On n-panelled spatial graphs, n-flat graphs and graph minors*

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1. Introduction

Let G be a finite graph. We consider G as a topological space in the usual way. A subgraph γ of G is called a *cycle* if it is homeomorphic to the 1-sphere. We denote the set of all cycles of G by $C(G)$. For an embedding $\phi : G \to S^3$, the image $\Gamma = \phi(G)$ is called a *spatial graph* of G. A simple closed curve C in Γ is called a *circuit*. We note that for any circuit C there is a unique cycle $\gamma \in C(G)$ such that $C = \phi(\gamma)$.

1.1. Panelled spatial graphs, flat graphs and Sachs' linkless embedding conjecture

For a spatial graph Γ of a graph G, an embedded disk D in S^3 is called a Γ-*panel for* C if $D \cap \Gamma = \partial D = C$. A spatial graph Γ is said to be *panelled* if every circuit C of Γ admits a Γ-panel. A graph G is said to be *flat* if there exists a panelled spatial graph Γ of G. The notion of panelled spatial graphs and flat graphs were introduced by T. Böhme [1] to solve what is called Sachs' linkless embedding conjecture. J. H. Conway and C. Gordon [2] and H. Sachs [12] proved that K_6 is *intrinsically linked*, namely any spatial graph of K_6 contains a non-splittable link, and Sachs also conjectured that a graph G is intrinsically linked if and only if G has a *minor* in the set of seven graphs obtained from K_6 and $K_{3,3,1}$ by $\Delta - Y$ exchanges (Figure 1.1), where a *minor* of G is a graph that is obtained from a subgraph of G by contracting edges. It can be checked that the set of seven graphs above coincides with the set of graphs obtained from the *Petersen graph* by $\Delta - Y$ and $Y - \Delta$ exchanges. So it is called the *Petersen family*. Böhme also conjectured in [1] that if a graph is not intrinsically linked then it is flat. Since each flat graph is not intrinsically linked and each graph in the Petersen family is intrinsically linked, his conjecture was a strengthening of Sachs' conjecture. In [11], N. Robertson, P. Seymour and R. Thomas proved the conjecture above in accordance with Böhme's proposal as follows.

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Theorem 1.1.1. ([11, (1.2)]) *For a graph G, the following are equivalent.*
(i) *G is not intrinsically linked,*
(ii) *G is flat,*
(iii) *G has no minor in the Petersen family.* $\square$

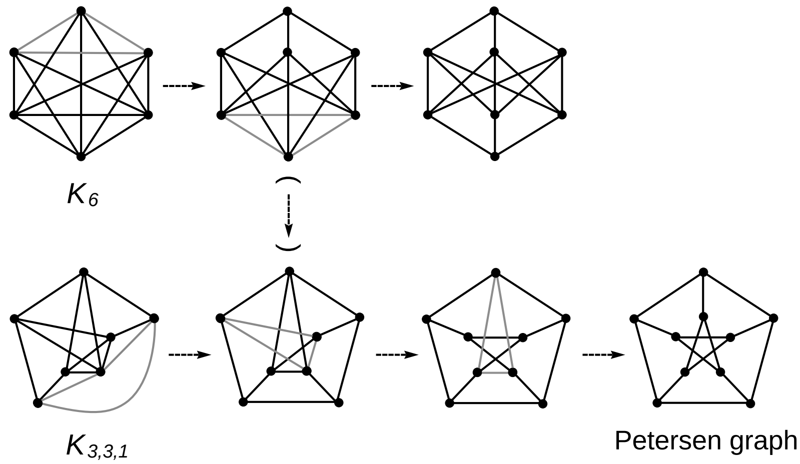

Fig. 1.1. The Petersen family

1.2. A filtration on flat graphs

One of our motivations is Böhme's lemma [1, (2.1)]. Let Γ be a panelled spatial graph and C_1, C_2, $\ldots$, C_n circuits of Γ such that $C_i \cap C_j$ is either empty or connected for $i \neq j$. Then he proved that there are Γ-panels D_1, D_2, $\ldots$, D_n for C_i's respectively whose interiors are mutually disjoint. But he did not mention to the case that $C_i \cap C_j$ is disconnected for some i and j. In fact, as we will say later, Böhme's lemma does not always hold in such a case. Therefore it is natural to consider the following property of a graph which gives a filtration on the set of flat graphs.

