Curves and Surfaces Design With Geometric Constraint By Subdivision Scheme

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Abstract - The aim of this paper is to investigate curve or surface design and modeling with geometric constraint by subdivision scheme. To construct a curve with a given length, constraint is employed on the free parameter of a newly proposed 4-point approximating subdivision scheme. Feasibility to this problem is investigated, convergence and smoothness analysis are presented, and more properties about the cases of degenerate or strong-degenerate polygon are discussed. It is shown that the problem of curve design with arc-length constraint can not be solved by using the subdivision scheme for strong-degenerate initial polygon. In the end of the paper, some geometric modeling examples are given to illustrate the efficiency of the mentioned method.

Keywords—**Subdivision; Arc length constraint; Degeneration**

I. Introduction

Common geometric modeling is usually to construct smooth curves and surfaces with given interpolating information, while in some practical applications such as CAD, simulating deformations of objects, animation etc, only interpolation and smoothness are not enough, and some geometrical or physical constraints are necessary. As to arc length constrained curve design, Damme & Wang proposed S_3^1 curve interpolation with constrained length, and gave its error estimation[1], Roulier & Piper investigated parametric curves constrained by boundary conditions and arc length[2]. In this paper, we are trying to design a discrete curve subject to arc length by subdivision schemes.
Subdivision schemes have become important tools in the fields of computer aided geometric design, digital human design, computer fashion design and image process in recent years because it provides a uniform and efficient way to describe smooth curves and surfaces. Its beauty lies in the elegant mathematical formulation and simple implementation. During the last 30 years, from Chakin's "cutting corners" algorithm [3], many subdivision schemes were proposed. Especially we notice that there is a free parameter in each 4-point subdivision scheme, such as 4-point interpolating scheme by Dyn [4], 4-point interpolating and ternary scheme [5], and also four newly proposed 4-point approximating subdivision schemes [6], [7], [8]. The free parameter plays a very important role in the determination of the smoothness of the limit curve and can be used to control the shape of the limit curve. Interpolating schemes have good shape-preserving, but they don't keep good fairing while the free parameter is ranging. So we choose an approximating scheme to generate a smooth curve which satisfies the length constraint and at the same time keeps good shape-preserving by adjusting the free parameter. A successive application of this method is that with human body data and suitable lengths, we can design his clothes. The organization of the paper is as follows. In section 2 we present our constraint problem and give its feasibility. Section 3 gives the convergence and smoothness analysis of the scheme. Some further discussions about strong degeneration are proposed in section 4. And some geometric examples are given in the last part to illustrate the efficiency of our method.

II. Problem and its Feasibility

Reference [7] proposed a 4-point approximating subdivision scheme that the polygon $P^k = (P_i^k)$ is mapped to a refined polygon $P^{k+1} = (P_i^{k+1})$ by applying the following rules,

$$\begin{cases} P_{2i}^{k+1} = -\omega p_{i-1}^k + (\frac{3}{4}+\omega)P_i^k \\ \qquad +(\frac{1}{4}+\omega)P_{i+1}^k - \omega P_{i+2}^k \\ P_{2i+1}^{k+1} = -\omega p_{i-1}^k + (\frac{1}{4}+\omega)P_i^k \\ \qquad +(\frac{3}{4}+\omega)P_{i+1}^k - \omega P_{i+2}^k \end{cases}, \tag{1}$$

and proved scheme (1) generates a limit curve of C^1 for $\omega \in (-\frac{1}{8}, \frac{1}{8})$. If ω changes at each recursion step, (1) will be:

$$\begin{cases} P_{2i}^{k+1} = -\omega_{k+1} p_{i-1}^k + (\frac{3}{4}+\omega_{k+1})P_i^k \\ \qquad +(\frac{1}{4}+\omega_{k+1})P_{i+1}^k - \omega_{k+1} P_{i+2}^k \\ P_{2i+1}^{k+1} = -\omega_{k+1} p_{i-1}^k + (\frac{1}{4}+\omega_{k+1})P_i^k \\ \qquad +(\frac{3}{4}+\omega_{k+1})P_{i+1}^k - \omega_{k+1} P_{i+2}^k \end{cases}. \tag{2}$$

Let $P^0 = \{P_i^0\}_{i=-1}^{n+2}$ be an initial control polygon on plane,

$P^k = \{P_i^k\}_{i=-1}^{2^k n+2}$ be control polygon after k recursive refinement by (2), denote

$$L^k = \sum_{i=0}^{2^k n-1} |P_i^k P_{i+1}^k|,$$

which is the perimeter of control polygon P^k. To design a curve of length l, we construct a monotone number series $\{l_n\}_{n=1}^{\infty}$ such that $lim_{n\to\infty} l_n = l$. And ω_k is determined by

$$L^k = l_k, k = 1, 2, \cdots \tag{3}$$

Constraint equation (3) can be solved by Newton-Raphson algorithm.

