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The Arc-Length Method from Controlling Load to Controlling Electric Potential

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Abstract. This paper generalizes the arc-length method, and proposes a new scheme of controlling equilibrium path including static equilibrium and electrostatic equilibrium. To take example for the thin plate electrode posing on the electrostatic field force, whose governing equations are the two fold nonlinear coupling equation, the method to compute the incremental factor is give in detain. The result is to show that not only the approximate critical voltage is obtained but also the trace after losing stability is found stably.

Introduction

Usually if a structure loses stability, it has a large deformation or displacement, and the carrying capacity of structure decreases. Hence it is important to find critical value of loading and to predict the displacement response of structure in the theory. According to the nonlinear finite element method [1], the nonlinear solution depends on the adopted control scheme. If the scheme of the controlled load is introduced, the snap-back can occur and the post-critical respond cannot be traced. Or if the system is controlled by a single displacement component, at a point close to the maximum load the displacement can drift away from the equilibrium path. In order to overcome these problems and obtain post-critical equilibrium states, the arc-length control method was proposed by Riks [2] in 1979. This method, which introduces an auxiliary constraint condition with respect to the length of the equilibrium path, solves the algebraic equations along equilibrium paths. All of the stable or unstable resolutions to satisfy the equilibrium equation are obtained, and the critical load is gained. Hence the method has widely been used to solve the problem of bifurcations and post-buckling.

To my knowledge, so far the arc-length control method is used to solve the linear elastic fracture problem and the geometrical nonlinear problem as well as simple plastic problem, but the application in the coupling problem of multi-fields is little referred [3,4]. In recent years microelectromechanical structures has been designed and investigated, in which when the large deformation or displacement is necessary, post critical point equilibrium states has practically applied value [5]. The kind of problem has characteristic of multi-fields coupling. As critical value is approached, the computing process is unstable. And it is difficult to gain the post-buckling trace so well. Based on the principle of the direct arc-length method and the idea of the indirect arc-length method, this paper generalizes the arc-length control method, which changes controlling displacement-load curve into controlling deformation-voltage curve. Motivation is to resolve the coupling problems of both deformation field and electrostatic field. At the same time, this provides a probability that the arc-length method can been generalized to solve a part of multi-fields nonlinear problem.

The Arc-Length Method

It is well known for the elastic mass an external load f produces a configuration $\boldsymbol{w}$, this pair $(\boldsymbol{f},\boldsymbol{w})$ holds a static equilibrium situation. We obtain generally this by establishing the incremental finite element model and resolving its governing equation. The form of the incremental iteration is written as [1, 6]

$$\boldsymbol{K}_m^{i-1}\Delta\boldsymbol{w}_m^i = \lambda_m\boldsymbol{Q} - \boldsymbol{P}_m^{i-1} = \boldsymbol{R}(\lambda_{\mathrm{m}}) \tag{1}$$

Here, superscript i indicates the i-th iteration step, subscript m denotes the m-th incremental step, $\boldsymbol{K}$ is the tangent matrix, $\boldsymbol{Q}$ is the nodal external force vector, $\Delta\boldsymbol{w}$ is the incremental displacement vector, λ_m is the scaling load factor, $\boldsymbol{P}$ denotes the nodal internal force vector, $\boldsymbol{R}(\lambda_m)$ is defined as the total residual force, as well as $\boldsymbol{K}$ and $\boldsymbol{P}$ are dependent on the displacement field. After every iteration step, a modified configuration, $\boldsymbol{w}_m^{i+1} = \boldsymbol{w}_m^i + \Delta\boldsymbol{w}_m^i$, are obtained. Once Eq. (1) satisfies $\boldsymbol{R}(\lambda_m)$ within an incremental step, an equilibrium configuration $\boldsymbol{w}_m$ is obtained. If the iteration procedure processes n steps, the total incremental displacement can be written as

$$\Delta\mathbf{w}_m = \sum_{i=1}^{n}\Delta\mathbf{w}_m^i \tag{2}$$

For next incremental step the incremental load factor is employed following as

$$\lambda_{m+1} = \lambda_m + \Delta\lambda \tag{3}$$

The above increment procedure is essentially of controlling load. But as structure become soft, to gain equilibrium solution paths after limit points is very difficult.