Definition 1.2.1. A spatial graph Γ is said to be *n-panelled* if for any mutually distinct n circuits C_1, C_2, $\ldots$, C_n of Γ, there are Γ-panels D_1, D_2, $\ldots$, D_n for C_i's respectively whose interiors are mutually disjoint. Such Γ-panels are said to be *good*.

Definition 1.2.2. A graph G is *n-flat* if there exists an n-panelled spatial graph or a $|C(G)|$-panelled spatial graph of G.

We note that if $|C(G)| < n$ then any spatial graph of G cannot be n-panelled. But we may regard G as an n-flat graph if there exists a $|C(G)|$-panelled spatial graph of G. A graph which admits a $|C(G)|$-panelled spatial graph is n-flat for any positive integer n. Such graphs were characterized by the first author and R. Shinjo [6].

We define the *flatting number of* G by

$$\mathrm{fl}(G) = \begin{cases} |C(G)| & \text{if } G \text{ is } |C(G)|\text{-flat}, \\ \max\{\ n \mid G \text{ is } n\text{-flat}\ \} & \text{otherwise.} \end{cases}$$

For any positive integer n, there exists a connected graph G with $\mathrm{fl}(G) = n$. For example, the flatting number of the bouquet with n loops is equal to n. Note that a spatial graph is 1-panelled if and only if it is panelled, namely a graph G is 1-flat if and only if it is flat. We also note that an n-panelled spatial graph is $(n-1)$-panelled, namely an n-flat graph is $(n-1)$-flat. Thus n-flatness gives a filtration on the set of 1-flat graphs.

Let $\mathcal{P}$ be a property of graphs inherited by minors. Let $\Omega(\mathcal{P})$ be the set of graphs that do not have $\mathcal{P}$ and whose proper minors have $\mathcal{P}$. We call $\Omega(\mathcal{P})$ the *obstruction set* for $\mathcal{P}$. According to Robertson-Seymour's Graph Minor Theorem [7], $\Omega(\mathcal{P})$ is a finite set. For n-flatness and graph minors, we show the following.

Proposition 1.2.3. *Each minor of an n-flat graph is n-flat.* $\square$

Therefore by Proposition 1.2.3 we are interested in the obstruction set $\Omega(\mathcal{F}_n)$, where $\mathcal{F}_n$ denotes the n-flatness. We note that $\Omega(\mathcal{F}_1)$ is the Petersen family by Theorem 1.1.1.

In our recent research [5], we discussed the n-flatness for $n \geq 2$ and could determine the elements of $\Omega(\mathcal{F}_n)$ for $n \geq 3$. The elements of $\Omega(\mathcal{F}_2)$ has not been determined yet. This is still open. Our purpose in this short report is to introduce the most impressive part in it. In section 2, we reveal the all elements of $\Omega(\mathcal{F}_n)$ for $n \geq 3$. In section 3, we give several elements of $\Omega(\mathcal{F}_2)$ and present a conjecture.

2. The obstruction sets $\Omega(\mathcal{F}_n)$ for $n \geq 3$

Let K_5, $K_{3,3}$, $K_{3,4}$, K_4, $\mathbf{D}_3$, $Y_{2,2,2}$, $\Lambda V_{2,2}$, $N_{2,2}$ and θ_m be graphs as illustrated in Figure 2.1. Let $V_{n_1,n_2,\dots,n_k}$ be a graph as illustrated in Figure 2.1, where the pair of two vertices v_0 and v_i are adjacent by n_i edges $e_{i,1}, e_{i,2}, \dots, e_{i,n_i}$ $(i = 1, 2, \dots, k)$. For a graph G that is isomorphic to $Y_{2,2,2}$, $\Lambda V_{2,2}$, $N_{2,2}$ or $V_{n_1,n_2,\dots,n_k}$, we denote the cycle that is determined uniquely by the pair of two edges e and e' by $\gamma(e, e')$.

