An Algebraic Characterization of First-Order Definability

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Abstract

We give a variable-free relational calculus which defines exactly all first-order definable relations in a arbitrary structure. We then show that, over an arbitrary class $\mathcal{C}$ of finite ordered structures with signature $\{\boldsymbol{<}, R_1, \ldots, R_\alpha\}$, the unary relations uniformly defined by this calculus over $\mathcal{C}$ are characterized by a another simplified variable-free calculus which we call $\mathcal{Q}$. $\mathcal{Q}$ is the least set of formal expressions such that:

$$\begin{aligned}\mathcal{Q} \supseteq\ & \{\varnothing, R_1, \ldots, R_\alpha\} \cup \\ & \{(Q \boldsymbol{\oplus} x) \mid Q \in \mathcal{Q}, x \in \omega \cup \{\infty\}\} \cup \{(Q \boldsymbol{\ominus} x) \mid Q \in \mathcal{Q}, x \in \omega \cup \{\infty\}\} \cup \\ & \{(\boldsymbol{\sim} Q) \mid Q \in \mathcal{Q}\} \cup \{(Q_1 \boldsymbol{\cap} Q_2) \mid Q_1, Q_2 \in \mathcal{Q}\} \cup \{(Q_1 \boldsymbol{\cup} Q_2) \mid Q_1, Q_2 \in \mathcal{Q}\} .\end{aligned}$$

where $\boldsymbol{\oplus}$ and $\boldsymbol{\ominus}$ are "shift" operators defined in Section 3.

1 Introduction

We consider various formalisms in this report: first-order logic in addition to several other logics especially tailored for our analysis. We compare these formalisms by the relations they define in relational structures $\mathfrak{A}$ (which are, in this report, first-order structures with finitely many underlying relations). If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two such formalisms, and φ_1 and φ_2 are expressions of $\mathcal{L}_1$ and $\mathcal{L}_2$ that define relations $\varphi_1^{\mathfrak{A}}$ and $\varphi_2^{\mathfrak{A}}$ in a relational structure $\mathfrak{A}$, respectively, then we write: $\varphi_1 \equiv \varphi_2$ just in case $\varphi_1^{\mathfrak{A}} = \varphi_2^{\mathfrak{A}}$ for every relational structure $\mathfrak{A}$. And we write $\mathcal{L}_1 \equiv \mathcal{L}_2$ iff:

$$(\forall \varphi_1 \in \mathcal{L}_1)(\exists \varphi_2 \in \mathcal{L}_2)[\varphi_1 \equiv \varphi_2] \qquad \text{and} \qquad (\forall \varphi_2 \in \mathcal{L}_2)(\exists \varphi_1 \in \mathcal{L}_1)[\varphi_1 \equiv \varphi_2] .$$

There is also considerable interplay between syntax and semantics, and in many places we need to explicitly distinguish between symbols and their interpretations. When the same character is used to denote both a symbol and its interpretation, we use two different fonts to distinguish them, boldface and regular, respectively (for example, $\boldsymbol{\cup}$ and $\cup$, $\boldsymbol{<}$ and $<$, $\boldsymbol{=}$ and $=$, etc.).

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2 An Algebraic Characterization of First-Order Definability

We develop a variable-free relational calculus which has the same expressive power as first-order logic. This development is similar but not identical to others in the literature (see for example [2], [4], and others cited in these two references dealing with cylindrical algebras). Our concerns are sufficiently different, just as are the details, to necessitate a separate and self-contained treatment.

2.1 Syntax

Let Σ be a first-order signature, $\Sigma = \{R_1, ..., R_\alpha\}$, where $\alpha \geq 0$ and every $R \in \Sigma$ is a relational symbol of arbitrary finite arity ≥ 0. We define the set $\mathcal{S}$ of *first-order definitional schemes* (FODS, for short) over the signature Σ.

For every $n \geq 1$, let Π_n be the collection of all permutations of $\{1, \ldots, n\}$, and let $\Pi = \bigcup_{n \geq 1} \Pi_n$. A FODS is a formal expression built up from the alphabet Σ by means of the following constructors:

- unary constructors: $\{\boldsymbol{\sim}, \mathbf{C}, \mathbf{P}\} \cup \Pi$
- binary constructors: $\{\boldsymbol{\cup}, \boldsymbol{\cap}\}$

Each FODS has an arity $n \geq 0$, so that if $\mathcal{S}_n$ is the set of all FODS's of arity n then $\mathcal{S} = \bigcup_{n \geq 0} \mathcal{S}_n$. Formally, $\mathcal{S}$ is defined by the following induction, where $n \geq 0$ throughout:

1. (Basis) $\varnothing \in \mathcal{S}_1$,
   $\Delta \in \mathcal{S}_2$,
   If $R \in \Sigma$ is of arity $n \geq 0$ then $R \in \mathcal{S}_n$.
2. (Complementation) If $S \in \mathcal{S}_n$ then $\boldsymbol{\sim} S \in \mathcal{S}_n$.
3. (Union) If $S_1, S_2 \in \mathcal{S}_n$ then $(S_1 \boldsymbol{\cup} S_2) \in \mathcal{S}_n$.
4. (Intersection) If $S_1, S_2 \in \mathcal{S}_n$ then $(S_1 \boldsymbol{\cap} S_2) \in \mathcal{S}_n$.
5. (Cylindrification) If $S \in \mathcal{S}_n$ then $\mathbf{C} S \in \mathcal{S}_{n+1}$.
6. (Projection) If $S \in \mathcal{S}_{n+1}$ then $\mathbf{P} S \in \mathcal{S}_n$.
7. (Permutation) If $S \in \mathcal{S}_{n+1}$ and $\pi \in \Pi_{n+1}$ then $\pi S \in \mathcal{S}_{n+1}$.

Δ in part 1 is a distinguished symbol, to be interpreted as the equality relation. The choice of our constructors is not optimal (e.g. we do not need both $\boldsymbol{\cup}$ and $\boldsymbol{\cap}$) and is largely dictated by convenience.

2.2 Semantics

A FODS S over signature Σ is interpreted in a relational structure $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \ldots, R_\alpha^{\mathfrak{A}})$, where $R^{\mathfrak{A}}$ is the interpretation of $R \in \Sigma$. If R is of arity $n \geq 1$ then $R^{\mathfrak{A}} \subseteq A^{(n)}$ where $A^{(n)} = A \times \cdots \times A$ (n times). If R is of arity 0 then $R^{\mathfrak{A}}$ is either tt (true) or $f\!f$ (false). The interpretation of S in $\mathfrak{A}$ is denoted $S^{\mathfrak{A}}$. Delaying the interpretation of FODS's of arity 0 to the next paragraph, we require that $n \geq 1$ throughout the induction below:

1. If $S = \varnothing$ then $S^{\mathfrak{A}} = \varnothing$,
   If $S = \Delta$ then $S^{\mathfrak{A}} = \{(a,a) | a \in A\}$,
   If $S = R \in \Sigma$ is of arity $n \geq 1$ then $S^{\mathfrak{A}} = R^{\mathfrak{A}}$.

