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Recovering Baire one functions on ultrametric spaces

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Abstract

We find a characterization of those Polish ultrametric spaces on which each Baire one function is first return recoverable. The notion of pseudo-convergence originating in the theory of valuation fields plays a crucial role in the characterization.

1. *Introduction*

A function $f : X \to \mathbb{R}$ defined on a metric separable space is said to be *Baire class one* if it is the limit of a pointwise converging sequence of continuous functions. There are several distinct approaches to Baire class one functions that give very diverse conditions equivalent to being Baire class one; see [**4**]. This is part of the reason for the wide applicability of this class of functions in, among others, topology and Banach space theory; see, for example, [**7**, **8**]. The notions dealt with in this paper showed themselves relevant in Banach space considerations and in theories of differentiation and integration; see [**2**, **3**, **5**] in which these connections were first pointed out.

In [**1**], Darji and Evans proposed a new approach to Baire class one functions based on some dynamical notions that came up in O'Malley's work [**5**] on differentiation of real functions. They formulated the definition of first return recoverable functions, or recoverable functions for short, which we now recall. Let (X, d) be a separable metric space and let (x_n) be a sequence dense in X. We call (x_n) a *trajectory*. Let $\xi(B(x, r))$ be the first element of the trajectory in $B(x, r)$. Then the *first return route to x based on (x_n)*, $(z^x_k)_{k\in\mathbb{N}}$, is defined recursively by

$$z^x_0 = x_0,$$
$$z^x_{k+1} = \begin{cases} x & \text{if } z^x_k = x; \\ \xi(B(x, d(x, z^x_k))) & \text{otherwise.} \end{cases}$$

It is easy to check, and was checked in [**1**], that $z^x_k \to x$ as $k \to \infty$. We say that $f : X \to \mathbb{R}$ is *first return recoverable with respect to the trajectory* (x_n) if for every x in X, $f(z^x_k) \to f(x)$ as $k \to \infty$. A function f is called *first return recoverable*, or *recoverable* for short, if there exists some trajectory in X with respect to which f is first return recoverable. Note that recoverability is a metric, not a topological, notion.

Darji and Evans showed in [**1**] that each recoverable function $f : X \to \mathbb{R}$ on a Polish metric space X is Baire one. It is now of some importance to see under what conditions on the Polish metric space X each Baire one function is recoverable. Darji and Evans proved in [**1**] that this is so if X is a compact metric space. The assumption in this last statement was relaxed in [**3**] to X being the union of countably many compact sets.

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The above problem on Polish *ultrametric* spaces was studied by Lecomte [**3**]. Recall that a metric space (X, d) is *ultrametric* if it fulfills the following strong triangle inequality

$$d(x, z) \leqslant \max(d(x, y), d(y, z)).$$

(This class of spaces was first considered in the theory of valuation fields, but has since come up naturally in various other contexts.) Lecomte proved in [**3**] that if the range of the metric on a Polish ultrametric space X is such that any strictly decreasing sequence in it converges to 0, then each Baire one function is recoverable. He also found an example of a Baire one function on a Polish ultrametric space that is not recoverable. (This was, in fact, the first example of a non-recoverable Baire one function on any Polish space.) It is recalled below to give the reader an idea of what kind of ultrametric spaces are relevant here. Consider the set

$$\left\{(q_n) \in \mathbb{Q}^{\mathbb{N}} \ : \ \forall n\ 0 \leqslant q_n < q_{n+1} \text{ and } \lim_{n\to\infty} q_n = \infty\right\},$$

equipped with the ultrametric

$$d((q_n), (q_n')) = \begin{cases} 2^{-\min\{q_{\min\{n:\ q_n\neq q_n'\}}, q'_{\min\{n:\ q_n\neq q_n'\}}\}} & \text{if } (q_n) \neq (q_n'); \\ 0 & \text{otherwise.} \end{cases}$$

Let the Polish ultrametric metric space X be the metric completion of thus defined ultrametric space. (In this case, taking of the metric completion amounts to adding countably many points.) Then, as proved in [**3**], the characteristic function of the closed set

$$\{(q_n) \in X : \forall n \in \mathbb{N}\ n < q_n < n + 1\},$$

which is Baire class one, is not first return recoverable.

In our main result, Theorem 2.2, we characterize those Polish ultrametric spaces on which each Baire one function is recoverable. In Example 3.1 we construct a Polish ultrametric space on which each Baire one function is recoverable, that is, the criterion from Theorem 2.2 applies to this space, but the sufficient conditions for recoverability of Baire one functions from [**1**, **3**] fail.

2. *The main theorem*

The notion of pseudo-convergence will be important in the main result. This notion originated in the theory of valuation fields (see [**6**, pp. 368–370, 374–378]) and was later carried over to the general setting of ultrametric spaces. Let (X, d) be an ultrametric space. We say that the sequence (x_n) of elements of X *pseudo-converges to* $x \in X$ if $d(x_n, x) > d(x_{n+1}, x)$ for all but finitely many n.