Therefore, Riks and Wempner put forward the method that controlled the length of equilibrium path. An extra constraint to the residual equation is added to limit the length of the incremental solution. In the first iteration within a given increment, a vector r is defined as

$$\mathbf{r}^1 = \left[\Delta\mathbf{w}^1, \Delta\lambda^1\right]^T \tag{4}$$

Similarly, other vector $\Delta\boldsymbol{r}$ after the first iteration is presented as

$$\Delta\boldsymbol{r}^i = \left[\Delta\boldsymbol{w}^i, \Delta\lambda^i\right]^T \tag{5}$$

Here, we introduce the enforcing form of Risk' procedure, which is two steps procedure. The constant arc-length constraint is imposed on the first iteration.

$$\left(\boldsymbol{r}^1\right)^T\left(\boldsymbol{r}^1\right) = l^2 \tag{6}$$

where l^2 is a pre-given value. This is accomplished by

$$\left(\Delta\boldsymbol{w}^1\right)^T\left(\Delta\boldsymbol{w}^1\right) + \Delta\lambda^1\Delta\lambda^1 = l^2 \tag{7}$$

In subsequent iterations ($i > 1$) the enforcing constraint satisfies the following equation

$$\left(\boldsymbol{r}^1\right)^T\left(\Delta\boldsymbol{r}^i\right) = 0 \qquad i > 1 \tag{8}$$

Substituting Eqs. (4) and (5) into Eq. (8), we arrive at

$$\left(\Delta\boldsymbol{w}^1\right)^T\left(\Delta\boldsymbol{w}^i\right) + \Delta\lambda^1\Delta\lambda^i = 0 \tag{9}$$

For the sake of getting $\Delta\lambda^1$ and $\Delta\lambda^i$, a system of equation with matrix coefficient $\boldsymbol{K}_m^{i-1}$ is solved twice as follows. Noticing that Eq. (1) may be rewritten as

$$\boldsymbol{K}_{m-1}^{i-1}\boldsymbol{w}_m^i = \left(\lambda_{m-1}^{i-1} + \Delta\lambda^i\right)\boldsymbol{Q} - \boldsymbol{P}_{m-1}^{i-1} \tag{10}$$

To resolve Eq. (10) is equal to solve the following two systems of equation with $\boldsymbol{K}_{m-1}^{i-1}$ as coefficient matrix

$$\boldsymbol{K}_{m-1}^{i-1}\Delta\boldsymbol{w}_m^{i(1)} = \boldsymbol{Q} \tag{11}$$

$$\boldsymbol{K}_{m-1}^{i-1}\Delta\boldsymbol{w}_m^{i(2)} = \lambda_{m-1}^{i-1}\boldsymbol{Q} - \boldsymbol{P}_{m-1}^{i-1} \tag{12}$$

Hence, the incremental displacement vector at iteration i can be expressed as

$$\Delta\boldsymbol{w}_m^i = \Delta\lambda^i\Delta\boldsymbol{w}_m^{i(1)} + \Delta\boldsymbol{w}_m^{i(2)} \tag{13}$$

After substituting Eq. (13) into Eq. (9), the expression of $\Delta\lambda^i$ is written as

$$\Delta\lambda^i = -\frac{\left(\Delta\boldsymbol{w}^1\right)^T\Delta\boldsymbol{w}_m^{i(2)}}{\left(\Delta\boldsymbol{w}^1\right)^T\Delta\boldsymbol{w}_m^{i(2)} + \Delta\lambda^1} \tag{14}$$

Finally, the new load factor is computed according to

$$\lambda^i = \lambda^{i-1} + \Delta\lambda^i \tag{15}$$

Hereinbefore, we have only introduced one of many numerically controlled methods. In general, the method is called the normal plane constraint method.