2. If $S \in \mathcal{S}_n$ then $(\boldsymbol{\sim} S)^{\mathfrak{A}} = \sim S^{\mathfrak{A}}$.

3. If $S_1, S_2 \in \mathcal{S}_n$ then $(S_1 \mathbf{U} S_2)^{\mathfrak{A}} = S_1^{\mathfrak{A}} \cup S_2^{\mathfrak{A}}$.

4. If $S_1, S_2 \in \mathcal{S}_n$ then $(S_1 \mathbf{\cap} S_2)^{\mathfrak{A}} = S_1^{\mathfrak{A}} \cap S_2^{\mathfrak{A}}$.

5. If $S \in \mathcal{S}_n$ then $(\mathbf{C} S)^{\mathfrak{A}} = S^{\mathfrak{A}} \times A$.

6. If $S \in \mathcal{S}_{n+1}$ then $(\mathbf{P} S)^{\mathfrak{A}} =$
   $\{ \ (a_1, \ldots, a_n) \mid \exists a_{n+1} \text{ such that } (a_1, \ldots, a_n, a_{n+1}) \in S^{\mathfrak{A}} \ \}$.

7. If $S \in \mathcal{S}_n$ and $\pi \in \Pi_n$ then $(\pi S)^{\mathfrak{A}} =$
   $\{ \ (a_{\pi^{-1}(1)}, \ldots, a_{\pi^{-1}(n)}) \mid (a_1, \ldots, a_n) \in S^{\mathfrak{A}} \ \} = \{ \ (a_1, \ldots, a_n) \mid (a_{\pi(1)}, \ldots, a_{\pi(n)}) \in S^{\mathfrak{A}} \ \}$.

In part 2 above, $\sim S^{\mathfrak{A}}$ means the complement of $S^{\mathfrak{A}}$, i.e. $A^{(n)} - S^{\mathfrak{A}}$. The interpretation of any $S \in \mathcal{S}_n$ in $\mathfrak{A}$ where $n \geq 1$ is therefore a subset of $A^{(n)}$.

The interpretation of any $S \in \mathcal{S}_0$ in $\mathfrak{A}$ is always in $\{tt, f\!f\}$, which is given by the following induction, where $\{\neg, \vee, \wedge\}$ are the usual boolean connectives {not,or,and}:

1. If $S = R \in \Sigma$ is of arity 0 then $S^{\mathfrak{A}} = R^{\mathfrak{A}} \in \{tt, f\!f\}$.

2. If $S \in \mathcal{S}_0$ then $(\boldsymbol{\sim} S)^{\mathfrak{A}} = \neg(S^{\mathfrak{A}})$.

3. If $S_1, S_2 \in \mathcal{S}_0$ then $(S_1 \mathbf{U} S_2)^{\mathfrak{A}} = S_1^{\mathfrak{A}} \vee S_2^{\mathfrak{A}}$.

4. If $S_1, S_2 \in \mathcal{S}_0$ then $(S_1 \mathbf{\cap} S_2)^{\mathfrak{A}} = S_1^{\mathfrak{A}} \wedge S_2^{\mathfrak{A}}$.

5. If $S \in \mathcal{S}_0$ then $(\mathbf{C} S)^{\mathfrak{A}} = \begin{cases} A, & \text{if } S^{\mathfrak{A}} = tt, \\ \varnothing, & \text{if } S^{\mathfrak{A}} = f\!f. \end{cases}$

6. If $S \in \mathcal{S}_1$ then $(\mathbf{P}S)^{\mathfrak{A}} = \begin{cases} tt, & \text{if } S^{\mathfrak{A}} \neq \varnothing, \\ ff, & \text{if } S^{\mathfrak{A}} = \varnothing. \end{cases}$

We define two composite constructors based on those introduced earlier. For every $n \geq 1$ and every $i \in \{1, \ldots, n\}$ we define the constructor $\mathbf{P}_{n,i}$ as:

$$\mathbf{P}_{n,i} \;=\; \mathbf{P}\pi \quad \text{where} \quad \pi \;=\; (1)(2)\cdots(i-1)(n, n-1, \ldots, i)\;.$$

If we write $\mathbf{P}_{n,i}S$ we mean therefore $\mathbf{P}\pi S$, with π as given above. For every $n \geq 1$ and every $i \in \{0, 1, \ldots, n\}$ we define the constructor $\mathbf{C}_{n,i}$ as:

$$\mathbf{C}_{n,i} \;=\; \pi\mathbf{C} \quad \text{where} \quad \pi \;=\; (1)(2)\cdots(i)(i+1, i+2, \ldots, n+1)\;.$$

If we write $\mathbf{C}_{n,i}S$ we mean $\pi\mathbf{C}S$, with π as given above.

Lemma 2.1 *Let $\mathfrak{A}$ be an arbitrary relational structure. For every $S \in \mathcal{S}_n$ where $n \geq 1$, every $i \in \{1, \ldots, n\}$, and every $j \in \{0, 1, \ldots, n\}$:*

1. $(\mathbf{P}_{n,i}S)^{\mathfrak{A}} = \{\ (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \mid \exists a_i \text{ such that } (a_1, \ldots, a_n) \in S^{\mathfrak{A}}\ \}$,
2. $(\mathbf{C}_{n,j}S)^{\mathfrak{A}} = \{\ (a_1, \ldots, a_j, b, a_{j+1}, \ldots, a_n) \mid b \in A \text{ and } (a_1, \ldots, a_n) \in S^{\mathfrak{A}}\ \}$.

Proof: For part 1, consider $\mathbf{P}_{n,i} = \mathbf{P}\pi$ where $\pi = (1)(2)\cdots(i-1)(n, n-1, \ldots, i)$. It follows that if $(a_1, \ldots, a_n) \in S^{\mathfrak{A}}$ then $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, a_i) \in (\pi S)^{\mathfrak{A}}$, i.e. the i-th component a_i is removed and placed at the right end of the n-tuple. Hence, $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in (\mathbf{P}\pi S)^{\mathfrak{A}} = (\mathbf{P}_{n,i}S)^{\mathfrak{A}}$.