Using the ultrametric properties of (X, d) one can easily prove the following lemma. Its proof is left to the reader as an exercise.

LEMMA 2.1. *Let (X, d) be an ultrametric space. Let (x_n) be a sequence of points in X and let $x \in X$. Then the following conditions hold.*

(i) *The sequence (x_n) pseudo-converges to x if and only if there are reals d_n such that for large enough n we have $d(x_n, x) = d_n$ and $d(x_n, x_p) = d_n$ for all $p > n$, and $d_n > d_{n+1}$.*

(ii) *If (x_n) pseudo-converges to x, then for any $y \in X$ we have $d(x_n, x) \leqslant d(x_n, y)$ for all but finitely many n.*

Now we state our main result.

THEOREM 2.2. *Let (X, d) be a Polish ultrametric space. The following conditions are equivalent.*

(i) *Each Baire one function from X to $\mathbb{R}$ is recoverable.*

(ii) *The indicator function χ_F is recoverable for each closed $F \subseteq X$.*

(iii) *We have $X = \bigcup_{n\in\mathbb{N}} X_n$, where, for each n, X_n is closed and such that every sequence of elements of X_n which is pseudo-convergent in X converges.*

One proves without difficulty that in an ultrametric space X if $Y \subseteq X$ has the property that each sequence of elements of Y which is pseudo-convergent in X converges, then the closure of Y also has this property. Thus, in point (iii) of Theorem 2.2, if we omit the assumption that the sets X_n are closed, we obtain an equivalent condition. Note further that condition (iii) concerns not only the sets X_n but also how these sets are situated inside of X. This is because it places restrictions on sequence of points in X_n that pseudo-converge in X not necessarily to elements in X_n.

An inspection of the proof of Theorem 2.2 shows that the arguments justifying the implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) do not use completeness of the metric d and, therefore, are valid for arbitrary separable ultrametric spaces (X, d).

Condition (iii) from Theorem 2.2 is weaker than (and easily seen to be so) the sufficient conditions found in [**1**, **3**] and discussed in Section 1 guaranteeing that each Baire one function is recoverable: σ-compactness of X and the property that each strictly decreasing sequence of elements of the range of the metric on X converges to 0. We say more about it in Section 3.

We turn now to the proof of Theorem 2.2. We recall first some relevant notions. Let $\mathbb{N}^{<\mathbb{N}}$ be the set of all finite sequences of natural numbers, such that $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$. For $s, t \in \mathbb{N}^{<\mathbb{N}}$ we write $s \subseteq t$ if t extends s and $s \perp t$ if neither $s \subseteq t$ nor $t \subseteq s$. For $s \in \mathbb{N}^{<\mathbb{N}}$ let $|s|$ be the length of s, that is, the unique n with $s \in \mathbb{N}^n$. If $i \leqslant |s|$ then let $s{\restriction}i$ be the initial segment of s of length i. For $i \in \mathbb{N}$, let $s^\frown i$ be the sequence of length $|s| + 1$ extending s with $(s^\frown i)(|s|) = i$. A set $S \subseteq \mathbb{N}^{<\mathbb{N}}$ is called a *tree* if for each $s \in S$ and $i \leqslant |s|$ we have $s{\restriction}i \in S$.

Let (X, d) be an ultrametric space. For $x \in X$ and $r > 0$, let $B(x, r)$ be the open ball $\{y \in X : d(x, y) < r\}$. We call a family of open balls $B_s \subseteq X$ and points $x_s \in X$ with $s \in S$, for some tree $S \subseteq \mathbb{N}^{<\mathbb{N}}$, a *refinement of X* if for $s \in S$ we have the following conditions:

(A) $B_\emptyset = X$;

(B) $B_s = \{x_s\} \cup \bigcup_{s^\frown i\in S} B_{s^\frown i}$;

(C) $B_{s^\frown i} = B(x_{s^\frown i}, d(x_s, x_{s^\frown i}))$ if $s^\frown i \in S$.

Note that (A) and (B) ensure that for any $y \in X \setminus \{x_s : s \in S\}$ there exists $\alpha \in \mathbb{N}^\mathbb{N}$ with $y \in \bigcap_n B_{\alpha\restriction n}$.

We describe now a procedure producing refinements. We shall refer to this procedure in subsequent arguments. Let $x_\emptyset \in X$ and let $B_\emptyset = X$. If B_s and $x_s \in B_s$ are chosen, and $B_s = \{x_s\}$, then we declare s to be a terminal node of S. If $B_s \neq \{x_s\}$, then we consider the relation $\sim$ on $B_s \setminus \{x_s\}$ defined by $x \sim y$ if and only if $d(x, y) < \min(d(x, x_s), d(y, x_s))$. By ultrametric properties this is an equivalence relation whose equivalence classes are balls of the form $B(x, d(x, x_s))$. Choose $x_{s^\frown i}$ with i ranging over a subset of $\mathbb{N}$, so that $B(x_{s^\frown i}, d(x_{s^\frown i}, x_s))$ are pairwise disjoint and list all these equivalence classes. The result of the above procedure depends on the choice of x_s, but no matter how this choice is made, we always have (A)–(C). In the applications of the procedure that follow, the choice of the points x_s will be important.