2.1. $3, 4$ and 5-flatness

Let us consider the graph K_4. For a spatial graph Γ of K_4, if there are Γ-panels D_1, D_2 and D_3 for three mutually distinct circuits C_1, C_2 and C_3 each of length four, then we can see that $H_1(\Gamma \cup D_1 \cup D_2 \cup D_3; \mathbf{Z}) \cong \mathbf{Z}_2$ by a direct calculation. Then by applying [3, Lemma 6.8], we have that P is never embedded in S^3. This implies the following.

Proposition 2.1.1. *For any spatial graph Γ of K_4, the three mutually distinct circuits each of length four do not admit good Γ-panels. In particular, K_4 is not 3-flat.* $\square$

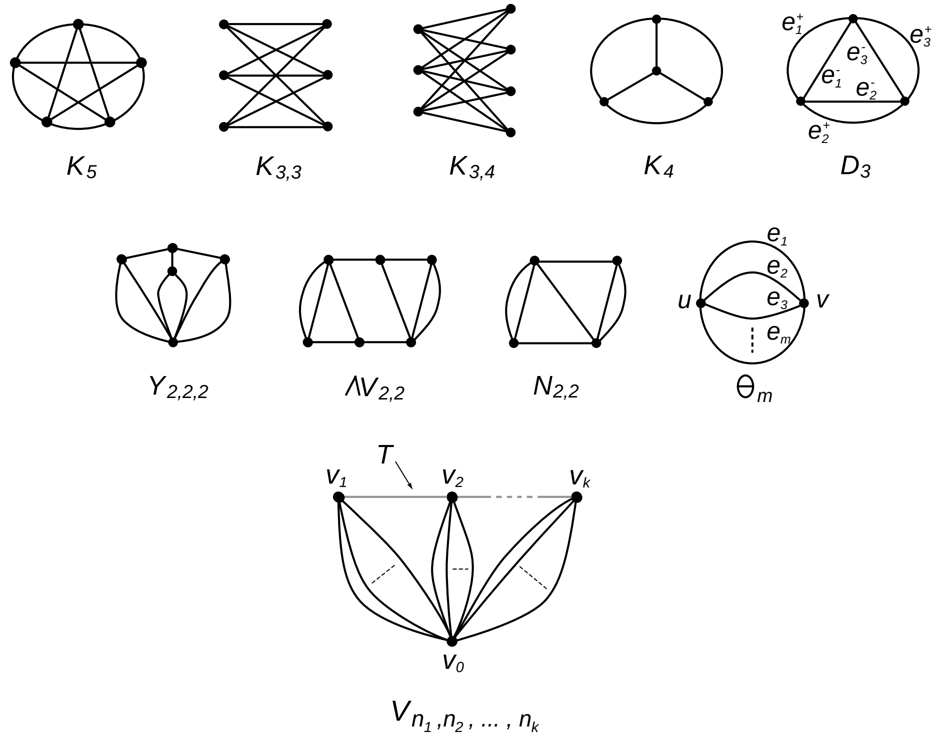

Fig. 2.1.

We note that each of $C_1 \cap C_2$, $C_2 \cap C_3$ and $C_1 \cap C_3$ is disconnected. Thus this is an example that does not let Böhme's lemma hold.

It is known that each 3-connected graph contains a subgraph that is homeomorphic to K_4. We note that each of K_5 and $K_{3,3}$ is 3-connected and it is well known that a graph G is planar if and only if it does not contain any subgraph that is homeomorphic to K_5 or $K_{3,3}$ [4]. Thus we have the following.

Corollary 2.1.2. *Each* 3*-connected graph is not* 3*-flat. In particular, each non-planar graph is not* 3*-flat.* $\square$

We also have that if $G \in \Omega(\mathcal{F}_3)$ then G is planar. In the end we have the following.

Theorem 2.1.3. $\Omega(\mathcal{F}_3) = \{K_4\}$. $\square$

Next let us consider the graph $\mathbf{D}_3$. In a similar way as Proposition 2.1.1, we have the following.