For part 2, consider $\mathbf{C}_{n,j} = \pi\mathbf{C}$ where $\pi = (1)(2)\cdots(j)(j+1, j+2, \ldots, n+1)$. If $(a_1, \ldots, a_n) \in S^{\mathfrak{A}}$ then $(a_1, \ldots, a_n, b) \in (\mathbf{C}S)^{\mathfrak{A}}$ for every $b \in A$. Hence, $(a_1, \ldots, a_j, b, a_{j+1}, \ldots, a_n) \in (\pi\mathbf{C}S)^{\mathfrak{A}} = (\mathbf{C}_{n,j}S)^{\mathfrak{A}}$. ■

It is clear that for all $n \geq 1$, $\mathbf{P}_{n,n}$ and $\mathbf{C}_{n,n}$ are equivalent to $\mathbf{P}$ and $\mathbf{C}$, respectively.

2.3 Expressive power

We show that FODS's define the same relations as first-order (FO) formulae in any relational structure $\mathfrak{A}$. Our conventions for FO formulae are standard, except for two (see for example [3]). First, it is customary to write $\varphi(x_1, \ldots, x_n)$ whenever one needs to mention that the free variables of the formula φ are *among* $x_1, \ldots, x_n$. Contrary to this convention, and with no loss of generality, if we write $\varphi(x_1, \ldots, x_n)$ we mean that the free variables of φ are *exactly* $x_1, \ldots, x_n$. As usual, the relation defined by φ in $\mathfrak{A}$ is:

$$\varphi^{\mathfrak{A}} = \{\ (a_1, \ldots, a_n) \mid \mathfrak{A} \models \varphi[a_1, \ldots, a_n]\ \}$$

This definition makes clear that, for the purpose of comparison with FODS's, we need to agree on another convention: There is exactly one order in which we can we list the free variables of φ — and, for definiteness, we choose this order to be that of increasing indexes.[1] (Formally, we take the set of all first-order variables to be $x_0, x_1, \ldots, x_i, \ldots,\ i \in \omega$.)

By the first convention above, if the free variables of φ are, say, exactly x_1 and x_2, then we can write $\varphi(x_1, x_2)$ but not $\varphi(x_1, x_2, x_3)$. Nor can we write $\varphi(x_2, x_1)$, by the second convention above; if we want to refer to the formula ψ obtained from φ by replacing x_1 by x_2 and x_2 by x_1, we write instead $\psi = \varphi[x_1 := x_2, x_2 := x_1]$.

To follow some of the details in the proofs of Propositions 2.2 and 2.3, we need to pay special attention to how the permutation π is used in order to define the relation $(\pi S)^{\mathfrak{A}}$ from the relation $S^{\mathfrak{A}}$. For example, if $S^{\mathfrak{A}} = \{(a, b, c, d, e)\}$ and $\pi = (1, 2, 3)(4)(5)$ then $(\pi S)^{\mathfrak{A}} = \{(c, a, b, d, e)\}$, so that if the formula $\varphi(x_1, \ldots, x_5)$ defines $S^{\mathfrak{A}}$ then the formula $\varphi[x_1 := x_{\pi(1)}, \ldots, x_5 := x_{\pi(5)}]$ defines $(\pi S)^{\mathfrak{A}}$.

Proposition 2.2 *For every FODS S there is a FO formula φ_S such that $S \equiv \varphi_S$.*

Proof: For every $S \in \mathcal{S}_n$ where $n \geq 0$, we define a FO formula φ_S whose free variables are exactly $x_1, \ldots, x_n$. We proceed by induction on the syntax of FODS's — where $S_0, S_1, S_2 \in \mathcal{S}_n$ throughout, except in part 6 where $S_0 \in \mathcal{S}_{n+1}$:

1. If $S = \varnothing$ then let φ_S be $\neg(x_1 \mathbf{=} x_1)$,
   if $S = \Delta$ then let φ_S be $(x_1 \mathbf{=} x_2)$,
   and if $S = R \in \Sigma$ is of arity $n \geq 0$ then let $\varphi_S = R(x_1, \ldots, x_n)$.
2. If $S = \boldsymbol{\sim} S_0$ then let $\varphi_S = \neg \varphi_{S_0}$.
3. If $S = (S_1 \mathbf{\cup} S_2)$ then let $\varphi_S = (\varphi_{S_1} \vee \varphi_{S_2})$.
4. If $S = (S_1 \mathbf{\cap} S_2)$ then let $\varphi_S = (\varphi_{S_1} \wedge \varphi_{S_2})$.
5. If $S = \mathbf{C} S_0$ then let $\varphi_S = (\varphi_{S_0} \wedge (x_{n+1} \mathbf{=} x_{n+1}))$.
6. If $S = \mathbf{P} S_0$ then let $\varphi_S = (\exists x_{n+1})\varphi_{S_0}$.
7. If $S = \pi S_0$ then let $\varphi_S = \varphi_{S_0}[x_1 := x_{\pi(1)}, \ldots, x_n := x_{\pi(n)}]$.

[1] Without these two conventions, the relation defined by a FO formula φ is unique only up to cylindrification (adding redundant free variables not explicitly mentioned in φ) and permutation of its components (listing the free variables of φ in a different order). But, in the presence of variables to name each of the components, the resulting ambiguity is harmless (e.g. the meaning of a FO sentence does not change when its bound variables are renamed).

Based on the semantics of FODS's, it is now straightforward to verify $S^{\mathfrak{A}} = \varphi_S^{\mathfrak{A}}$ for every relational structure $\mathfrak{A}$. ∎

Proposition 2.3 *For every FO formula φ there is a FODS S_φ such that $\varphi \equiv S_\varphi$.*

Proof: There is no loss of generality in restricting the syntax of FO formulae as follows — where $i, j, j_1, j_2, \ldots, j_n \in \omega$ throughout the induction:

1. $(x_i = x_j)$ is a FO formula,
   and if $R \in \Sigma$ is of arity $n \geq 0$ and $j_1 < j_2 < \cdots < j_n$ then $R(x_{j_1}, x_{j_2}, \ldots, x_{j_n})$ is a FO formula.

2. If φ is a FO formula then so is $\neg\varphi$.

3. If φ_1 and φ_2 are FO formulae such that:

   — either φ_1 and φ_2 have exactly the same set of free variables,

   — or φ_2 is $(x_i = x_i)$ and x_i is not free in φ_1,

   then $(\varphi_1 \wedge \varphi_2)$ is a FO formula.