LEMMA 2.3. *Let (X, d) be an ultrametric space. Let $f : X \to \mathbb{R}$ be a Baire one function. Then f is recoverable if and only if there exists a refinement B_s, x_s for $s \in S$, of X such that for any $y \in X \setminus \{x_s : s \in S\}$ and $\alpha \in \mathbb{N}^\mathbb{N}$ with $y \in \bigcap_n B_{\alpha\restriction n}$ we have $x_{\alpha\restriction n} \to y$ and $f(x_{\alpha\restriction n}) \to f(y)$ as $n \to \infty$.*

Proof. ($\Leftarrow$) Assume that we have a refinement as in the statement. Let x^n ($n \in \mathbb{N}$), be an enumeration of $\{x_s : s \in S\}$ such that if $x_s = x^m$, $x_t = x^n$ and $s \subseteq t$, then $m \leqslant n$. We claim that (x^n) is a trajectory, that is, it is dense, and that it recovers f. These two points will be proved simultaneously.

If $y \in \{x_s : s \in S\}$, then the route to y with respect to the trajectory (x^n) is equal to y from some point on, and thus the conclusion holds in this case. If $y \in X \setminus \{x_s : s \in \mathbb{N}^{<\mathbb{N}}\}$, then let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be such that $y \in \bigcap B_{\alpha\restriction n}$. Let z_k ($k \in \mathbb{N}$) be the route to y taken with respect to (x^n) described above. We claim that for each k, we have

$$z_k = x_{\alpha\restriction k}, \tag{1}$$

which will prove the conclusion by our assumption on x_s, B_s for $s \in S$. The claim is proved by induction on k. Since $x^0 = x_\emptyset$, if follows that (1) holds for $k = 0$. Assume that it holds for k. Then $z_k = x_{\alpha\restriction k}$ and z_{k+1} is the first element in (x^n) in

$$\begin{aligned} B(y, d(z_k, y)) &= B(y, d(x_{\alpha\restriction k}, y)) \\ &= B(x_{\alpha\restriction k+1}, d(x_{\alpha\restriction k}, x_{\alpha\restriction k+1})) = B_{\alpha\restriction k+1}. \end{aligned} \tag{2}$$

In (2), the first equality holds since $z_k = x_{\alpha\restriction k}$, the second one holds since $y \in B(x_{\alpha\restriction k+1}, d(x_{\alpha\restriction k}, x_{\alpha\restriction k+1}))$ and by ultrametric properties, and the third one holds by (C). By (B) and (C), if $x_s \in B_{\alpha\restriction k+1}$, then $\alpha\restriction k+1 \subseteq s$. Thus, by the definition of the enumeration (x^n), the first element of it in the last ball in (2) is $x_{\alpha\restriction k+1}$, therefore z_{k+1} is equal to that element, and (1) holds for $k+1$.

($\Rightarrow$) Let (x^n) be a trajectory recovering f. Define a refinement B_s, x_s for $s \in S$ using the procedure producing refinements described above requiring additionally that $x_{s^\frown i}$ be the first element of (x^n) in $B_{s^\frown i}$. Note that if for some $\alpha \in \mathbb{N}^{\mathbb{N}}$, $y \in \bigcap_n B_{\alpha\restriction n}$, then the sequence $(x_{\alpha\restriction n})$ is the route to y with respect to the trajectory (x^n) since, by ultrametric properties, y is the center of each ball $B_{\alpha\restriction n}$. Thus, the required properties of our refinement follow from the fact that (x^n) recovers f. □

Proof of Theorem 2.2. Call a subset of X *good* if each sequence of its elements that pseudo-converges in X converges.

Since (i) implies (ii), it will suffice to show that (iii) implies (i) and that (ii) implies (iii).

(iii) $\Rightarrow$ (i). Assume that (iii) holds. We show that (i) holds. We recall the definition of the lexicographic order on $\mathbb{N}^{<\mathbb{N}}$. For $s, t \in \mathbb{N}^{<\mathbb{N}}$, set $s \leqslant_{\text{lex}} t$ if $s \subseteq t$ or there exists $i < \min\{|s|, |t|\}$ with $s\restriction i = t\restriction i$ and $s(i) < t(i)$. Assume that we are given a family F_r ($r \in \mathbb{N}^{<\mathbb{N}}$), of closed sets such that:

(a) $F_r \supseteq F_s$ if $r \subseteq s$;
(b) $F_r \cap F_s = \emptyset$ if $r \perp s$;
(c) $F_r = \bigcup_i F_{r^\frown i}$;
(d) $F_\emptyset = X$;
(e) each F_r with $r \neq \emptyset$ is good.

