\mathbf{I}_{[\widetilde{N}\in\widetilde{C}_L^C]}\mathbf{I}_{[\int_{-\infty}^{x}\int_{x-\eta}^{\infty}(\widetilde{N}+\delta_{(w_1,w_2)})(d\eta,dy)\le C,-L\le x\le L]}\kappa\, dw_1 d\mu_G(w_2) \\
&\qquad - \int_{\mathrm{R}\times[0,\infty)} g(w_1,w_2)\mathbf{I}_{[\widetilde{N}\in\widetilde{C}_L^C]}\widetilde{N}((dw_1,dw_2))\Big] \\
&= \frac{1}{\widetilde{\eta}(\widetilde{C}_L^C)}\mathrm{E}\Big[\int_{\mathrm{R}\times[0,\infty)} g(w_1,w_2)\mathbf{I}_{[\widetilde{N}+\delta_{(w_1,w_2)}\in\widetilde{C}_L^C]}\kappa\, dw_1 d\mu_G(w_2) \\
&\qquad - \int_{\mathrm{R}\times[0,\infty)} g(w_1,w_2)\mathbf{I}_{[\widetilde{N}\in\widetilde{C}_L^C]}\widetilde{N}((dw_1,dw_2))\Big].
\end{aligned}
$$

For each m, let $\{A_i^m\}$ be a partition of $\mathrm{R}\times[0,\infty)$ into rectangles, with the mesh of the partition going to zero as $m\to\infty$, and let $\mathcal{G}_i^m = \sigma(\widetilde{N}(B); B\subset A_i^m)$. Partitioning the expectations and conditioning on $\mathcal{G}_i^m$, eventually the partition will isolate the points and since $\widetilde{N}+\delta_{(w_1,w_2)}$ has the same distribution as $\widetilde{N}$ conditioned on having a point sited at (w_1,w_2), the result follows.

Uniqueness follows from a regeneration argument similar to that used by Lotwick and Silverman (1981). Let m_1^L be the loss network evolving in time as described by system (2.13) and initial distribution $\delta_\emptyset$, that is, no calls at time 0. Let m_2^L be a similar loss network started in equilibrium, i.e., having as its initial distribution $\widetilde{\eta}_L$. We would like to investigate the convergence of the process m_1^L by considering the double process (m_1^L, m_2^L). We can construct both processes m_1^L and m_2^L by using the same underlying Poisson process N by the projection method. Let N_0 be a Poisson point process on $\mathrm{R}\times[0,\infty)^3$ with mean measure $\widetilde{\eta}_L\times\delta_{\{0\}}\times\mu_e$ and denote by N_0+N the superposition of the two processes, then

$$
\begin{aligned}
m_1^L(x,t) &= \int_{-\infty}^{x\wedge(-L)}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty} N(d\eta,dy,dz,ds) \\
&\quad + \int_{x\wedge(-L)}^{x\wedge L}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty} \mathbf{I}_{\{m^L(\eta,z)\le C\}}N(d\eta,dy,dz,ds) \qquad (2.17)\\
&\quad + \int_{x\wedge -L}^{x}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty} N(d\eta,dy,dz,ds)
\end{aligned}
$$

and

$$
m_2^L(x,t) = \int_{-\infty}^{x\wedge(-L)}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}(N_0+N)(d\eta,dy,dz,ds)
$$

$$+\int_{x\wedge(-L)}^{x\wedge L}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}\mathbf{I}_{\{m^L(\eta,z)\le C\}}(N_0+N)(d\eta,dy,dz,ds) \tag{2.18}$$
$$+\int_{x\wedge -L}^{x}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}(N_0+N)(d\eta,dy,dz,ds).$$

Note that there exists T_0 such that

$$m_1(x,T_0)=m_2(x,T_0)=0,\ \text{for all } x\in[-L,L]$$

since

$$\begin{aligned}
&\mathrm{E}[\int_{x\wedge(-L)}^{x\wedge L}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}\mathbf{I}_{\{m^L(\eta,z)\le C\}}(N_0+N)(d\eta,dy,dz,ds)] = \\
&= \quad \int_{x\wedge(-L)}^{x\wedge L}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}\mathbf{I}_{\{m^L(\eta,z)\le C\}}(\widetilde{\eta_L}\times\delta_{\{0\}}\times\mu_e)(d\eta,dy,dz,ds) \\
&\quad + \int_{x\wedge(-L)}^{x\wedge L}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}\mathbf{I}_{\{m^L(\eta,z)\le C\}}(m\times\mu_G\times m\times\mu_e)(d\eta,dy,dz,ds) \\
&\quad < +\infty.
\end{aligned}$$