Generalizing Procedure

Within a coupling system consisting of deformation field and electrostatic field, the potential must satisfy the Laplace equation in the region between the conductors. The charge density on each conductor can be obtined by solving a integral equation [7, 8] in moment method followed as

$$\boldsymbol{C}(\boldsymbol{w})\boldsymbol{D} = \boldsymbol{U} \tag{16}$$

here, $\mathbf{C} \in \mathbf{R}^{N\times N}$, $\boldsymbol{D}$ is the vector of the surface charge density, $\mathbf{U} \in \mathbf{R}^N$ is the vector of known panel potential, and $\mathbf{C}_{ij} = \int_{S_j}\frac{1}{4\pi\varepsilon_0\left|\mathbf{x}_i - \mathbf{x}'_j\right|}dS'$, in which $\mathbf{x}_i$ is the center position vector of the i-th panel, $\mathbf{x}'_j$ is the center position vector of the j-th panel, and S_j is the area of the j-th panel.

According to the electrostatic theory [7], the electric force q^e exerted on the plate can be formulated in terms of the electric charge density D_e of the form as

$$\overset{\rightarrow}{q^e} = \frac{D_e^2}{2\varepsilon_0}\vec{n} \tag{17}$$

Here, $\vec{n}$ denotes the unit vector normal to the surface of the deformed plate.

Similar to Eq. (1), the incremental electric potential is applied to Eq. (16) to lead to $\boldsymbol{C}(\boldsymbol{w})\boldsymbol{D} = \boldsymbol{U}_0 + \lambda\Delta\boldsymbol{U}$, in which λ is a scale factor, $\boldsymbol{U}_0$ denotes the potential vector of some a configuration which satisfies the electrical equilibrium, and $\Delta\boldsymbol{U}$ is a column of reference increment of potential at nodes. We adopt the Riks's law [1, 2] at every incremental step, which is rewritten as

$$(\Delta\boldsymbol{w}^1)^{\boldsymbol{T}}(\Delta\boldsymbol{w}^1) + \lambda^1\lambda^1 = l^2 \tag{18}$$

$$(\Delta\boldsymbol{w}^1)^{\boldsymbol{T}}(\Delta\boldsymbol{w}^i) + \lambda^1\lambda^i = 0 \quad \text{For} \quad i > 1 \tag{19}$$

Signification of Eqs. (18) and (19) is different from that of Eqs. (7) and (9), in the classical arc length method the enforcing constraint equation gives a relationship between load and deformation, but in Eqs. (18) and (19) the relationship of the potential and of the deformation is conferred.

Similarly, in order to obtain the iteration incremental potential factor, $\lambda^{\boldsymbol{i}}$, the split-step technique is employed as Eqs. (11) and (12). In the following context the index *i* and *m* denote the *i*-th iteration step and the *m*-th incremental step, respectively. The following two systems of equations with $\boldsymbol{C}$ as coefficient matrix are resolved

$$\boldsymbol{C}(\boldsymbol{w})^i_m \boldsymbol{D}^i_m = \boldsymbol{U}^i \text{ , } \boldsymbol{C}(\boldsymbol{w})^i_m \Delta\boldsymbol{D}^i_m = \Delta\boldsymbol{U} \tag{20}$$