Definition 1 The polygon $\{P_i\}_{i=-1}^{n+2}$ is called a degenerate control polygon, if

$$P_{i-1} - 2P_i + 2P_{i+2} - P_{i+3} = 0, i = 0, 1, \cdots, n-1;$$

Otherwise, $\{P_i\}_{i=-1}^{n+2}$ is non-degenerated.

Definition 2 The polygon $\{P_i\}_{i=-1}^{n+2}$ is called a strong-degenerate control polygon, if

$$P_{i-1} - 3P_i + 3P_{i+1} - P_{i+2} = 0, i = 0, 1, \cdots, n.$$

Theorem 1 Suppose $P^0 = \{P_i^0\}_{i=-1}^{n+2}$ is an initial control polygon, P^k is the refined control polygon after k subdivisions by (2). If P^k is non-degenerated, then for arbitrary $l_{k+1} \geq L^k$, their exists ω_{k+1} such that $L^{k+1} = l_{k+1}, k = 1, 2, \cdots$

Proof. From (1) we get

$$\begin{cases} P_{2i}^{k+1}P_{2i+1}^{k+1} = \frac{1}{2}P_i^kP_{i+1}^k \\ P_{2i+1}^{k+1}P_{2i+2}^{k+1} = (\frac{1}{4}+2\omega)P_i^kP_{i+2}^k + \omega p_{i+3}^k p_{i-1}^k \end{cases},$$

$i = 0, 1, \cdots, 2^k n - 1$, then,

$$\begin{aligned} L^{k+1} &= \sum_{i=0}^{2^{k+1}n-1} |P_i^{k+1}P_{i+1}^{k+1}| = \sum_{i=0}^{2^k n-1} |P_{2i}^{k+1}P_{2i+1}^{k+1}| \\ &+ \sum_{i=0}^{2^k n-1} |P_{2i+1}^{k+1}P_{2i+2}^{k+1}| \end{aligned}$$

$$= \sum_{i=0}^{2^k n-1} \frac{1}{2}|P_i^kP_{i+1}^k| + \sum_{i=0}^{2^k n-1} |(\frac{1}{4}+2\omega)P_i^kP_{i+2}^k + \omega p_{i+3}^k p_{i-1}^k|$$

$$= \frac{1}{2}L^k + \sum_{i=0}^{2^k n-1} |(\frac{1}{4}+2\omega)P_i^kP_{i+2}^k + \omega p_{i+3}^k p_{i-1}^k| \leq g^{k+1}(\omega)$$

.
It is obvious that $g^{k+1}(\omega)$ is continuous on $(-\infty, \infty)$. Further we consider

$$g_i^{k+1}(\omega) = |(\frac{1}{4}+2\omega)P_i^kP_{i+2}^k + \omega P_{i+3}^kP_{i-1}^k|^2$$

Owing to the facts that the map from point $P(x, y)$ on plane to vector OP is one-to-one, and that the map from point $P(x, y)$ on complex plane to complex $x + yi$ is one-to-one, we can substitute length of vector OP with modulus of complex $x + yi$. denote $r_1 = |P_i^kP_{i+2}^k|$, $r_2 = |P_{i+3}^kP_{i-1}^k|$, and θ the angle of $P_i^kP_{i+2}^k$ and $P_{i+3}^kP_{i-1}^k$, then

$$\begin{aligned} g_i^{k+1}(\omega) &= |(\tfrac{1}{4}+2\omega)r_1 + \omega r_2 e^{i\theta}|^2 \\ &= [(2r_1 + r_2 cos\theta)^2 + {r_2}^2 sin^2\theta]\omega^2 \\ &+ ({r_1}^2 + \tfrac{1}{2}r_1r_2cos\theta)\omega + \tfrac{1}{16}{r_1}^2. \end{aligned} \quad (4)$$