Since at any time t there are only finitely many calls in the system

$$\mathrm{P}(m_1^L(x,t)=m_2^L(x,t)=0,\forall x\in[-L,L])>0$$

and $\mathrm{P}(T_0<+\infty)=1$. Moreover, for t sufficiently large, since N_0 has finitely many points

$$\int_{-\infty}^{x\wedge(-L)}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}(N_0+N)(d\eta,dy,dz,ds)=\int_{-\infty}^{x\wedge(-L)}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}N(d\eta,dy,dz,ds)$$

and

$$\int_{x\wedge L}^{x}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}(N_0+N)(d\eta,dy,dz,ds)=\int_{x\wedge L}^{x}\int_{x-\eta}^{\infty}\int_0^t\int_{t-z}^{\infty}N(d\eta,dy,dz,ds).$$

Therefore we can couple the process m_1^L and m_2^L after some time T_0 and conclude that $\widetilde{\eta}_L$ is the stationary distribution for m^L.

Note that the above argument is valid no matter what the initial distribution for the process m_1^L. □

3 Quasi-stationary distributions for Markov processes

Let S be a denumerable space and $S_1\subset S$ be finite and strictly contained in S. We consider an irreducible ergodic Markov process ξ_r on S with stationary distribution α. We assume that there exists $x \in S_1$ and $y \in S \setminus S_1$ with a positive rate of jump from x to y. Let $Q = \{Q(x,y) : x, y \in S_1\}$ be the transition rates matrix of ξ_r restricted to the states of S_1. Here $Q(x,y) \geq 0$ for $x \neq y$ and our hypotheses imply

$$Q(x,x) < -\sum_{y \in S_1 : y \neq x} Q(x,y) \tag{3.1}$$

Let ρ be the eigenvalue of Q with maximal real part. Under these hypotheses, ρ is real, simple and $\rho < 0$. Let v' (the transpose of v) and w be the left and right eigenvectors of Q corresponding to the eigenvalue ρ. In correspondence, for any fixed t, v' and w are the left and right eigenvectors of e^{tQ} corresponding to the eigenvalue $e^{t\rho}$. Without loss of generality we can assume

$$\sum_{x \in S_1} v(x) = 1; \quad \sum_{x \in S_1} v(x)w(x) = 1 \tag{3.2}$$

In the following lemma we put in a convenient form a straightforward extension of results by Darroch and Seneta (1967) and Pollett (1988).

Lemma 3.1 *If the rates matrix Q of the process ξ_r restricted to S_1 satisfies (3.1) and (3.2), then for $t(1) < t(2) < \ldots < t(n)$ and $y, x_1, \ldots, x_n \in S_1$ the following limit holds:*

$$\begin{aligned}
& \lim_{L \to \infty} \mathrm{P}(\xi_{t(i)} = x_i, 1 \leq i \leq n \mid \xi_r \in S_1, r \in [-L, L], \xi_{-L} = y) \\
= \;& \frac{v(x_1) \prod_{i=1}^{n-1} \exp\left[\sum_i Q(t(i+1) - t(i))\right](x_i, x_{i+1}) w(x_n)}{\exp[\rho(t(n) - t(1))]} \\
= \;& v(x_1)w(x_1) \prod_{i=1}^{n-1} \exp\left[\widetilde{Q}(t(i+1) - t(i))\right](x_i, x_{i+1}) \\
= \;& v(x_n)w(x_n) \prod_{i=1}^{n-1} \exp\left[Q^*(t(n-i+1) - t(n-i))\right](x_{n-i+1}, x_{n-i})
\end{aligned} \tag{3.3}$$

where

$$\widetilde{Q}(x,y) = -\frac{Q(x,y)w(y)}{w(x)} \quad \textit{and} \quad Q^*(x,y) = -\frac{Q(y,x)v(y)}{v(x)}$$

are transition rates matrices for honest Markov processes $\widetilde{X}(n)$ and $X^*(n)$ on S_1. Furthermore, the measure $v(\cdot)w(\cdot)$ is invariant for both $\widetilde{Q}$ and Q^*.