4. If φ is a FO formula containing x_i as a free variable then $(\exists x_i)\varphi$ is a FO formula.

It is easy to justify the restrictions introduced in parts 1, 3 and 4, above, because we can always replace:

— $R(\ldots, x_i, \ldots, x_i, \ldots)$ by $(\exists x_j)(R(\ldots, x_i, \ldots, x_j, \ldots) \wedge (x_i = x_j))$;

— $R(\ldots, x_i, \ldots, x_j, \ldots)$, where $i > j$, by $(\exists x_k)(R(\ldots, x_i, \ldots, x_k, \ldots) \wedge (x_j = x_k))$ where $i < k$;

— $(\varphi_1 \wedge \varphi_2)$, where x_i is a free variable in φ_1 but not in φ_2, by $(\varphi_1 \wedge (\varphi_2 \wedge (x_i = x_i)))$;

— $(\exists x_i)\varphi$, where x_i is not free in φ, by φ.

By induction on the syntax of FO formulae, as restricted above, we define for every formula φ a FODS S_φ:

1. If φ is $(x_i = x_i)$ then let $S_\varphi = \sim\varnothing$,
   if φ is $(x_i = x_j)$ with $i \neq j$ then let $S_\varphi = \Delta$,
   and if $\varphi = R(x_{j_1}, x_{j_2}, \ldots, x_{j_n})$ where $j_1 < j_2 < \cdots < j_n$ then let $S_\varphi = R$.

2. If $\varphi = \neg\varphi_0$ then $S_\varphi = \sim S_{\varphi_0}$.

3. If $\varphi = (\varphi_1 \wedge \varphi_2)$ and:

   — if φ_1 and φ_2 have exactly the same set of free variables then let $S_\varphi = (S_{\varphi_1} \mathbf{\cap} S_{\varphi_2})$,

   — if φ_2 is $(x_k \mathbf{=} x_k)$, and the free variables of φ_1 are exactly $x_{i_1}, \ldots, x_{i_n}$ such that $i_j < k < i_{j+1}$ for some $0 \leq j \leq n$, then let $S_\varphi = \mathbf{C}_{n,j} S_{\varphi_1}$.

4. If $\varphi = (\exists x_{i_j})\varphi_0$ and the free variables of φ_0 are exactly $x_{i_1}, \ldots, x_{i_j}, \ldots, x_{i_n}$ then let $S_\varphi = \mathbf{P}_{n,j} S_{\varphi_0}$.

Using the semantics of FODS's defined in Section 2.2, and Lemma 2.1, it is readily verified $\varphi^{\mathfrak{A}} = S_\varphi^{\mathfrak{A}}$ for every relational structure $\mathfrak{A}$. ∎

2.4 Prenex forms

We say that $S \in \mathcal{S}$ is in *prenex form* if $S = \rho_1 \cdots \rho_k S'$ where $k \geq 0$, S' is a FODS not mentioning the constructor $\mathbf{P}$, and:

$$\rho_1 = [\boldsymbol{\sim}]\mathbf{P}\pi_1 \ , \ \ldots \ , \ \rho_k = [\boldsymbol{\sim}]\mathbf{P}\pi_k$$

for some $\pi_1, \ldots, \pi_k \in \Pi$ and the notation $[\boldsymbol{\sim}]$ means that the constructor $\boldsymbol{\sim}$ may or may not occur.

There are obvious conditions of well-formedness in this definition. If S' is of arity 0, then $k = 0$ and $S = S'$. If S' is of arity $n \geq 1$, then $k \leq n$ and $\pi_k \in \Pi_n, \ldots, \pi_1 \in \Pi_{n-k+1}$, and the arity of S is $(n - k)$.

FODS's in prenex form are the counterpart of FO formulae in prenex form. The proof of the following proposition is identical to the proof of 2.3 and therefore omitted; the only difference is the restriction that "$\exists$" and "$\neg\exists$" are introduced last in the syntax of FO formulae in prenex form, so that their translation into FODS's are also in prenex form.

Proposition 2.4 *For every FO formula φ in prenex form there is a FODS S_φ in prenex form such that $\varphi \equiv S_\varphi$.*

Corollary 2.5 *For every FODS S_1 there is a FODS S_2 in prenex form such that $S_1 \equiv S_2$.*

Proof: First use 2.2 to obtain a FO formula φ equivalent to S_1, then transform φ into prenex form, then use 2.4 to obtain the desired S_2. ∎

3 First-Order Definability in Finite Ordered Structures

Henceforth, we restrict our attention to finite ordered structures of the form $\mathfrak{A} = (A, <, R_1^{\mathfrak{A}}, \ldots, R_\alpha^{\mathfrak{A}})$ where $<$ is a total order on the domain A, the relations $R_1^{\mathfrak{A}}, \ldots, R_\alpha^{\mathfrak{A}}$ are unary, and $\alpha \geq 0$. The signature is therefore $\Sigma = \{\boldsymbol{<}, R_1, \ldots, R_\alpha\}$ where all the relation symbols (except for $\boldsymbol{<}$) are unary.

Let $\mathcal{C} = \{\mathfrak{A}_i\}_{i \in \omega}$ be an infinite class of such finite ordered structures. By a *property over* $\mathcal{C}$ we mean a family of sets $\{X_i\}_{i\in\omega}$ where every $X_i \subseteq |\mathfrak{A}_i|$. We say that $\{X_i\}_{i\in\omega}$ is a *first-order* property over $\mathcal{C}$ just in case $\{X_i\}_{i\in\omega} = \{\varphi^{\mathfrak{A}_i}\}_{i\in\omega}$ for some (unary) first-order formula φ in the signature Σ. We wish to formulate an algebraic condition which is satisfied by a property $\{X_i\}_{i\in\omega}$ over $\mathcal{C}$ if and only if it is first-order.