P^k is non-degenerated, so there exists at least one i_0 such that $P_{i_0-1}^k - 2P_{i_0}^k + 2P_{i_0+2}^k - P_{i_0+3}^k \neq 0$, that is to say the coefficient of the square term of $g_{i_0}^{k+1}(\omega)$ is positive, hence there exists a sufficiently large $\hat{\omega}$ such that $g_{i_0}^{k+1}(\hat{\omega}) > l_{k+1}$. Noticing that (1) will be Chaikin's algorithm if $\omega = 0$, we get $g^{k+1}(0) < L^k \leq l_{k+1}$ from the geometric rule of Chaikin's algorithm(see Fig. 1). Hence there exists $\omega_{k+1} \in (0, \hat{\omega})$, such that $L^{k+1} = g^{k+1}(\omega_{k+1}) = l_{k+1}$.

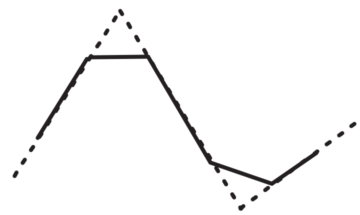

Fig. 1. Chaikin's Cutting Corner algorithm. The length of P^{k+1}(solid lines) is less than the length of P^k(dashed lines).

III. Convergence and Smoothness Analysis

The analysis of the subdivision scheme determines the smoothness of each component of the generated curve/surface. Since each component is a scalar function generated by the same subdivision scheme, it is sufficient to analyze control points on R^1 [9]. Here we will give a proof similar to [10]. Let $\{f_i^0\}_{i=-1}^{n+2}$ be a number series, and $\{f_i^k\}_{i=-1}^{2^k n+2}$ be the number series after k recursive refinement by (2). Define the piecewise linear function interpolating $(\frac{i}{2^k}, f_i^k)$ by:

$$f^k(t) = (1 - 2^k t + i)f_i^k + (2^k t - i)f_{i+1}^k,$$

$$\frac{i}{2^k} \leq t < \frac{i+1}{2^k}, i = 0, 1, \cdots, 2^k n - 1,$$

we have the following theorem:

Theorem 2 If there exists an integer $K \in Z^+$, such that for every $k > K, |\omega_k| < \frac{1}{8}$, then there exists a function $f \in C^1[0, n]$, and $lim_{k\to\infty} f^k = f$.

Proof. While $t \in [\frac{i}{2^k}, \frac{i+\frac{1}{2}}{2^k})$,

$$\begin{array}{rcl} f^{k+1}(t) & - & f^k(t) = -\omega_{k+1} f^k_{i-1} + (-\frac{1}{4} + \omega_{k+1}) f^k_i \\ & + & (\frac{1}{4} + \omega_{k+1}) f^k_{i+1} - \omega_{k+1} f^k_{i+2} \\ & = & \omega_{k+1}(f^k_i - f^k_{i-1}) + \frac{1}{4}(f^k_{i+1} - f^k_i) \\ & - & \omega_{k+1}(f^k_{k+2} - f^k_{i+1}); \end{array}$$

While $t \in [\frac{i+\frac{1}{2}}{2^k}, \frac{i+1}{2^k})$, denote $x = 2^k t - i$,

$$\begin{array}{rcl} f^{k+1}(t) & - & f^k(t) = \omega_{k+1}(1-2x)(f^k_i - f^k_{i-1}) \\ & + & [(2\omega_{k+1} - \frac{1}{2})x + \frac{3}{4}](f^k_{i+1} - f^k_i) \\ & + & [(2\omega_{k+1} + \frac{1}{2})x - \omega_{k+1}](f^k_{i+2} \\ & - & f^k_{i+1}) - 2\omega_{k+1} x (f^k_{i+3} - f^k_{i+2}), \end{array}$$

so,

$$|f^{k+1}(t) - f^k(t)| \leq (6|\omega_{k+1}| + \frac{3}{4}) max_i |f^k_{i+1} - f^k_i|.$$

Further, we estimate $|f^k_{i+1} - f^k_i|$.