Proposition 2.1.4. *For each spatial graph* $\Gamma \subset S^3$ *of* $\mathbf{D}_3$*, the four circuits corresponding to the four cycles* $e_1^- \cup e_2^- \cup e_3^-$, $e_1^+ \cup e_2^+ \cup e_3^-$, $e_1^- \cup e_2^+ \cup e_3^+$ *and* $e_1^+ \cup e_2^- \cup e_3^+$ *do not admit good* Γ*-panels* D_i*. In particular,* $\mathbf{D}_3$ *is not* 4*-flat.* $\square$

We can see that K_4 is also an element in $\Omega(\mathcal{F}_4)$, and K_4 and $\mathbf{D}_3$ are also elements in $\Omega(\mathcal{F}_5)$. Moreover we can determine the obstruction set for $4, 5$-flatness as follows.

Theorem 2.1.5. $\Omega(\mathcal{F}_4) = \Omega(\mathcal{F}_5) = \{K_4, \mathbf{D}_3\}$. $\square$

2.2. $6, 7, 8, 9$ and 10-flatness

For the obstruction set for $6, 7, 8, 9$ and 10-flatness, we have the following.

Theorem 2.2.1. (i) $\Omega(\mathcal{F}_6) = \{Y_{2,2,2}\}$.
(ii) $\Omega(\mathcal{F}_7) = \{V_{2,2,2}, \Lambda V_{2,2}\}$.
(iii) $\Omega(\mathcal{F}_8) = \{V_{4,2}, V_{2,1,2}, N_{2,1,2}\}$.
(iv) $\Omega(\mathcal{F}_9) = \{V_{3,2}, \theta_6\}$.
(v) $\Omega(\mathcal{F}_{10}) = \{\theta_5\}$. $\square$

In the following we give several examples. Let us consider the graph θ_5. For any spatial graph Γ of θ_5, let $N(u)$ be a regular neighbourhood of $\phi(u)$. Assume that θ_5 is 10-flat (namely $|C(G)|$-flat). We may assume that $\partial N(u)$ and Γ-panels $D_1, D_2, \ldots, D_{10}$ of Γ are in general position and $\partial N(u) \cap D_i$ consists of an arc. Then by observing $\partial N(u) \cap (\Gamma \cup (\bigcup_{i=1}^{10} D_i))$, we have that K_5 can be embedded in S^2. This is a contradiction. So we have that θ_5 is not 10-flat. In the same way, we have that for each spatial graph Γ of θ_6, the nine circuits corresponding to $e_1 \cup e_4$, $e_1 \cup e_5$, $e_1 \cup e_6$, $e_2 \cup e_4$, $e_2 \cup e_5$, $e_2 \cup e_6$, $e_3 \cup e_4$, $e_3 \cup e_5$ and $e_3 \cup e_6$ do not admit good Γ-panels. So θ_6 is not 9-flat. Figure 2.2 illustrates these observations.

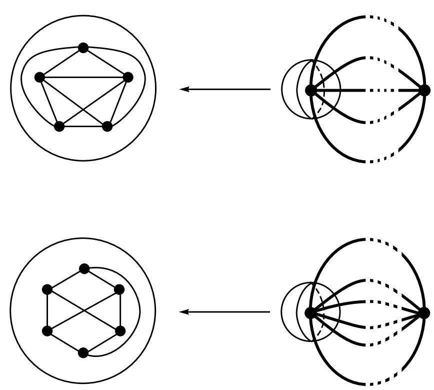

Fig. 2.2.