Note that some of the sets F_s may be empty.

Let B_s, x_s for $s \in S$ be a refinement of X constructed so that x_s is chosen from $F_{r(s)}$, where $r(s) \in \mathbb{N}^{|s|}$ is the lexicographically least element of the set

$$\{r \in \mathbb{N}^{|s|} : F_r \cap B_s \neq \emptyset\}.$$

This is done by the procedure whose description was given above following the definition of refinement.

Now let $y \in X \setminus \{x_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ and let $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ be such that $y \in \bigcap_m B_{\alpha\restriction m}$ and $y \in \bigcap_n F_{\beta\restriction n}$. Such α and β exist by (A) and (B) of the definition of refinement and by (c) and (d).

Claim 2.4. *For each n, for all but finitely many m, $x_{\alpha\restriction m} \in F_{\beta\restriction n}$, and $x_{\alpha\restriction m} \to y$ as $m \to \infty$.*

Proof. We will need the following two facts whose proofs, of which the first one uses ultrametric properties, we leave to the reader.

Fact 1. If $y \in \bigcap_n B_{\alpha\restriction n}$ for some $\alpha \in \mathbb{N}^{\mathbb{N}}$, then $d(y, x_{\alpha\restriction n}) > d(y, x_{\alpha\restriction n+1})$ for each n.

Fact 2. Let $\beta \in \mathbb{N}^{\mathbb{N}}$ and let $r_m \in \mathbb{N}^m$ for $m \in \mathbb{N}$ be such that $r_m \leqslant_{\mathrm{lex}} \beta\restriction m$. Then one of the following two possibilities holds:

(I) for some i, for infinitely many m, $r_m\restriction i = \beta\restriction i$ and $r_m(i) < \beta(i)$;
(II) for each i, for all but finitely many m, $r_m(i) = \beta(i)$.

For $m \in \mathbb{N}$, let $r_m \in \mathbb{N}^m$ be such that $x_{\alpha\restriction m} \in F_{r_m}$. Since $y \in F_{\beta\restriction m} \cap B_{\alpha\restriction m}$ makes the intersection $F_{\beta\restriction m} \cap B_{\alpha\restriction m}$ non-empty, we have, from the choice of $x_{\alpha\restriction m}$, that $r_m \leqslant_{\mathrm{lex}} \beta\restriction m$. Now we apply Fact 2 to the sequence (r_m) and to β. Assume first that (I) holds, that is, we can fix i such that for infinitely many m we have $r_m\restriction i = \beta\restriction i$ and $r_m(i) < \beta(i)$. Fix a sequence $m_0 < m_1 < \dots$ of indices for which this holds. We can of course assume that $r_{m_0}(i) = r_{m_1}(i) = \dots$. Let r be the common value of $r_{m_j}\restriction i+1$. Now, by property (a), we see that

$$x_{\alpha\restriction m_0}, x_{\alpha\restriction m_1}, \dots \in F_r$$

and by Fact 1, we have

$$d(y, x_{\alpha\restriction m_0}) > d(y, x_{\alpha\restriction m_1}) > \dots .$$

Thus the sequence $x_{\alpha\restriction m_0}, x_{\alpha\restriction m_1}, \dots$ pseudo-converges to y. Since F_r is good, the sequence $x_{\alpha\restriction m_0}, x_{\alpha\restriction m_1}, \dots$ actually converges. It converges to an element of F_r, since this set is closed, and, at the same time, it converges to y since it pseudo-converges to it. However, $y \notin F_r$ by property (b) since $y \in F_{\beta\restriction i+1}$ and $\beta(i) \neq r(i)$. This contradiction proves that (I) fails. Thus, (II) holds. However, point (II) easily implies, by property (a), the first part of the conclusion of Claim 2.4.

To prove the second part of the conclusion, note that, by what was proved above, for m large enough, $x_{\alpha\restriction m} \in F_{\beta(0)}$ and that, by Fact 1, $d(y, x_{\alpha\restriction m}) > d(y, x_{\alpha\restriction m+1})$. Since $F_{\beta(0)}$ is good, it follows that the sequence $(x_{\alpha\restriction m})$ converges. Thus, since it pseudo-converges to y, it also converges to y, and the claim follows. □