Our formulation is in terms of another formal language $\mathcal{Q}$. The main result of this section (Theorem 3.13) states that, over classes of finite ordered structures (as specified above), *unary first-order properties are exactly expressed by* $\mathcal{Q}$. Thus, $\mathcal{Q}$ gives another algebraic characterization of first-order definability, but now especially adapted to the case when only finite ordered monadic structures are considered.[2]

3.1 $\mathcal{Q}$ and some of its properties

It is convenient to introduce two operations $\oplus$ ("right shift") and $\ominus$ ("left shift") on sets of integers. If $A \subseteq \omega^* + \omega$ and $x \in \omega$, we define:

$$A \oplus x \;=\; \{a + x \mid a \in A\} \qquad \text{and} \qquad A \ominus x \;=\; \{a - x \mid a \in A\}\ .$$

We extend the shift operations by allowing x to be the special value ∞, for which we define:

$$A \oplus \infty \;=\; \bigcup\{A \oplus x \mid x \in \omega\} \qquad \text{and} \qquad A \ominus \infty \;=\; \bigcup\{A \ominus x \mid x \in \omega\}\ .$$

The new symbols in the syntax of $\mathcal{Q}$ are $\boldsymbol{\oplus}$ and $\boldsymbol{\ominus}$, later interpreted as $\oplus$ and $\ominus$ respectively. Formally, $\mathcal{Q}$ is the least set of formal expressions such that:

$$\begin{aligned}\mathcal{Q} \;\supseteq\;\; & \{\varnothing, R_1, \ldots, R_\alpha\} \;\cup \\ & \{(Q\boldsymbol{\oplus}x) \mid Q \in \mathcal{Q}, x \in \omega \cup \{\infty\}\} \;\cup\; \{(Q\boldsymbol{\ominus}x) \mid Q \in \mathcal{Q}, x \in \omega \cup \{\infty\}\} \;\cup \\ & \{(\boldsymbol{\sim}Q) \mid Q \in \mathcal{Q}\} \;\cup\; \{(Q_1\boldsymbol{\cap}Q_2) \mid Q_1, Q_2 \in \mathcal{Q}\} \;\cup\; \{(Q_1\boldsymbol{\cup}Q_2) \mid Q_1, Q_2 \in \mathcal{Q}\}\ .\end{aligned}$$

The semantics of $\mathcal{Q}$ are specified relative to a finite ordered structure $\mathfrak{A} = ([0,u], <, R_1^{\mathfrak{A}}, \ldots, R_\alpha^{\mathfrak{A}})$, where the domain $[0,u] = \{0, 1, \ldots, u\}$ for some $u \in \omega$. By induction on $\mathcal{Q}$:

[2] For conciseness we omit the qualifier "monadic" in all later references to these structures, simply calling them "finite ordered structures".

1. If $Q = R \in \Sigma$ then $Q^{\mathfrak{A}} = R^{\mathfrak{A}}$, and if $Q = \varnothing$ then $Q^{\mathfrak{A}} = \varnothing$.

2. If $Q = (Q_0 \boldsymbol{\oplus} x)$ then $Q^{\mathfrak{A}} = (Q_0^{\mathfrak{A}} \oplus x) \cap [0, u]$.

3. If $Q = (Q_0 \boldsymbol{\ominus} x)$ then $Q^{\mathfrak{A}} = (Q_0^{\mathfrak{A}} \ominus x) \cap [0, u]$.

4. If $Q = (\boldsymbol{\sim} Q_0)$ then $Q^{\mathfrak{A}} = [0, u] - Q_0^{\mathfrak{A}}$.

5. If $Q = (Q_1 \boldsymbol{\cap} Q_2)$ then $Q^{\mathfrak{A}} = Q_1^{\mathfrak{A}} \cap Q_2^{\mathfrak{A}}$.

6. If $Q = (Q_1 \boldsymbol{\cup} Q_2)$ then $Q^{\mathfrak{A}} = Q_1^{\mathfrak{A}} \cup Q_2^{\mathfrak{A}}$.

As we need to carry out proofs by induction on $\mathcal{S}$, the members of which can be of any arity ≥ 1, we extend $\mathcal{Q}$ to arities greater than 1. We pose $\mathcal{Q}_1 = \mathcal{Q}$ and, for every $n \geq 2$, we define $\mathcal{Q}_n$ as the least set of formal expressions such that:

$$\mathcal{Q}_n \quad \supseteq \quad \mathcal{Q} \boldsymbol{\times} \cdots \boldsymbol{\times} \mathcal{Q} \quad (n \text{ times}) \quad \cup \quad \{(Q_1 \boldsymbol{\cup} Q_2) \mid Q_1, Q_2 \in \mathcal{Q}_n\} \ .$$

While our goal in this section is to prove that $\mathcal{Q} = \mathcal{Q}_1$ has the same expressive power as $\mathcal{S}_1$, it turns out that $\mathcal{Q}_n$ is strictly weaker than $\mathcal{S}_n$ for $n \geq 2$, as shown by part 3 of Lemma 3.11.

We also need to consider a particular well-behaved subset $\mathcal{P}_n$ of $\mathcal{Q}_n$. First, we define $\mathcal{P} = \mathcal{P}_1$ as the least set of formal expressions such that:

$$\begin{aligned} \mathcal{P} \quad \supseteq \quad & \{\varnothing, R_1, \ldots, R_\alpha\} \ \cup \\ & \{(\boldsymbol{\sim} P) \mid P \in \mathcal{P}\} \ \cup \ \{(P_1 \boldsymbol{\cap} P_2) \mid P_1, P_2 \in \mathcal{P}\} \ \cup \ \{(P_1 \boldsymbol{\cup} P_2) \mid P_1, P_2 \in \mathcal{P}\} \ . \end{aligned}$$

And for every $n \geq 2$, we define $\mathcal{P}_n$ as the least set of formal expressions such that:

$$\mathcal{P}_n \quad \supseteq \quad \mathcal{P} \boldsymbol{\times} \cdots \boldsymbol{\times} \mathcal{P} \quad (n \text{ times}) \quad \cup \quad \{(P_1 \boldsymbol{\cup} P_2) \mid P_1, P_2 \in \mathcal{P}_n\} \ .$$

The semantics of $\mathcal{Q}$ are extended to $\mathcal{Q}_n$ in the obvious way. The only new symbol in the syntax of $\mathcal{Q}_n$ is $\boldsymbol{\times}$. If $\mathfrak{A} = ([0, u], <, R_1^{\mathfrak{A}}, \ldots, R_\alpha^{\mathfrak{A}})$ and we define $(Q_1 \boldsymbol{\times} Q_2)^{\mathfrak{A}}$ as $Q_1^{\mathfrak{A}} \times Q_2^{\mathfrak{A}}$, then the interpretation of any $Q \in \mathcal{Q}_n$ in $\mathfrak{A}$, denoted $Q^{\mathfrak{A}}$, is a subset of the n-dimensional cube $[0, u]^{(n)}$.

Lemma 3.1 *Let A be an arbitrary set and $X_1, X_2, X_3, X_4 \subseteq A$:*

1. $X_1 \times X_2 \ = \ \varnothing$ *iff* $X_1 = \varnothing$ *or* $X_2 = \varnothing$.

2. $(X_1 \times X_2) \cup (X_3 \times X_2) \ = \ (X_1 \cup X_3) \times X_2$.