$$\begin{array}{ll} f^k_{2i+1} & -f^k_{2i} = \frac{1}{2}(f^{k-1}_{i+1} - f^{k-1}_i); \\ f^k_{2i+2} & -f^k_{2i+1} \\ & = -\omega_k(f^{k-1}_i - f^{k-1}_{i-1}) + (\frac{1}{4} + \omega_k)(f^{k-1}_{i+1} \\ & -f^{k-1}_i) \\ & +(\frac{1}{4} + \omega_k)(f^{k-1}_{i+2} \\ & -f^{k-1}_{i+1}) - \omega_k(f^{k-1}_{i+3} - f^{k-1}_{i+2}). \end{array}$$

so,

$$|f^k_{i+1} - f^k_i| \leq (4|\omega_k| + \frac{1}{2}) max_i |f^{k-1}_{i+1} - f^{k-1}_i|,$$

So the following inequation holds

$$\|f^{k+1}(t) - f^k(t)\|_\infty = max_{x\in[0,n]} |f^{k+1}(t) - f^k(t)|$$

$$\leq (6|\omega_{k+1}| + \frac{3}{4}) \prod_{j=1}^{k} (4|\omega_j| + \frac{1}{2}) max_i |f^0_{i+1} - f^0_i|.$$

while $j > K, 4|\omega_j| + \frac{1}{2} < 1$, then,

$lim_{k\to\infty} \prod_{j=K+1}^{k} (4|\omega_j| + \frac{1}{2}) = 0$, hence $\{f^k(t)\}_{k=0}^{\infty}$ is a Cauchy series, so there exists $f(t) \in C[0,n]$, such that $lim_{k\to\infty} f^k(t) = f(t)$, and $f^k(t)$ converges to $f(t)$ uniformly.

Define $d^k_i = 2^k(f^k_{i+1} - f^k_i), i = 0, 1, 2, \cdots, 2^k n - 1$, then,

$$\left\{ \begin{array}{rcl} d^{k+1}_{2i} & = & 2^{k+1}(f^{k+1}_{2i+1} - f^{k+1}_{2i}) = d^k_i \\ d^{k+1}_{2i+1} & = & 2^{k+1}(f^{k+1}_{2i+2} - f^{k+1}_{2i+1}) \\ & = & -2\omega_{k+1} d^k_{i-1} + (\frac{1}{2} + 2\omega_{k+1}) d^k_i \\ & + & (\frac{1}{2} + 2\omega_{k+1}) d^k_i - 2\omega_{k+1} d^k_{i+2} \end{array} \right. . \quad (5)$$

Define a piecewise linear function interpolating $(\frac{i}{2^k}, d^k_i)$ as

$$d^k(t) = (1 - 2^k t + i) d^k_i + (2^k t - i) d^k_{i+1},$$

$$\frac{i}{2^k} \leq t < \frac{i+1}{2^k}, i = 0, 1, \cdots, 2^k n - 2$$

while $t \in [\frac{i}{2^k}, \frac{i+\frac{1}{2}}{2^k})$, denote $x = 2^k t - i$ for simplicity, then,

$$\begin{array}{rcl} d^{k+1} - d^k(t) & = & [(1-2x) d^{k+1}_{2i} + 2x d^{k+1}_{2i+1}] \\ & & -[(1-x) d^k_i + x d^k_{i+1}] \\ & = & 4\omega_{k+1}(d^k_i - d^k_{i-1}) - 4\omega_{k+1}(d^k_{i+2} \\ & & -d^k_{i+1}); \end{array}$$

While $t \in [\frac{i+\frac{1}{2}}{2^k}, \frac{i+1}{2^k})$, denote $x = 2^k t - i$ for simplicity,

$$\begin{array}{rcl} d^{k+1} - d^k(t) & = & [(1-2x) d^{k+1}_{2i+1} + 2x d^{k+1}_{2i+2}] \\ & & -[(1-x) d^k_i + x d^k_{i+1}] \\ & = & 2\omega_{k+1}(1-2x)(d^k_i - d^k_{i-1}) \\ & & +\frac{1}{2}(d^k_{i+1} - d^k_i) - 2\omega_{k+1}(1-2x) \\ & & (d^k_{i+2} - d^k_{i+1}). \end{array}$$

So,

$$|d^{k+1}(t) - d^k(t)| \leq (4|\omega_{k+1}| + \frac{1}{2}) max_i |d^k_{i+1} - d^k_i|.$$