Next let us consider the graph $V_{3,2}$. For any spatial graph Γ of $V_{3,2}$, let $N(T)$ be a regular neighbourhood of $\phi(T)$, where T is a path which consists of gray edges as illustrated in Figure 2.1. Assume that $V_{3,2}$ is 9-flat. In a similar way as the case of θ_5 and θ_6, we observe $\partial N(T) \cap (\Gamma \cup (\bigcup_{i=1}^{9} D_i))$. Then we can see that the nine circuits corresponding to $\gamma(e_{1,1}, e_{1,2})$, $\gamma(e_{1,2}, e_{1,3})$, $\gamma(e_{1,3}, e_{1,1})$, $\gamma(e_{1,1}, e_{2,1})$, $\gamma(e_{1,2}, e_{2,1})$, $\gamma(e_{1,3}, e_{2,1})$, $\gamma(e_{1,1}, e_{2,2})$, $\gamma(e_{1,2}, e_{2,2})$ and $\gamma(e_{1,3}, e_{2,2})$ do not admit good Γ-panels. So $V_{3,2}$ is not 9-flat. Figure 2.3 illustrates this observation.

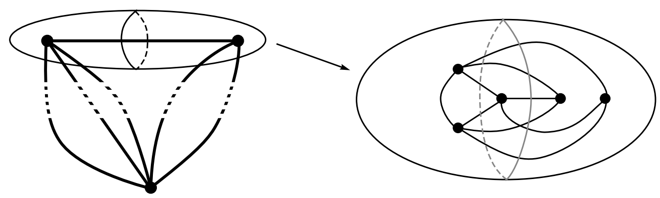

Fig. 2.3.

2.3. On graphs of higher flatting numbers

To consider the obstruction set for n-flatness for $n \geq 11$, we determine the flatting numbers of θ_m. By Theorem 2.2.1 (iv) and (v), we have that $\mathrm{fl}(\theta_5) = 9$ and $\mathrm{fl}(\theta_6) = 8$. On the other hand, it is known that each of θ_2, θ_3 and θ_4 has a $|C(G)|$-panelled spatial embedding [6]. So we have the following.

Theorem 2.3.1. $\mathrm{fl}(\theta_2) = 1$, $\mathrm{fl}(\theta_3) = 3$, $\mathrm{fl}(\theta_4) = 6$, $\mathrm{fl}(\theta_5) = 9$ *and* $\mathrm{fl}(\theta_{m\geq 6}) = 8$. $\square$

Let G be a graph and T a subgraph of G that is a tree. We denote a minor of G obtained by contracting all edges of T by G/T. The following lemma plays an important role to determine the obstruction set for n-flatness.

Lemma 2.3.2. *Let G be a 2-connected graph. If G does not have a minor homeomorphic to $\mathbf{D}_3$, then G contains disjoint trees T^+ and T^- such that $G/T^+/T^- = \theta_{\beta_1(G)+1}$.* $\square$

By using Lemma 2.3.2, we can also give the upper bound of the flatting number of 2-connected graphs as follows. For a 2-connected graph G, it is known that G has an edge e such that $G\backslash e$ is 2-connected. Namely there is a sequence $G_2 = \theta_3$, G_3, $\ldots$, G_{n-1}, $G_n = G$ of 2-connected graphs such that G_{i+1} is obtained by adding an arc e_{i+1} to G_i and $\beta_1(G_i) = i$. Suppose that $\mathrm{fl}(G) \geq 10$. Since G_i is a minor of G, by Proposition 2.1.4 we have that each minor of G_i is not homeomorphic to $\mathbf{D}_3$.

By Theorem 2.3.1, we have that $\mathrm{fl}(G) < 9$ unless $\beta_1(G) = 4$. If $\beta_1(G) = 4$, then G has a minor which is homeomorphic to θ_5. It is easy to see that $V_{3,2}$ is the only critical 2-connected graph that contains a minor which is homeomorphic to θ_5, Namely, if $\beta_1(G) = 4$ and $G \neq \theta_5$ then G has a minor homeomorphic to $V_{3,2}$. Thus $\mathrm{fl}(G) < 9$ by Theorem 2.2.1 (iv). So we have the following.