3. $(X_1 \times X_2) \cap (X_3 \times X_4) \ = \ (X_1 \cap X_3) \times (X_2 \cap X_4)$.

4. $\sim(X_1 \times X_2) = (\sim X_1 \times \sim X_2) \cup (X_1 \times \sim X_2) \cup (\sim X_1 \times X_2)$.

Proof: Straightforward. ■

We also consider expressions which are, strictly speaking, not part of the formalisms defined above, e.g. the expressions $(Q_1 \mathbf{\cap} Q_2)$, $(\mathbf{\sim} Q_1)$ and $(Q_1 \mathbf{\times} Q_2)$ in the next lemma are not in $\mathcal{Q}_n$ if $n \neq 1$, but their interpretation is clear. We do not bother to define them formally for the sake of brevity.

Lemma 3.2

1. $\forall Q_1, Q_2 \in \mathcal{Q}_n$ *(resp.* $\mathcal{P}_n$*)*, $\exists Q \in \mathcal{Q}_n$ *(resp.* $\mathcal{P}_n$*)*, $(Q_1 \mathbf{\cup} Q_2) \equiv Q$.

2. $\forall Q_1, Q_2 \in \mathcal{Q}_n$ *(resp.* $\mathcal{P}_n$*)*, $\exists Q \in \mathcal{Q}_n$ *(resp.* $\mathcal{P}_n$*)*, $(Q_1 \mathbf{\cap} Q_2) \equiv Q$.

3. $\forall Q_1 \in \mathcal{Q}_n$ *(resp.* $\mathcal{P}_n$*)*, $\exists Q \in \mathcal{Q}_n$ *(resp.* $\mathcal{P}_n$*)*, $(\mathbf{\sim} Q_1) \equiv Q$.

4. $\forall Q_1 \in \mathcal{Q}_{n_1}$ *(resp.* $\mathcal{P}_{n_1}$*)*, $\forall Q_2 \in \mathcal{Q}_{n_2}$ *(resp.* $\mathcal{P}_{n_2}$*)*, $\exists Q \in \mathcal{Q}_{n_1+n_2}$ *(resp.* $\mathcal{P}_{n_1+n_2}$*)*, $(Q_1 \mathbf{\times} Q_2) \equiv Q$.

Proof: Part 1 is immediate, as $\mathcal{Q}_n$ (resp. $\mathcal{P}_n$) is syntactically closed under $\mathbf{\cup}$, i.e. we take $Q = (Q_1 \mathbf{\cup} Q_2)$. For parts 2 and 3, we "push in" $\mathbf{\cap}$ and $\mathbf{\sim}$ using 3 and 4 of Lemma 3.1, and the boolean identities (DeMorgan's law and distributivity of $\mathbf{\cap}$ over $\mathbf{\cup}$); the desired conclusion follows from the closure of $\mathcal{Q}$ (resp. $\mathcal{P}$) itself under the boolean operations. Part 4 follows from 2 of Lemma 3.1. Details omitted. ■

In general, equality between $(X_1 \times X_2) \cup (X_3 \times X_4)$ and $(X_1 \cup X_3) \times (X_2 \cup X_4)$ holds only if $X_2 = X_4$ (part 2 of Lemma 3.1). Hence, $\mathcal{Q}_n$ which is closed under $\mathbf{\cup}$ is more expressive, i.e. it defines more relations, than $\mathcal{Q} \mathbf{\times} \cdots \mathbf{\times} \mathcal{Q}$ (n times). On the other hand, Lemma 3.2 shows that the closure of $\mathcal{Q}_n$ under $\mathbf{\cap}$ and $\mathbf{\sim}$ does not increase its expressive power.

3.2 Arrangements and some of their properties

We temporarily re-introduce first-order variables x_i, $i \in \omega$, and consider particular formulae over them. The formulae under consideration are called "arrangements", generalizing formulae by the same name in [1], and we use the letter μ to denote them. For any $n \geq 2$, an *n-arrangement* is a finite conjunction of $(n-1)$ *basic* formulae:

$$\mu = \nu_1 \wedge \nu_2 \wedge \cdots \wedge \nu_{n-1}$$

where every basic formula ν_i is either $(y_i + d_i \mathbf{<} y_{i+1})$ or $(y_i + d_i \mathbf{=} y_{i+1})$, where $d_i \in \omega$ and $i \in \{1, \ldots, n-1\}$, and the sequence $y_1, \ldots, y_n$ is a permutation of the sequence $x_1, \ldots, x_n$. In particular, every n-arrangement is an open first-order formula whose free variables are exactly $\{x_1, \ldots, x_n\}$.

If ν is the basic formula $(x + d \# y)$ where $\# \in \{\mathbf{=}, \mathbf{<}\}$ and $d \in \omega$, we define $\mathit{left}(\nu) = x$, $\mathit{right}(\nu) = y$, $\mathit{degree}(\nu) = d$, and $\mathit{symbol}(\nu) = \#$.

If $\mu = \nu_1 \wedge \cdots \wedge \nu_{n-1}$ is an n-arrangement, we define $\mathit{degree}(\mu) = \mathit{max}\{\mathit{degree}(\nu_1), \ldots, \mathit{degree}(\nu_{n-1})\}$. We say that the n-arrangement $\mu = \nu_1 \wedge \cdots \wedge \nu_{n-1}$ is *simple* if there is $d \in \omega$ such that for all $i \in \{1, \ldots, n-1\}$:

- if ν_i is $(y_i + d_i \mathbf{<} y_{i+1})$ then $d_i = d$,
- if ν_i is $(y_i + d_i \mathbf{=} y_{i+1})$ then $d_i \leq d$,

where $\{y_1, \ldots, y_n\} = \{x_1, \ldots, x_n\}$ and $d_1, \ldots, d_{n-1} \in \omega$. In such a case it is also clear that d is also the degree of μ, and we can say that μ is *simple of degree d*. (The "arrangements" in [1] are, in our sense, "simple arrangements of degree 0".)

A special subset of all n-arrangements of degree d is the subset of all *simple* n-arrangements of degree d, denoted $\mathcal{M}_{n,d}$. It is easy to see that for fixed $n \geq 2$ and $d \in \omega$, the set of all n-arrangements of degree d is finite, and so is therefore the subset $\mathcal{M}_{n,d}$.