Further, we estimate $|d^k_{i+1} - d^k_i|$.

$$\begin{array}{rcl} d^k_{2i+1} & - & d^k_{2i} = -2\omega_k d^{k-1}_{i-1} + (-\frac{1}{2} + 2\omega_k) d^{k-1}_i \\ & & +(\frac{1}{2} + 2\omega_k) d^{k-1}_{i+1} - 2\omega_k d^{k-1}_{i+2} \\ & = & 2\omega_k(d^{k-1}_i - d^{k-1}_{i-1}) + \frac{1}{2}(d^{k-1}_{i+1} - d^{k-1}_i) \\ & & -2\omega_k(d^{k-1}_{i+2} - d^{k-1}_{i+1}); \\ d^k_{2i+2} & - & d^k_{2i+1} = 2\omega_k d^{k-1}_{i-1} - (\frac{1}{2} + 2\omega_k) d^{k-1}_i \\ & & -(-\frac{1}{2} + 2\omega_k) d^{k-1}_{i+1} - 2\omega_k d^{k-1}_{i+2} \\ & = & -2\omega_k(d^{k-1}_i - d^{k-1}_{i-1}) + \frac{1}{2}(d^{k-1}_{i+1} - d^{k-1}_i) \\ & & +2\omega_k(d^{k-1}_{i+2} - d^{k-1}_{i+1}). \end{array}$$

So,

$$|d^k_{i+1} - d^k_i| \leq (4|\omega_k| + \frac{1}{2}) max_i |d^{k-1}_{i+1} - d^{k-1}_i|,$$

Hence,

$$\|d^{k+1}(t)-d^k(t)\|_\infty = max_{x\in[0,n]}|d^{k+1}(t)-d^k(t)|$$

$$\leq \prod_{j=1}^{k+1}(4|\omega_j|+\frac{1}{2})max_i|d^0_{i+1}-d^0_i|.$$

while $j > K, 4|\omega_j| + \frac{1}{2} < 1$, then, $lim_{k\to\infty}\prod_{j=K+1}^{k+1}(4|\omega_j|+\frac{1}{2}) = 0$, hence $\{f^k(t)\}_{k=0}^\infty$ is a Cauchy series, so there exists $d(t)\in C[0,n]$, such that $lim_{k\to\infty}d^k(t) = d(t)$, and $d^k(t)$ converges to $d(t)$ uniformly.

Next we proof $f'(t) = d(t)$.

Denote

$$g_k(t) = \sum_{i=0}^{2^k n}\binom{2^k n}{i}(\frac{t}{n})^i(1-\frac{t}{n})^{2^k n-i}f_i^k,$$

then,

$$g_k{}'(t) = \sum_{i=1}^{2^k n}\binom{2^k n}{i}\frac{i}{n}(\frac{t}{n})^{i-1}(1-\frac{t}{n})^{2^k n-i}f_i^k$$

$$+\sum_{i=0}^{2^k n-i-1}\binom{2^k n}{i}(\frac{t}{n})^i(-\frac{2^k n-i}{n})(1-\frac{t}{n})^{2^k n-i-1}f_i^k$$

$$=\sum_{i=0}^{2^k n-1}\binom{2^k n-1}{i}(\frac{t}{n})^i(1-\frac{t}{n})^{2^k n-i-1}d_i^k,$$

Noticing that (5) is an interpolating subdivision scheme, by Weierstrass theorem we get that $g_k(t)$ converges to $f(t)$, and that $g'_k(t)$ converges to $d(t)$ uniformly, hence $f'(t) = d(t)$.

The following phenomenon seems to be very interest from lots of numerical tests that the free parameter ω_k converges to $\frac{1}{32}$ in the subdivision scheme (2) with arc length constraint. The full proof of the above statement for general case has not been completed yet. Here, we shall give the proof for a special case.

proposition 1 Let an initial control polygon be a regular $n-$polygon P^0 and let $l_k = na$ in (3), where a is the length of each edge of P^0. Then $\lim_{k\to\infty}\omega_k = \frac{1}{32}$.