Assume now that $f : X \to \mathbb{R}$ is a Baire one function. Since X is zero-dimensional, being an ultrametric, it is easy to check that it has the following property: for any family E_n ($n \in \mathbb{N}$) of F_σ sets there exists a family of F_m ($m \in \mathbb{N}$) of closed sets such that $\bigcup_n E_n = \bigcup_m F_m$; for each m there exists n with $F_m \subseteq E_n$; and for any distinct m, m' we have $F_m \cap F_{m'} = \emptyset$. Using this property and the fact that preimages via f of open intervals are F_σ, we can define a family of closed sets F_r ($r \in \mathbb{N}^{<\mathbb{N}}$) fulfilling (a)–(d) and additionally for $r \neq \emptyset$ we have

$$\operatorname{diameter}(f(F_r)) < 1/|r|. \tag{3}$$

In fact, since X can be covered by countably many closed good sets, we can arrange the construction such that for each $i \in \mathbb{N}$ the set $F_{\langle i\rangle}$ is contained in one of those sets. Thus, we can guarantee that (e) holds as well for the family $\{F_r : r \in \mathbb{N}^{<\mathbb{N}}\}$.

Now choose a refinement of X as was done at the beginning of this proof for the family $\{F_r : r \in \mathbb{N}^{<\mathbb{N}}\}$. It follows from Lemma 2.3 and from Claim 2.4 and inequality (3) that f is recoverable.

(ii) $\Rightarrow$ (iii). Assume that (iii) fails. We prove that (ii) fails. For countable ordinals α define a closed set $F_\alpha \subseteq X$ as follows. Let $F_0 = X$, $F_\lambda = \bigcap_{\alpha<\lambda} F_\alpha$ if λ is a limit ordinal, and we have

$$F_{\alpha+1} = F_\alpha \setminus \bigcup\{U : U \text{ is open and } F_\alpha \cap U \text{ is good}\}.$$

Since X is separable, there exists a countable ordinal α_0 with $F_{\alpha_0+1} = F_{\alpha_0}$. Note further that for each ordinal α, $F_\alpha \setminus F_{\alpha+1}$ is covered by countably many closed good sets. Since X cannot be covered by countably many such sets, we have that F_{α_0} is non-empty. Set

$$Z = F_{\alpha_0}.$$

This is a non-empty closed set with the following property: each relatively open non-empty subset of Z contains a pseudo-convergent in X sequence (z_n) that is not convergent. After deleting finitely many elements of (z_n), this condition gives us $y \in X$ and reals d_n such that for each $n \in \mathbb{N}$ we have the following conditions:

(a) $d(y, z_n) = d_n$;
(b) $d(z_p, z_n) = d_n$ for all $p > n$;
(c) $d_n > d_{n+1}$ and $\inf_n d_n > 0$.

Now we produce $y_s \in X$ and real numbers $d_s^i, \epsilon_s > 0$ for $s \in \mathbb{N}^{<\mathbb{N}}$ and $i \in \mathbb{N}$. We do it so that:

(α) $\epsilon_{s\restriction|s|-1} > d_s^i > d_s^{i+1} > \epsilon_s$;
(β) $d(y_s, y_{s^\frown i}) = d_s^i$;
(γ) $0 < \epsilon_s \leqslant 1/(|s|+1)$;

where, in (α), $\epsilon_{s\restriction|s|-1}$ is set to be ∞ for $s = \emptyset$. The construction is done in stages. At stage $n \in \mathbb{N}$ we have $y_s \in X$ for $|s| \leqslant n$, $z_t \in X$ for $0 < |t| \leqslant n+1$, and $\epsilon_s > 0$ for $|s| \leqslant n$ with the following properties:

$$\begin{aligned} z_t &\in Z \quad \text{for } 0 < |t| \leqslant n+1;\\ \epsilon_s &\leqslant \frac{1}{|s|+1} \quad \text{for } |s| \leqslant n;\\ d(y_s, y_{s^\frown i}) &= d(y_s, z_{s^\frown i}) \quad \text{for } |s| < n;\\ \epsilon_{s\restriction|s|-1} > d(y_s, z_{s^\frown i}) &> d(y_s, z_{s^\frown(i+1)}) > \epsilon_s \quad \text{for } |s| = n, \end{aligned} \tag{4}$$

where, by convention, the expression $\epsilon_{s\restriction|s|-1}$ is set to equal ∞ for $s = \emptyset$. Note that once these objects are constructed, (4) implies that the points y_s and the real numbers ϵ_s with d_s^i defined by (β) fulfill (α)–(γ).

Stage $n = 0$: By assumption Z contains a non-convergent sequence (z_i) that pseudo-converges to a point y. (Note that y may not be an element of Z.) Set

$$y_\emptyset = y, \quad z_{\langle i\rangle} = z_i \quad \text{and} \quad \epsilon_\emptyset = \min(1, \inf_i d(y, z_i)).$$

From properties (a)–(c) of pseudo-convergent, non-convergent sequences listed above, we obtain (4).