If the total charge density at the i-th iteration are expressed as

$$\boldsymbol{D} = \boldsymbol{D}^i_m + \lambda^i \Delta\boldsymbol{D}^i_m \tag{21}$$

substitution of Eq. (21) into Eq. (17) yields the electrostatic distribution force of the i-th iteration step, $\boldsymbol{q}^e_i = \boldsymbol{q}^e_{(1)} + \lambda^i \boldsymbol{q}^e_{(2)} + \lambda^{i2} \boldsymbol{q}^e_{(3)}$, where $\boldsymbol{q}^{\mathrm{e}}_{(1)}$, $\boldsymbol{q}^{\mathrm{e}}_{(2)}$ and $\boldsymbol{q}^{\mathrm{e}}_{(3)}$ are equal to $\boldsymbol{q}^e_{(1)} = \frac{1}{2\varepsilon_0} \boldsymbol{D}^{i\,2}_m$, $\boldsymbol{q}^e_{(2)} = \frac{1}{\varepsilon_0} \Delta\boldsymbol{D}^i_m \boldsymbol{D}^i_m$, $\boldsymbol{q}^e_{(3)} = \frac{1}{2\varepsilon_0} \Delta\boldsymbol{D}^{i\,2}_m$. Noting that Eq. (10) may be rewritten as

$$\boldsymbol{K}\Delta\boldsymbol{w}^i = \boldsymbol{Q}_{(1)} + \lambda^i \boldsymbol{Q}_{(2)} + \lambda^{i^2} \boldsymbol{Q}_{(3)} - \boldsymbol{P} \tag{22}$$

where $\boldsymbol{Q}_{(1)}$, $\boldsymbol{Q}_{(2)}$ and $\boldsymbol{Q}_{(3)}$, which are obtained by standard finite element procedure, are produced by $\boldsymbol{q}^{\mathrm{e}}_{(1)}$, $\boldsymbol{q}^{\mathrm{e}}_{(2)}$ and $\boldsymbol{q}^{\mathrm{e}}_{(3)}$. Again, the method treating Eq. (10) is applied to Eq. (22) to result in $\boldsymbol{K}\Delta\boldsymbol{w}_{(1)} = \boldsymbol{Q}_{(1)} - \boldsymbol{P}$, $\boldsymbol{K}\Delta\boldsymbol{w}_{(2)} = \boldsymbol{Q}_{(2)}$, and $\boldsymbol{K}\Delta\boldsymbol{w}_{(3)} = \boldsymbol{Q}_{(3)}$. Hence, the total displacement increment at an iteration step is expressed as

$$\Delta\boldsymbol{w}^i = \Delta\boldsymbol{w}_{(1)} + \lambda^i \Delta\boldsymbol{w}_{(2)} + \lambda^{i^2} \Delta\boldsymbol{w}_{(3)} \tag{23}$$

Finally, by substituting (23) into (19) we obtain a quadratic equation, $a\lambda^{i2} + b\lambda^i + c = 0$, where a , b and c are defined as $a = (\Delta\boldsymbol{w}^1)^{\boldsymbol{T}} \Delta\boldsymbol{w}_{(3)}$, $b = (\Delta\boldsymbol{w}^1)^{\boldsymbol{T}} \Delta\boldsymbol{w}_{(2)} + \lambda^1$, and $c = (\Delta\boldsymbol{w}^1)^{\boldsymbol{T}} \Delta\boldsymbol{w}_{(1)}$. For the first iteration step is based on some equilibrium state, where the nodal internal forces equal to the nodal internal forces, so $\boldsymbol{K}\Delta\boldsymbol{w}_{(1)} = \boldsymbol{Q}_{(1)} - \boldsymbol{P} = 0$. And the expression of $\Delta\boldsymbol{w}^1$ is

$$\Delta\boldsymbol{w}^1 = \lambda^1 \Delta\boldsymbol{w}_{(2)} + \lambda^{1^2} \Delta\boldsymbol{w}_{(3)} \tag{24}$$

If substitution of (24) into (18) yields a quartic equation, to obtain and to select λ^1 become slightly difficult. But, owing to an equal position to assign λ^1 or l in Eq. (18), λ^1 can be gained by Eq. (18) after l is selected.

It should be noted that the arc-length method employed here is some distinct from that in Ref. [2]. In the typical arc-length method the incremental factor λ^i is a linear incremental load factor, and directly applied to the external nodal force vector Q , whereas in this paper λ^i is the linear incremental factor for the electric potential and the nonlinear factor for the external nodal force. A nonlinear relationship is embodied between the potential and the electrostatic force also.