Proof The limit curve is a circle of perimeter na. In fact, from the symmetry of control polygon and (2) we get that

$$|P_0^1P_1^1| = |P_2^1P_3^1| = \cdots|P_{2n-2}^1P_{2n-1}^1|;$$

$$|P_1^1P_2^1| = |P_3^1P_4^1| = \cdots|P_{2n-1}^1P_{2n}^1|.$$

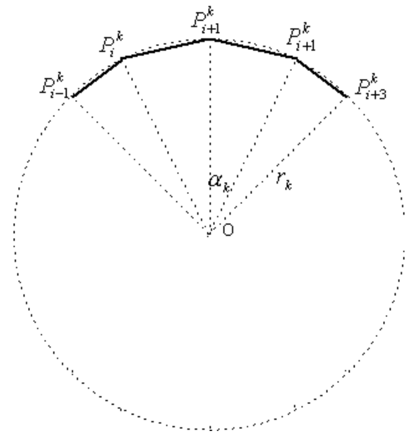

Fig. 2.

Notice that $P_{2i}^{k+1}P_{2i+1}^{k+1} = \frac{1}{2}P_i^kP_{i+1}^k$, we get

$$|P_{2i}^1P_{2i+1}^1| = |P_{2i+1}^1P_{2i+2}^1| = \frac{a}{2}.$$

With the recursion, P^k is a regular 2^kn-polygon, so the limit is circle.

Next consider the value of ω. Since P^k and P^{k+1} are regular polygons, so $|P_{2i}^{k+1}P_{2i+1}^{k+1}| = \frac{a}{2^{k+1}}$, $|P_i^kP_{i+1}^k| = \frac{a}{2^k}$.

Let $\odot O$ be the circumcircle of P^k (See Fig. 2), and r_k be the radius of $\odot O$, $\alpha_k = \angle P_i^kOP_{i+1}^k = \frac{2\pi}{2^kn}$, we can easily calculate that $|P_i^kP_{i+2}^k| = 2r_k\sin(\alpha_k), |P_{i-1}^kP_{i+3}^k| = 2r_k\sin(2\alpha_k), r_k = \frac{a}{2^k\sin(\frac{\alpha_k}{2})}$.

From (3) and the proof of theorem 1, we obtain

$$\frac{a}{2^{k+1}} = (\frac{1}{4}+2\omega_{k+1})2r_k\sin(\alpha_k) - \omega_{k+1}2r_k\sin(2\alpha_k).$$

Hence,

$$\omega_{k+1} = \frac{\frac{1}{2}\sin(\frac{\alpha_k}{2})-\frac{1}{4}\sin(\alpha_k)}{2\sin(\alpha_k)-\sin(2\alpha_k)}.$$

So,

$$\omega = \lim_{k\to\infty}\omega_k = \lim_{\alpha\to 0}\frac{\frac{1}{2}\sin(\frac{\alpha}{2})-\frac{1}{4}\sin(\alpha)}{2\sin(\alpha)-\sin(2\alpha)} = \frac{1}{32}.$$

IV. Discussion about Strong Degeneration

In this section we will give a special discussion about strong degeneration.

Theorem 3 Suppose $P^0 = \{P_i^0\}_{i=-1}^{n+2}$ is an initial control polygon on plane, and $P^l = \{P_i^l\}_{i=-1}^{2^l n+2}$ is the control polygon after l refinement by (2), $l \in Z^+$, if P^k is strongly degenerated, then P^k is degenerated.

Proof. If $P_{i-1}^k - 3P_i^k + 3P_{i+1}^k - P_{i+2}^k = 0, i = 0, 1, \cdots, 2^k n$, then

$$P_{i-1}^k - 2P_i^k + 2P_{i+2}^k - P_{i+3}^k =$$

$$(P_{i-1}^k - 3P_i^k + 3P_{i+1}^k - P_{i+2}^k) + (P_i^k - 3P_{i+1}^k + 3P_{i+2}^k - P_{i+3}^k)$$

$$= 0, \ \ i = 0, 1, \cdots, 2^k n - 1$$

. That is to say P^k is degenerated.