Proof. First write the conditional probability in the left hand side of (3.3) as

$$\frac{\mathrm{P}(\xi_{t(i)} = x_i, 1 \leq i \leq n;\ \xi_r \in S_1, r \in [-L, L] \mid \xi_{-L} = y)}{\mathrm{P}(\xi_r \in S_1, r \in [-L, L] \mid \xi_{-L} = y)} \tag{3.4}$$

Then use the Markov property to write the numerator as

$$
\begin{aligned}
& \mathrm{P}(\xi_r \in S_1, r \in [-L, t(1)]; \xi_{t(1)} = x_1 \mid \xi_{-L} = y) \\
& \times \mathrm{P}(\xi_{t(i)} = x_i,\ 1 \leq i \leq n \mid \xi_{t(1)} = x_1) \\
& \times \mathrm{P}(\xi_r \in S_1, r \in [t(n), L] \mid \xi_{t(n)} = x_n) \\
= \; & \mathrm{P}(\xi_r \in S_1, r \in [-L, t(1)]; \xi_{t(1)} = x_1 \mid \xi_{-L} = y) \\
& \times \prod_{i=1}^{n-1} \exp\left[Q(t(i+1) - t(i))\right](x_i, x_{i+1}) \\
& \times \mathrm{P}(\xi_r \in S_1, r \in [t(n), L] \mid \xi_{t(n)} = x_n)
\end{aligned} \tag{3.5}
$$

We compute the denominator by partitioning on all possible values of $\xi_{t(1)}, \xi_{t(n)}$:

$$
\begin{aligned}
& \mathrm{P}(\xi_r \in S_1, r \in [-L, L] \mid \xi_{-L} = y) \\
= \; & \sum_{y_1} \sum_{y_n} \mathrm{P}(\xi_r \in S_1, r \in [-L, t(1)], \xi_{t(1)} = y_1 \mid \xi_{-L} = y) \\
& \times \mathrm{P}(\xi_r \in S_1, r \in [t(1), t(n)], \xi_{t(n)} = y_n) \mid \xi_{t(1)} = y_1) \\
& \times \mathrm{P}(\xi_r \in S_1, r \in [t(n), L] \mid \xi_{t(n)} = y_n) \\
= \; & \sum_{y_1} \sum_{y_n} \mathrm{P}(\xi_r \in S_1, r \in [-L, t(1)], \xi_{t(1)} = y_1 \mid \xi_{-L} = y) \\
& \times \exp\left[(t(n) - t(1))Q\right](y_1, y_n) \\
& \times \mathrm{P}(\xi_r \in S_1, r \in [t(n), L] \mid \xi_{t(n)} = y_n)
\end{aligned}
$$

Darroch and Seneta (1967) gives

$$
\lim_{L \to \infty} \frac{\mathrm{P}(\xi_r \in S_1, r \in [-L, t(1)], \xi_{t(1)} = y_1 \mid \xi_{-L} = y)}{\mathrm{P}(\xi_r \in S_1, r \in [-L, t(1)], \xi_{t(1)} = x_1 \mid \xi_{-L} = y)} = \frac{v(y_1)}{v(x_1)} \tag{3.6}
$$

and that

$$
\lim_{L \to \infty} \frac{\mathrm{P}(\xi_r \in S_1, r \in [t(n), L] \mid \xi_{t(n)} = y_n)}{\mathrm{P}(\xi_r \in S_1, r \in [t(n), L] \mid \xi_{t(n)} = x_n)} = \frac{w(y_n)}{w(x_n)} \tag{3.7}
$$

Then, since the sums are finite, (3.3) reads

$$
\begin{aligned}
& \frac{v(x_1) \prod_{i=1}^{n} \exp\left[Q(t(i+1) - t(i))\right](x_i, x_{i+1}) w(x_n)}{v \exp[(t_n - t_1)Q] w} \\
= \; & \frac{v(x_1) \prod_{i=1}^{n} \exp\left[Q(t(i+1) - t(i))\right](x_i, x_{i+1}) w(x_n)}{\exp[(t_n - t_1)\rho]}
\end{aligned} \tag{3.8}
$$

because v and w are left and right eigenvectors of Q with eigenvalue ρ and $v'w = 1$. The other two identities follow from Lemma 3.3 in Pollett (1988). □