Stage $n+1$: We have already defined $z_t \in Z$ with $|t| = n+1$. The relatively open in Z, non-empty set $B(z_t, \epsilon_{t\restriction n}) \cap Z$ contains a non-convergent sequence (z_i) that pseudo-converges to some $y \in X$. After deleting finitely many elements of (z_i), we can assume that $d(y, z_i) > d(y, z_{i+1})$ for each i. Note that since $z_i, z_{i+1} \in B(z_t, \epsilon_{t\restriction n})$, we have

$$d(y, z_i) = d(z_i, z_{i+1}) < \epsilon_{t\restriction n}. \tag{5}$$

Since, by (5), $d(y, z_0) < \epsilon_{t\restriction n}$ and since $z_0 \in B(z_t, \epsilon_{t\restriction n})$, we see that $d(y, z_t) < \epsilon_{t\restriction n}$. Thus, since, by (4), we have $d(y_{t\restriction n}, z_t) > \epsilon_{t\restriction n}$, by considering the triangle with vertices $y_{t\restriction n}, y, z_t$, we obtain

$$d(y_{t\restriction n}, y) = d(y_{t\restriction n}, z_t). \tag{6}$$

Let

$$y_t = y,\ z_{t^\frown i} = z_i \quad \text{and} \quad \epsilon_t = \min(1/(|t|+1), \inf_i d(y, z_i)).$$

By (5) and (6) and the choice of z_i and ϵ_t, we see that (4) holds (where equality (6) is used to see the third line in (4), while point (5) is used to see the first inequality in the fourth line of (4)).

For $r \in \mathbb{N}^{<\mathbb{N}}$ with $r \neq \emptyset$, let

$$C^r = B(y_r, \epsilon_{r\restriction |r|-1}).$$

Consider also $C^\emptyset = X$. It is routine to check from (α)–(γ) and the ultrametric properties that for $r \in \mathbb{N}^{<\mathbb{N}}$ and distinct $i, j \in \mathbb{N}$ we have

$$C^r \supseteq C^{r^\frown i}, C^{r^\frown j} \quad \text{and} \quad C^{r^\frown i} \cap C^{r^\frown j} = \emptyset \tag{7}$$

and that the set

$$F = \bigcap_n \bigcup_{r \in \mathbb{N}^n} C^r$$

is closed.

We make a couple of remarks for future reference. By the definition of F, (α) and (β), and by ultrametric properties, we have

$$F \cap C^r \subseteq \{x \in X : \exists i \in \mathbb{N}\ d(x, y_r) = d_r^i\}. \tag{8}$$

Furthermore, sets of the form $F \cap C^r$ are non-empty and hence, applying this observation to $F \cap C^{r^\frown k}$, we see that for each $k \in \mathbb{N}$

$$\exists x \in F d(x, y_{r^\frown k}) < \epsilon_r. \tag{9}$$

We claim that χ_F cannot be recovered. Assume toward a contradiction that χ_F can be recovered and, using Lemma 2.3, fix a refinement B_s, x_s, with $s \in S$, of X with the properties from the lemma.

Claim 2.5. *Let $t \in \mathbb{N}^{<\mathbb{N}}$ and let $\alpha \in \mathbb{N}^\mathbb{N}$. If $\alpha\restriction n \in S$ and $B_{\alpha\restriction n}$ intersects $\bigcup_i C^{t^\frown i}$ for infinitely many n, then there exists i_0 with $B_{\alpha\restriction n} \subseteq C^{t^\frown i_0}$ for all but finitely many n.*

Proof. Assume that $B_{\alpha\restriction n}$ intersects $C^{t^\frown i}$ and $C^{t^\frown j}$ for some $i < j$. Then by (α) and (β)

$$d(y_{t^\frown i}, y_{t^\frown j}) = d_t^i > \epsilon_t$$

and each of the balls $C^{t^\frown i}$ and $C^{t^\frown j}$, centered at $y_{t^\frown i}$ and $y_{t^\frown j}$, respectively, has radius ϵ_t. Thus, since $B_{\alpha\restriction n}$ intersects both of them, it has radius at least d_t^i. Since it contains a point at distance less than ϵ_t that is also less than d_t^i from $y_{t^\frown j}$, it contains $y_{t^\frown j}$. Since this point is at distance $d_t^j < d_t^i$ from y_t, we see that y_t belongs to the ball $B_{\alpha\restriction n}$. Thus, the ball $B_{\alpha\restriction n}$ contains y_t and its radius is greater than ϵ_t.