Given a finite ordered structure $\mathfrak{A} = ([0, u], <, R_1^{\mathfrak{A}}, \ldots, R_\alpha^{\mathfrak{A}})$, the interpretation of an n-arrangement μ in $\mathfrak{A}$ is a subset of the n-dimensional cube $[0, u]^{(n)}$ defined by:

$$\mu^{\mathfrak{A}} \;=\; \{\ (a_1, \ldots, a_n) \in [0, u]^{(n)} \mid (\mathfrak{A}, a_1, \ldots, a_n) \models \mu\ \}\ .$$

Convention 3.3 For simplicity in enumerating a finite set of n-arrangements we adopt the following convention. Let μ be an arbitrary n-arrangement. If μ mentions as a substring the following conjunction of basic formulae:

$$(x_{i_1} \mathbf{=} x_{i_2}) \ \wedge\ (x_{i_2} \mathbf{=} x_{i_3}) \ \wedge\ \cdots \ \wedge\ (x_{i_k} \mathbf{=} x_{i_{k+1}})$$

where $i_1, i_2, \ldots, i_{k+1} \in \{1, \ldots, n\}$ and $k \geq 1$, then we agree that $i_1 < i_2 < \cdots < i_{k+1}$. With this convention it is easy to see that two n-arrangements μ_1 and μ_2 are equivalent (i.e. they define the same set of n-tuples in every structure) iff μ_1 and μ_2 are syntactically identical. ∎

There is a lot we can say and prove about arrangements. We refrain from developing a comprehensive analysis and restrict ourselves to results we need in the proofs of Lemmas 3.10 and 3.11.

Lemma 3.4 *For given $n \geq 2$ and $d \in \omega$, let $\mathcal{M}_{n,d} = \{\mu_1, \ldots, \mu_m\}$ for some $m \geq 1$. Then for every finite ordered structure $\mathfrak{A} = ([0,u], <, R_1^{\mathfrak{A}}, \ldots, R_\alpha^{\mathfrak{A}})$:*

1. $\mu_i^{\mathfrak{A}} \cap \mu_j^{\mathfrak{A}} = \varnothing$ *if* $1 \leq i < j \leq m$,
2. $\mu_1^{\mathfrak{A}} \cup \cdots \cup \mu_m^{\mathfrak{A}} = [0,u]^{(n)}$.

Hence, $\mathcal{M}_{n,d}$ represents a partition of the n-dimensional cube $[0,u]^{(n)}$. This is not the case if we consider the set of all n-arrangements, simple or not, of degree d.

Proof: For part 1, consider arbitrary $\mu_i, \mu_j \in \mathcal{M}_{n,d}$. Let $\mu_i = \nu_{i,1} \wedge \cdots \wedge \nu_{i,n-1}$ and $\mu_j = \nu_{j,1} \wedge \cdots \wedge \nu_{j,n-1}$. Suppose there is a finite ordered structure $\mathfrak{A}$ such that $\mu_i^{\mathfrak{A}} \cap \mu_j^{\mathfrak{A}} \neq \varnothing$, say $(a_1, \ldots, a_n) \in \mu_i^{\mathfrak{A}} \cap \mu_j^{\mathfrak{A}}$. In the particular case when all the entries in $(a_1, \ldots, a_n)$ are pairwise unequal, if $a_{p_1} < \cdots < a_{p_n}$ where $\{p_1, \ldots, p_n\} = \{1, \ldots, n\}$ then

$$\mathit{left}(\nu_{i,q}) = \mathit{left}(\nu_{j,q}) = x_{p_q} \qquad \text{and} \qquad \mathit{right}(\nu_{i,q}) = \mathit{right}(\nu_{j,q}) = x_{p_{q+1}}$$

for every $q \in \{1, \ldots, n-1\}$. Because μ_i and μ_j are simple of the same degree d, it then follows that $\mu_i = \mu_j$, as desired. The same reasoning remains valid in the general case, using now convention 3.3 in addition, when some of the entries in $(a_1, \ldots, a_n)$ are equal, which happens when some of the basic formulae in μ_i and μ_j are of the form $(y \mathbin{\pmb{=}} z)$ for some $y, z \in \{x_1, \ldots, x_n\}$.

For part 2, consider a finite ordered structure $\mathfrak{A}$ with universe $[0,u]$ and an arbitrary $(a_1, \ldots, a_n) \in [0,u]^n$. Let $a_{p_1} \leq \cdots \leq a_{p_n}$ where $\{p_1, \ldots, p_n\} = \{1, \ldots, n\}$. We need to show there is an n-arrangement $\mu = \nu_1 \wedge \cdots \wedge \nu_{n-1}$ in $\mathcal{M}_{n,d}$ such that $(\mathfrak{A}, a_1, \ldots, a_n) \models \mu$. For every $q \in \{1, \ldots, n-1\}$, define the basic formula ν_q as follows. Let $d_q = a_{p_{q+1}} - a_{p_q}$. If $d_q \leq d$ then let ν_q be $(x_{p_q} + d_q \mathbin{\pmb{=}} x_{p_{q+1}})$. If $d_q > d$ then let ν_q be $(x_{p_q} + d \mathbin{\pmb{<}} x_{p_{q+1}})$. It is now easy to check that μ satisfies the required condition. ∎

Lemma 3.5 *Let μ be an n-arrangement (possibly not simple), $n \geq 2$, and let $d = \mathrm{degree}(\mu)$. For every $d' \geq d$ there are simple n-arrangements of degree d', say $\mu_1, \ldots, \mu_m \in \mathcal{M}_{n,d'}$ for some $m \geq 1$, such that $\mu \equiv \mu_1 \vee \cdots \vee \mu_m$.*

Proof: Suppose $\mu = \nu_1 \wedge \cdots \wedge \nu_{n-1}$ is an arbitrary n-arrangement of degree d. Let $d' \geq d$. For every $i \in \{1, \ldots, n-1\}$, we define a finite disjunction ν_i' of basic formulae such that $\nu_i' \equiv \nu_i$. If ν_i is $(y + d_i \mathbin{\pmb{=}} z)$ where $y, z \in \{x_1, \ldots, x_n\}$ and $d_i \leq d \leq d'$, then let $\nu_i' = \nu_i$. If ν_i is $(y + d_i \mathbin{\pmb{<}} z)$ where $y, z \in \{x_1, \ldots, x_n\}$ and $d_i \leq d \leq d'$, then define ν_i' by:

$$\nu_i' \;=\; (y + [d_i + 1] \mathbin{\pmb{=}} z) \;\vee\; (y + [d_i + 2] \mathbin{\pmb{=}} z) \;\vee\; \cdots \;\vee\; (y + [d_i + (d' - d_i)] \mathbin{\pmb{=}} z) \;\vee\; (y + d' \mathbin{\pmb{<}} z)$$