Corollary. *Let $\tilde{\alpha}$ be the law induced on S^{R} by the process $(\xi_s, s \in \mathrm{R})$ with (marginal) invariant distribution α. Let $\tilde{\alpha}_L = \tilde{\alpha}(\cdot \mid \xi_s \in S_1, s \in [-L, L])$. Then $\tilde{\alpha}_L$ converges weakly to a unique distribution $\tilde{\alpha}_\infty$.*

Proof. Let f be a measurable function on S_1^{R} depending on the values of ξ_z for z in a finite interval. Then for L sufficiently large so that the interval determining f is contained in $[-L, L]$,

$$\tilde{\alpha}_L f = \sum_{y \in S_1} \alpha(y) \mathrm{E}(f(\xi) \mid \xi_{-L} = y)$$

Take limits as $L \to \infty$ to both members of the above identity. By Lemma 3.1 the limits of the conditional expectations in the right hand side do not depend on y and converge to the same value that we define as being $\alpha_\infty f$. $\square$

4 Infinite volume limit.

We write $m(r)$ as a function of a Markov process in some state space S, which will be conditioned to stay at certain finite set S_1 during the time interval $[-L, L]$. The idea is to represent the $M/G/\infty$ queue in a Markovian way. We are able to do so when the general distribution G corresponds to the absorption time of a finite state Markov process. Let $U = \{1, \ldots, k\}$. We assume that there exists a Markov process $Y_r \in U \cup \{\emptyset\}$, such that $\emptyset$ is absorbing and the rates are given by a matrix $q = \{q(i,j)\}_{i,j \in U}$, $q(i, \emptyset) = \Phi(i)$ and $q(\emptyset, i) \equiv 0$ and that the service time cumulating distribution function G is given by

$$G(r) = \sum_{i \in U} \varphi(i) \mathrm{P}(Y_r \in U \mid Y_0 = i) \tag{4.1}$$

for some probability distribution φ on U.

Remark. The above phase-type condition includes the Gamma distribution and mixtures of exponentials for instance. For the Gamma distribution with parameters k and λ, we fix $U = \{1, \ldots, k\}$, $\varphi(1) = 1, \varphi(x) = 0$ for $x > 1$, $\Phi(x) = 0$ for $1 \leq x \leq k-1$, $q(x, x+1) = \lambda$ and $\Phi(k) = \lambda$. For the mixture of k exponentials we fix $q(x, y) = 0$ for $y \neq \emptyset$. In this way we have that the exponential with rate $\Phi(x)$ is chosen with probability $\varphi(x)$. Many other distributions can be constructed as the absorption times of a finite state Markov process.

We now define a Markov process ξ_t on $S = U^{\mathrm{N}}$. A typical configuration $\xi \in S$ is a vector with k non negative naturals. $\xi(i)$ is the number of calls in progress whose service time is at state i. Define

$$\xi^{i+}(j) = \begin{cases} \xi(j) & \text{if } j \neq i \\ \xi(i)+1 & \text{if } j = i \end{cases} \qquad \xi^{i-}(j) = \begin{cases} \xi(j) & \text{if } j \neq i \\ \xi(i)-1 & \text{if } j = i \end{cases}$$

Let

$$c(\xi, \xi^{i+}) = \sum_{j \in U} \xi(j) q(j,i) + \varphi(i) \qquad c(\xi, \xi^{i-}) = \xi(i)\Phi(i)\mathbf{1}\{i \geq 1\}$$

The generator of the process is the following

$$\mathcal{L}f(\xi) = \sum_{i \in U} \sum_{\zeta \in \{\xi^{i+}, \xi^{i-}\}} c(\xi, \zeta)[f(\zeta) - f(\xi)] \tag{4.2}$$

The process with the above generator is ergodic and has a unique stationary state. We leave the proof of the next two lemmas to the reader.