It follows that if for infinitely many n, $B_{\alpha\restriction n}$ intersects more than one set $C^{t^\frown i}$, with $i \in \mathbb{N}$, then, for each n, $B_{\alpha\restriction n}$ has radius greater than ϵ_t and $\bigcap_n B_{\alpha\restriction n}$ is non-empty (as y_t is in this intersection). This conclusion contradicts our choice of B_s, x_s for $s \in S$; more precisely, it contradicts their properties listed in Lemma 2.3 and in point (C) of the definition of refinement. Thus, since the sequence of sets $(B_{\alpha\restriction n})$ descends as $n \to \infty$, there exists i_0 such that for all but finitely many n, and hence for all n, $B_{\alpha\restriction n}$ intersects $C^{t^\frown i_0}$. In an ultrametric space if two balls intersect each other, then one of them includes the other. It follows that either $C^{t^\frown i_0}$ is contained in each $B_{\alpha\restriction n}$, which contradicts the properties from Lemma 2.3 and property (C) in the definition of refinement as by (α) and (β) the diameter of $C^{t^\frown i}$ is greater than $\epsilon_{t^\frown i}$, or for all but finitely many n we have $B_{\alpha\restriction n} \subseteq C^{t^\frown i_0}$, as required. The claim is proved. □

We show now that there exists $s_0 \in S$ such that $B_{s_0} \cap F \neq \emptyset$ and for each $t \supseteq s_0$, with $t \in S$, if $B_t \cap F \neq \emptyset$, then $x_t \in F$. Otherwise, we would be able to produce $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha\restriction n \in S$ for each n and

$$B_{\alpha\restriction n} \cap F \neq \emptyset \quad \text{and} \quad x_{\alpha\restriction n} \notin F \ \text{ for infinitely many } n. \tag{10}$$

We claim that there exists $y \in \bigcap_n B_{\alpha\restriction n} \cap F$. To justify this claim, we construct $\beta \in \mathbb{N}^{\mathbb{N}}$ such that for each k, $B_{\alpha\restriction n} \subseteq C^{\beta\restriction k}$ for all but finitely many n. Assume that we have constructed $\beta\restriction k$. (Note that $\beta\restriction 0 = \emptyset$.) Since, by (7), $F \cap C^{\beta\restriction k} \subseteq \bigcup_i C^{(\beta\restriction k)^\frown i}$, by the first part of (10), for infinitely many n we have that,

$$B_{\alpha\restriction n} \cap \bigcup_i C^{(\beta\restriction k)^\frown i} \neq \emptyset.$$

By Claim 2.5, there exists an i_0 with $B_{\alpha\restriction n} \subseteq C^{(\beta\restriction k)^\frown i_0}$ for all but finitely many n. We define $\beta(k) = i_0$. Having defined β, we see that the element y such that $y \in \bigcap_k C^{\beta\restriction k}$ is in each $B_{\alpha\restriction n}$ and in F. The existence of such a y contradicts recoverability of χ_F since $\chi_F(y) = 1$ while, by the second part of (10), we have $\chi_F(x_{\alpha\restriction n}) = 0$ for infinitely many n. Thus, the existence of $s_0 \in S$ is shown.

Claim 2.6. *Let $s \supseteq s_0$ for $s \in S$, and let $r \in \mathbb{N}^{<\mathbb{N}}$ be such that*

$$y_r \in B_s \quad \text{and} \quad x_s \in F \cap C^r.$$

Then there exists $m \in \mathbb{N}$ such that $s^\frown m \in S$ and

$$y_r \in B_{s^\frown m} \quad \text{and} \quad x_{s^\frown m} \in F \cap C^r.$$

Proof. Since $x_s \in F \cap C^r$, by (8), we can fix $i \in \mathbb{N}$ with

$$d(y_r, x_s) = d_r^i. \tag{11}$$

Since $y_r \in B_s$ and $y_r \neq x_s$, from property (B) of refinements there exists m with $s^\frown m \in S$ and

$$y_r \in B_{s^\frown m}. \tag{12}$$

Point (12) gives $d(y_r, x_{s^\frown m}) < d(y_r, x_s)$. Thus, since $x_s \in C^r$, we have

$$x_{s^\frown m} \in C^r. \tag{13}$$

Take now $k > i$ such that $d(y_r, y_{r^\frown k}) = d_r^k < d(y_r, x_s)$ by (11). From this inequality, (12), and property (C) of refinements, we obtain

$$y_{r^\frown k} \in B_{s^\frown m}. \tag{14}$$

Formulas (11) and (12) give that the radius of $B_{s^\frown m}$ is d_r^i. From this observation and from (14) we obtain $B_{s^\frown m} \cap F \neq \emptyset$ since, by (9), there is a point in F at distance less than ϵ_r that is also less than d_r^i from $y_{r^\frown k}$. Since $s^\frown m$ extends s_0, this implies that

$$x_{s^\frown m} \in F. \tag{15}$$

The conclusion of the claim follows from (12), (13) and (15). □

Now note that since $x_{s_0} \in F$, we can find a $\beta \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{x_{s_0}\} = \bigcap_n C^{\beta\restriction n}.$$

If n is large enough, then $y_{\beta\restriction n} \in B_{s_0}$. Thus, for large enough n the assumptions of Claim 2.6 hold with $s = s_0$ and $r = \beta\restriction n$. Fix such an r and call it r_0.