Theorem 2.3.3. *The flatting number of a 2-connected graph other than θ_5 is at most eight.* $\square$

Corollary 2.3.4. $\Omega(\mathcal{F}_n) = \emptyset$ *for* $n \geq 11$. $\square$

3. On the obstruction set for 2-flatness

3.1. 2-flatness of planar graphs

Let G be a planar graph and Γ be a trivial spatial graph, namely there is an embedded 2-sphere F in S^3 containing a spatial graph Γ of G. Then, F separates S^3 into two 3-balls B_1, B_2. We note that any circuit of Γ can be naturally regarded as a circle on the boundary of each B_i. For any two distinct circuits C_1, C_2 of Γ, we see that C_i bounds a properly embedded disk D_i in B_i $(i = 1, 2)$ such that $D_1 \cap \Gamma = C_1$, $D_2 \cap \Gamma = C_2$ and $\mathrm{int}D_1 \cap \mathrm{int}D_2 = \emptyset$. Thus we have the following.

Proposition 3.1.1. *Each trivial spatial graph is 2-panelled. In particular, each planar graph is 2-flat.* $\square$

The converse of Proposition 3.1.1 is not true. In fact, each of K_5 and $K_{3,3}$ is 2-flat.

3.2. A 1-flat graph which is not 2-flat

Next we consider the graph which is not 2-flat. It is clear that each graph which has a minor in the Petersen family is not 2-flat. So we want to detect the 1-flat graph which is not 2-flat. Let us consider the graph $K_{3,4}$ (Figure 2.1).

Theorem 3.2.1. $K_{3,4}$ *is 1-flat but not 2-flat. Moreover,* $K_{3,4} \in \Omega(\mathcal{F}_2)$.

In the following we give a sketch of a proof of Theorem 3.2.1. We can see that a spatial graph Γ_0 of $K_{3,4}$ as illustrated in Figure 3.1 is 1-panelled, so $K_{3,4}$ is 1-flat. We can see also that each 1-panelled spatial graph of $K_{3,4}$ can be transformed into Γ_0 by exchanging the labels of vertices and ambient isotopies. So we have that 1-panelled spatial graphs of $K_{3,4}$ are unique up to ambient isotopy forgetting the labels of vertices.

Let C_1 and C_2 be two circuits of Γ_0 as in Figure 3.1. Then it can be proved that Γ_0-panels for C_1 are unique up to ambient isotopy. Then by taking a simple arc on the Γ_0-panel D_1 for C_1, there is a Hopf link containing C_2 as a component (Figure 3.2). Thus there does not exist any Γ_0-panel for C_2. So we have that Γ_0 is not 2-panelled. This implies that $K_{3,4}$ is not 2-flat.[1] We remark here that this is a simple example that does not let Böhme's lemma hold. We can check that each of proper minors of $K_{3,4}$ is 1-flat. Therefore we have that $K_{3,4} \in \Omega(\mathcal{F}_2)$.

[1] K. Taniyama suggested that this can be proved by using the *Simon invariant* of spatial subgraph of Γ which is homeomophic to $K_{3,3}$.

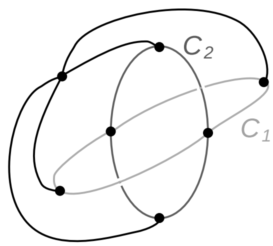

Fig. 3.1. A panelled spatial graph Γ_0 of $K_{3,4}$

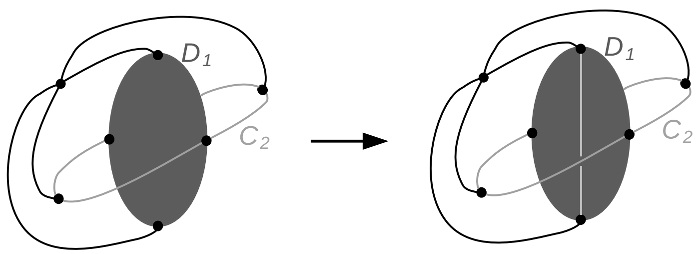

Fig. 3.2.

We can see that each graph in the Petersen family which is not K_6 has a minor which is homeomorphic to $K_{3,4}$. Moreover we can check that each of proper minors of K_6 is 1-flat. Thus we have that $\Omega(\mathcal{F}_2) \cap$ the Petersen family $= \{K_6\}$.

Conjecture 3.2.2. $\Omega(\mathcal{F}_2) = \{K_6,\ K_{3,4}\}$.

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