Numerical Example

In this part we present an example [5] to illustrate the availability of the above method. Fig.1 shows the sectional view of the driving plate in MEMS, and where other two opposite sides are free. When a voltage is applied between the plate and the ground plane, a charge distribution is induced on the plate to deflect and the charge distribution on the plate below surface is changed due to the plate deflection, and an equilibrium state, which is consisted of both static equilibrium and electrostatic equilibrium, is obtained when the forces owing to the plate deflection and the surface each other. Here is 200 μ m long, 100 μ m wide, and 0.5 μ m thick, the original gap g is 2.1 μ m, and Δd denotes the shuffling displacement. We employ the theory of geometrical nonlinear thin plate with the van Karman type deformation for plate deformation. The plate is discretized into 40×20 (along length, width, respectively) eight-node Serendipity rectangular elements and the ground plane is discretized into 800 rectangular elements.

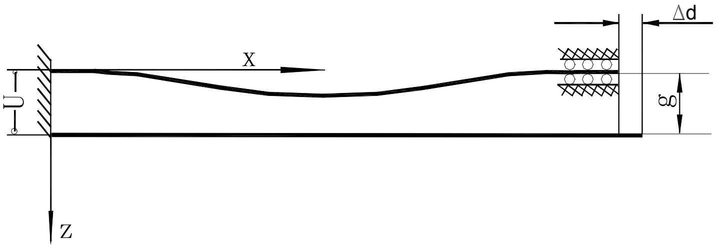

Fig.1 the sectional view of the driving plate

Fig.2 shows the characteristic curves of the maximum deflection versus the applied control voltage. It is obviously found that the critical value of instability and the post-buckling phenomena are captured by this paper method, and that near the critical voltage tracing process is very stable. A curve of the maximum distribution force versus the applied control is plot in Fig.3. Comparing Fig.2 and Fig.3, we noticed after passed over critical point the electrostatic force is to increase, which is different from that only considered the geometric nonlinear the force decreases with the deformation becoming large.

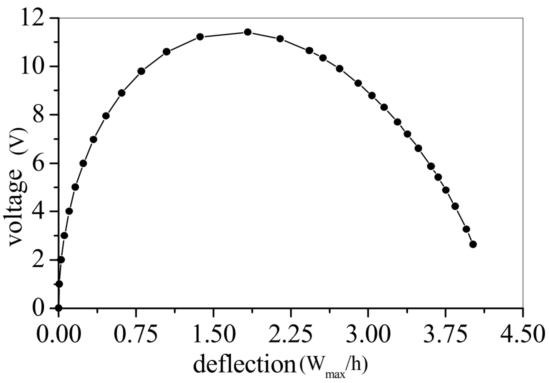

Fig.2 The response curves of the maxim transverse deflection

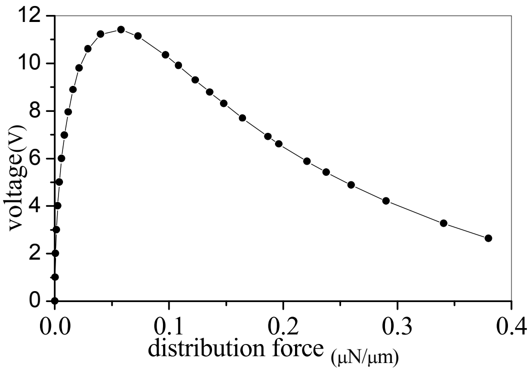

Fig.3 The applied voltage versus maximum distribution electric force

Conclusion

In this paper we extend the applied area of the arc-length control method, and offer the equilibrium path computation scheme with respect to the coupling problem of electrostatic field and deformation field. The incremental factor is deduced in detail. And we find a phenomenon also, in which, after passing over the critical point, both the deformation and the electrostatic field force increase.

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