Repeated application of Claim 2.6 allows us to construct $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that for each n, with $\alpha\restriction n \in S$, $y_{r_0} \in B_{\alpha\restriction n}$ and for $n \geqslant |s_0|$, with $x_{\alpha\restriction n} \in F$, which contradicts the assumption that B_s, x_s for $s \in S$ witnesses recoverability of χ_F, since $y_{r_0} \notin F$ (as no point of the form y_r is in F by (8)) and yet $x_{\alpha\restriction n} \in F$ for large n. □

3. *An example*

As previously noted, Lecomte showed in [**3**] that if the metric on an ultrametric Polish space has the property that each strictly decreasing sequence of elements of its range converges to 0, then every Baire one function is recoverable. As shown in [**1**, **3**], this is also the case when the metric space is σ-compact. In the following example, we produce a Polish ultrametric space which is not σ-compact and in which Lecomte's condition does not hold, but condition (iii) of Theorem 2.2 does hold and therefore all Baire one functions on the space are recoverable.

EXAMPLE 3.1. There exists a Polish ultrametric space (X, d) in which every open ball contains a sequence of points (x_k) with

$$d(x_k, x_{k+1}) > d(x_{k+1}, x_{k+2}) \quad \text{and} \quad \inf_k d(x_k, x_{k+1}) > 0, \tag{16}$$

and yet every pseudo-convergent sequence in (X, d) converges, and in particular, point (iii) of Theorem 2.2 holds.

Note that (16) and the Baire category theorem imply that X is not σ-compact and that (16) implies that Lecomte's condition fails for X. In fact, using (16) and the Baire category theorem one sees that X cannot be covered by countably many sets on which the metric fulfills Lecomte's condition. (This uses the easy-to-check fact that if the restriction of a metric to a set fulfills Lecomte's condition, then so does the restriction of the metric to the closure of that set.)

We make the range of the metric be a subset of $\mathbb{R}$ with the least element 0 and which is in a bijective order reversing correspondence with $\omega^2 + 1$. This correspondence is assumed to be continuous if $\omega^2 + 1$ is taken with the order topology. Designate the non-zero numbers in the range of the metric by λ^n_m for $n, m \in \mathbb{N}$ so that for all m_1, m_2 and all n we have $\lambda^n_{m_1} > \lambda^{n+1}_{m_2}$ and if $m_1 < m_2$, then $\lambda^n_{m_1} > \lambda^n_{m_2}$. Consider $\mathbb{N} \cup \{\infty\}$ with the usual convention that $n < \infty$ for each $n \in \mathbb{N}$. The underlying set of our metric space is

$$X = \{(n_m) \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}} : \forall m\, n_m \leqslant n_{m+1} \text{ and } \lim_m n_m = \infty\}.$$

Define the distance between two distinct sequences (n_m) and (n'_m) in X by letting

$$d((n_m), (n'_m)) = \lambda^{n_0}_{m_0},$$

where m_0 is the largest natural number such that $n_m = n'_m$ for all $m < m_0$ and $n_0 = \min(n_{m_0}, n'_{m_0})$.

We claim that (X, d) is a Polish ultrametric space. We leave the checking that d is an ultrametric to the reader. It is not difficult to see, and we also leave the proof to the reader, that X with the topology induced by d is a subspace of the compact metrizable space $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$, where $\mathbb{N} \cup \{\infty\}$ gets the order topology. Separability of (X, d) follows immediately. Furthermore, each sequence in X has a subsequence converging in $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ to a point that is an element of X or else is eventually equal (as a sequence with entries in $\mathbb{N} \cup \{\infty\}$) to some fixed natural number. Now to see the completeness of the metric d, we need to show that each Cauchy sequence has a convergent subsequence. Given a Cauchy sequence (x_k), choose a subsequence (y_k) converging in $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ to y. It is straightforward to check that if y were eventually equal to a natural number, then the sequence (y_k) would not be Cauchy. Thus, y is in X and (y_k) converges to it.

Furthermore, X has the following properties.

(i) Any sequence (x_k) of points from X pseudo-converging to $x \in X$ converges to x. This property is easily seen by considering a subsequence (y_k) of (x_k) that converges in $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ to some y and using the inequality $d(x, y_{k+1}) < d(x, y_k)$ for each k to show that $y = x$.

(ii) Any non-empty open set U in X contains a sequence of points the distances between which decrease to a real number greater than 0 as in (16). This property is witnessed by sequences (x_k) of elements of X constructed as follows. Let $n_m \in \mathbb{N}$ for $m \leqslant m_0$ be given. Fix k. For $m \in \mathbb{N}$, define $n^k_m = n_m$ if $m \leqslant m_0$, $n^k_m = n_{m_0}$ if $m_0 \leqslant m \leqslant m_0 + k$ and $n^k_m = \infty$ if $m_0 + k < m$. Let $x_k = (n^k_m)_m \in X$.

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