Lemma 4.1 *Let $\xi_{(\cdot)} = \{\xi_r : r \in \mathrm{R}\}$ be the process with generator (4.2) in equilibrium, that is, such that the (marginal) distribution of ξ_r does not depend on r. Then, defining $M(\xi_r) = \sum_{i \in U} \xi_r(i)$, the process $M(\xi_{(\cdot)})$ has distribution $\tilde{\eta}$ as defined by (2.12).*

Notice that since the distribution of ξ_r is the corresponding to a stationary Markov process, the equilibrium distribution of the $(M|G|\infty)$ queue is a function of a Markov process.

Lemma 4.2 *The process $M(\xi_{(\cdot)})$ conditioned to the set $\{M(\xi_r) \leq C, r \in [-L, L]\}$ has distribution $\tilde{\eta}_L$.*

Lemma 4.3 *The process $\xi_{(\cdot)}$ conditioned to the event $\{M(\xi_r) \leq C, r \in [-L, L]\}$ converges weakly to a unique distribution. Consequently, $\tilde{\eta}_L$ converges to a unique distribution $\tilde{\eta}_\infty$. Furthermore the limiting conditioned process is also Markovian.*

Proof. Defining $S_1 = \{\xi \in S : M(\xi) \leq C\}$, we can apply the Corollary of Lemma 3.1 to the process ξ_r to obtain the desired convergence. The convergence of $\tilde{\eta}_L$ to $\tilde{\eta}_\infty$ is then a consequence of the previous lemma. Markovianity follows from Lemma 3.1. $\square$

Remark. A similar argument can be applied for discrete $s \in \mathrm{Z}$. It can also be extended to any distribution G with finite support.

5 Percolation argument

For small κ a percolation argument can be used. Let us use a different graphical representation of the process. Let $\zeta = \{\zeta_1, \zeta_2, \ldots\}$ be a κ-homogeneous Poisson process on $\mathrm{R} \times [0, \infty)$, $H_1, H_2, \ldots$ be i.i.d. exponential random variables with $\mathrm{E}[H_1] = 1$ and $W_1, W_2, \ldots$ i.i.d. random variables with common distribution G, the variables H's, W's and the Poisson process being independent. Consider the random rectangles

$$R_i = \{(x, y); \zeta_{i1} \leq x \leq \zeta_{i1} + W_i, \zeta_{i2} \leq y \leq \zeta_{i2} + H_i\},$$

then $\{R_i, i \geq 1\} = \{\zeta_i + S_i, i \geq 1\}$ is a boolean model in the continuum $\mathrm{R} \times [0, \infty)$ (Hall, 1988, p.43) where $S_i = [0, W_i] \times [0, H_i]$ and it represents the independent process of attempted calls. Fix $(x, y) \in \mathrm{R} \times [0, \infty)$, then

$$\mathrm{P}((x, y) \text{ is not covered}) = \mathrm{P}(\text{ for all } i, (x, y) \notin R_i) = e^{-\kappa\mu^{-1}}.$$

Let $v(S_i)$ the content of the largest sphere contained in S_i and $V(S_i)$ the content of the smallest sphere containing S_i, then

$$v(S_i) = \pi(\frac{W_i}{2} \wedge \frac{H_i}{2})^2 \quad \text{and} \quad V(S_i) = \pi(\frac{W_i}{2} \vee \frac{H_i}{2})^2$$

since $\mathrm{E}[v(S_i)] > 0$, if we assume $\mathrm{E}[(\frac{W_i}{2} \vee \frac{H_i}{2})^3] < \infty$, there exists a critical value κ_c such that there is no continuum percolation (Theorem 4.11, Hall, 1988). That is, the number of rectangles in each clump is finite with probability 1. In this case, each $(s, t) \in \mathrm{R} \times [0, \infty)$ belongs to a finite number of random rectangles R_i and the loss network process can be constructed from the independent process by "erasing" the rectangles which lead to more than C calls.

Therefore, we can define

$$T_0(L) = \inf\{t \geq 0; \text{there is no path connecting } [-L, L] \text{ to } \mathrm{R} \times \{t\}\}$$

and if we couple two processes m_1 and m_2, m_1 beginning with an arbitrary measure for point inside $[-L, L]$ and m_2 beginning with $M/G/\infty$ queue restricted to C calls on $[-L, L]$, the two processes will coincide after time $T_0(L)$.

When H corresponds to an exponential distribution it has all moments. Hence, for this argument it is sufficient to ask that the random variable W with distribution G have a finite third moment.

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