Theorem 4 Suppose $P^0 = \{P_i^0\}_{i=-1}^{n+2}$ is an initial control polygon on plane, and $P^l = \{P_i^l\}_{i=-1}^{2^l n+2}$ is the control polygon after l refinement by (2), $l \in Z^+$, then P^{k+1} is strongly degenerated if and only if P^k is strongly degenerated, $k = 0, 1, \cdots$

Proof. If $P_{i-1}^k - 3P_i^k + 3P_{i+1}^k - P_{i+2}^k = 0, i = 0, 1, \cdots, 2^k n$, then

$$P_{2i}^{k+1} - 3P_{2i+1}^{k+1} + 3P_{2i+2}^{k+1} - P_{2i+3}^{k+1}$$

$$= 2\omega_{k+1}(P_{i-1}^k - 2P_i^k + 2P_{i+2}^k - P_{i+3}^k)$$

$$= 2\omega_{k+1}[(P_{i-1}^k - 3P_i^k + 3P_{i+1}^k - P_{i+2}^k)$$

$$+(P_i^k - 3P_{i+1}^k + 3P_{i+2}^k - P_{i+3}^k)] = 0;$$

$$\begin{aligned}
P_{2i+1}^{k+1} \quad - & \quad 3P_{2i+2}^{k+1} + 3P_{2i+3}^{k+1} - P_{2i+4}^{k+1} = \\
- & \quad \omega_{k+1}P_{i-1}^k + (\frac{1}{4} + \omega_{k+1})P_i^k \\
+ & \quad (-\frac{3}{4} + 2\omega_{k+1})P_{i+1}^k + (\frac{3}{4} - 2\omega_{k+1})P_{i+2}^k \\
- & \quad (\frac{1}{4} + \omega_{k+1})P_{i+3}^k + \omega_{k+1}P_{i+4}^k \\
= & \quad -\omega_{k+1}(P_{i-1}^k - 3P_i^k + 3P_{i+1}^k - P_{i+2}^k) \\
+ & \quad (\frac{1}{4} - 2\omega_{k+1})(P_i^k - 3P_{i+1}^k + 3P_{i+2}^k - P_{i+3}^k) \\
- & \quad \omega_{k+1}(P_{i+1}^k - 3P_{i+2}^k + 3P_{i+3}^k - P_{i+4}^k) = 0.
\end{aligned}$$

So, $P_{i-1}^{k+1} - 3P_i^{k+1} + 3P_{i+1}^{k+1} - P_{i+2}^{k+1} = 0, i = 0, 1, \cdots, 2^{k+1}n$.

Now consider the converse direction. If $P_{i-1}^{k+1} - 3P_i^{k+1} + 3P_{i+1}^{k+1} - P_{i+2}^{k+1} = 0, i = 0, 1, \cdots, 2^{k+1}n$, then

$$\begin{cases} P_{2i}^{k+1} - 3P_{2i+1}^{k+1} + 3P_{2i+2}^{k+1} - P_{2i+3}^{k+1} = 0 \\ P_{2i+1}^{k+1} - 3P_{2i+2}^{k+1} + 3P_{2i+3}^{k+1} - P_{2i+4}^{k+1} = 0 \end{cases},$$

$i = 0, 1, \cdots, 2^k n.$

By (2), we get

$$\begin{cases} \omega_{k+1}(P_{i-1}^k - 2P_i^k + 2P_{i+2}^k - P_{i+3}^k) = 0; \\ \omega_{k+1}(P_{i-2}^k - P_{i-1}^k - 2P_i^k + 2P_{i+1}^k \\ +P_{i+2}^k - P_{i+3}^k) + \frac{1}{4}(-P_{i-1}^k + 3P_i^k - 3P_{i+1}^k \\ +P_{i+2}^k) = 0. \end{cases} \tag{6}$$

From the first expression of (6) we get

$$\omega_{k+1}(P_{i-2}^k - 2P_{i-1}^k + 2P_{i+1}^k - P_{i+2}^k) +$$

$$\omega_{k+1}(P_{i-1}^k - 2P_i^k + 2P_{i+2}^k - P_{i+3}^k)$$

$$= \omega_{k+1}(P_{i-2}^k - P_{i-1}^k - 2P_i^k + 2P_{i+1}^k + P_{i+2}^k - P_{i+3}^k) = 0,$$

From the above expression and the second expression of (6) we get

$$-P_{i-1}^k + 3P_i^k - 3P_{i+1}^k + P_{i+2}^k = 0.$$

Hence we obtain P^k is strongly degenerated.

By Theorem(3),(4) we get to know that if the initial control polygon P^0 is strongly degenerated, then for every $k \in Z^+$, P^k is strongly degenerated. In this case, there is no solution for constraint equation (3).

In fact, if P^k is strongly degenerated, then P_{2+i}^k can be expressed by P_0^k, P_1^k, P_2^k,

$$\begin{aligned}
P_{2+i}^k \quad = & \quad \frac{i(i+1)}{2}P_0^k + [-3 - (i+3)(i-1)]P_1^k \\
+ & \quad [3 + \frac{(i+4)(i-1)}{2}]P_2^k.
\end{aligned}$$

By (1) the recursive process can be expressed in matrix formalism as follow

$$\begin{pmatrix} P_0^k \\ P_1^k \\ P_2^k \end{pmatrix} = \begin{pmatrix} \frac{3}{4} - 2\omega & \frac{1}{4} + 4\omega & -2\omega \\ \frac{1}{4} - 2\omega & \frac{3}{4} + 4\omega & -2\omega \\ -2\omega & \frac{3}{4} + 4\omega & \frac{1}{4} - 2\omega \end{pmatrix}^k \begin{pmatrix} P_0^0 \\ P_1^0 \\ P_2^0 \end{pmatrix},$$

So,

$$\begin{cases} P_0^k P_1^k = \frac{1}{2^k} P_0^0 P_1^0; \\ P_i^k P_{i+1}^k = (\frac{1}{2^k} - \frac{i}{4^k}) P_0^0 P_1^0 + \frac{i}{4^k} P_1^0 P_2^0 . i = 1, 2, \cdots \end{cases}$$

Hence,

$$L^k = \sum_{i=0}^{2^k n-1} |(\frac{1}{2^k} - \frac{i}{4^k}) P_0^0 P_1^0 + \frac{i}{4^k} P_1^0 P_2^0|$$

$$< (\frac{n^2}{2} - n - 1)|P_0^0 P_1^0| + \frac{n^2}{2}|P_1^0 P_2^0|.$$

We can see from the above that in this case L^k is independent of ω and that L^k is limited uniformly.

V. Geometric Examples

Given a concave initial control polygon of perimeter 14, scheme (2) generates a limit curve of length 16 (Fig. 2); If the initial control polygon is a regular polygon of perimeter l, and in (3) $l^k = l, k = 1, 2, \cdots$, then we obtain a circle of perimeter l (Fig. 3.); Given a unit cylinder and a generatrix, scheme (2) can generate rotary surfaces as Fig. 4 and Fig. 5. Fig. 6 is a human jacket based on virtual human body, the initial data are human slices coming from visible human server in Peripheral System Laboratory. For each slice we extract its contour which serves as the control polygon, and let the arc length be 1.1 times of the perimeter of the contour. This is just our primary attempt on this problem, there are some further work to do for industry design.

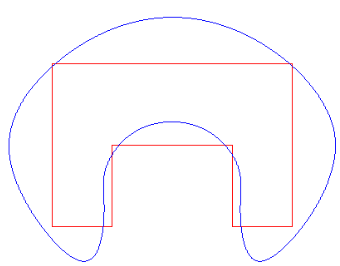

Fig. 3.

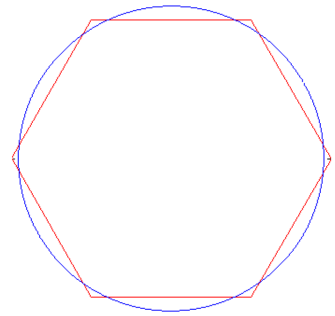

Fig. 4.

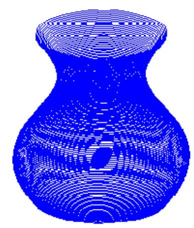

Fig. 5.

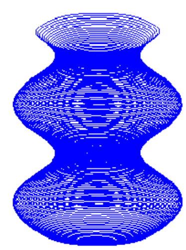

Fig. 6.

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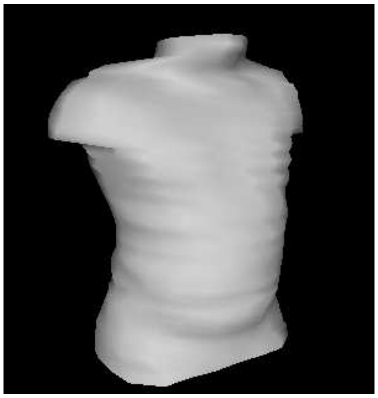

Fig. 7.

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