Clearly $\mu \equiv \nu'_1 \wedge \cdots \wedge \nu'_{n-1}$. By transforming $\nu'_1 \wedge \cdots \wedge \nu'_{n-1}$ into disjunctive normal form, i.e. a disjunction of some $m \geq 1$ conjuncts containing each exactly $(n-1)$ basic formulae, we get the desired result. ■

We need to define two operations on an n-arrangement μ; the first, denoted $\mu - \{n\}$, results in a $(n-1)$-arrangement and the second, denoted $\mu + \{n+1\}$, results in a set of $(n+1)$-arrangements. We introduce convenient notation towards this end. Let ν and ν' be the basic formulae $(x + d \# y)$ and $(y + d' \#' z)$, where $\#, \#' \in \{\mathbf{=}, \mathbf{<}\}$. The *merge* of ν and ν', denoted $[\nu, \nu']$, eliminates y as an intermediary variable. More precisely:

$$[\nu, \nu'] = \begin{cases} (x + [d + d'] \mathbf{=} z), & \text{if } \mathit{symbol}(\nu) = \mathit{symbol}(\nu') = \mathbf{=}\,, \\ (x + [d + d'] \mathbf{<} z), & \text{if } \mathit{symbol}(\nu) \neq \mathit{symbol}(\nu')\,, \\ (x + [d + d' + 1] \mathbf{<} z), & \text{if } \mathit{symbol}(\nu) = \mathit{symbol}(\nu') = \mathbf{<}\,. \end{cases}$$

Suppose μ is an n-arrangement for some $n \geq 3$. By omitting x_n in μ we obtain an $(n-1)$-arrangement, denoted $\mu - \{n\}$. Specifically, if $\mu = \nu_1 \wedge \cdots \wedge \nu_{n-1}$ and $n \geq 3$, then:

$$\mu - \{n\} = \begin{cases} \nu_2 \wedge \cdots \wedge \nu_{n-1}, & \text{if } \mathit{left}(\nu_1) = x_n, \\ \nu_1 \wedge \cdots \wedge \nu_{n-2}, & \text{if } \mathit{right}(\nu_{n-1}) = x_n, \\ \nu_1 \wedge \cdots \wedge [\nu_m, \nu_{m+1}] \wedge \cdots \wedge \nu_{n-1}, & \text{if } \mathit{right}(\nu_m) = \mathit{left}(\nu_{m+1}) = x_n \text{ for } 1 \leq m \leq n-2. \end{cases}$$

One complication to be aware of is the following: If μ is a simple n-arrangement, the $(n-1)$-arrangement $\mu - \{n\}$ is not necessarily simple. This complication is related to the fact that the degree of μ is not always preserved, and in general $0 \leq \mathit{degree}(\mu - \{n\}) \leq 1 + 2 \cdot \mathit{degree}(\mu)$. However, it is easy to see that if we consider the set $\mathcal{M}_{n,d}$ of all simple n-arrangements of degree d, then for at least one $\mu \in \mathcal{M}_{n,d}$ it is the case that $\mathit{degree}(\mu - \{n\}) = 1 + 2 \cdot \mathit{degree}(\mu) = 1 + 2 \cdot d$.

If μ_1 and μ_2 are n-arrangements, we write $\mu_1 \subseteq \mu_2$ iff for every finite ordered structure $\mathfrak{A}$, $\mu_1^{\mathfrak{A}} \subseteq \mu_2^{\mathfrak{A}}$. Let $n \geq 3$, $d \in \omega$, and $\mu \in \mathcal{M}_{n-1,d}$. We define the set $\mu + \{n\}$ by:

$$\mu + \{n\} = \{ \tilde{\mu} \in \mathcal{M}_{n,d} \mid (\tilde{\mu} - \{n\}) \subseteq \mu \}\ .$$

Note that $\mu + \{n\}$ is defined provided μ is a simple arrangement, and if it is, the resulting set is also a set of simple arrangements, which all have the same degree as μ. The relationship between $\mu - \{n\}$ and $\mu + \{n\}$ is more complicated than suggested by the notation; for example, it is not always the case that $\mu \in ((\mu - \{n\}) + \{n\})$ because $\mu - \{n\}$ is not generally a simple arrangement and $((\mu - \{n\}) + \{n\})$ is not therefore defined.

In Lemmas 3.6, 3.7, 3.8 and 3.9, and in their proofs, we mix various formalisms once more; specifically, we use **C**, **P** and arrangements with the syntax of $\mathcal{Q}_n$. For the sake of brevity, we do not bother to formally introduce these hybrid expressions. This should cause no problem, as the interpretations of **C**, **P** and arrangements remain unchanged.

Lemma 3.6 *Let $n \geq 3$, $d \in \omega$, and $\mu \in \mathcal{M}_{n-1,d}$. If $\mu + \{n\}$ is the set $\{\mu_1, \ldots, \mu_m\} \subseteq \mathcal{M}_{n,d}$ then $(\mathbf{C}\mu) \equiv \mu_1 \vee \cdots \vee \mu_m$.*

Proof: Let $\mathfrak{A}$ be an arbitrary finite ordered structure. Given a sequence $(a_1, \ldots, a_n) \in (\mathbf{C}\mu)^{\mathfrak{A}}$, we prove $(a_1, \ldots, a_n) \in \widetilde{\mu}^{\mathfrak{A}}$ for some $\widetilde{\mu} \in (\mu + \{n\})$. Let $a_{p_1} \leq \cdots \leq a_{p_{n-1}}$ where $\{p_1, \ldots, p_{n-1}\} = \{1, \ldots, n-1\}$. Hence, if $\mu = \nu_1 \wedge \cdots \wedge \nu_{n-2}$ then for every $q \in \{1, \ldots, n-2\}$:

$$\mathit{left}(\nu_q) \;=\; x_{p_q} \qquad \text{and} \qquad \mathit{right}(\nu_q) \;=\; x_{p_{q+1}} \;.$$

Let $p_q \in \{1, \ldots, n-1\}$ be the largest such that $a_{p_q} \leq a_n < a_{p_{q+1}}$. Define $e = a_n - a_{p_q}$ and $f = a_{p_{q+1}} - a_n$. The desired $\widetilde{\mu}$ depends on the values of e and f:

1. If $e > d$ and $f > d$, let
   $\widetilde{\mu} = \nu_1 \wedge \cdots \wedge \nu_{q-1} \wedge (x_{p_q} + d \mathbf{<} x_n) \wedge (x_n + d \mathbf{<} x_{p_{q+1}}) \wedge \nu_{q+1} \wedge \cdots \wedge \nu_{n-2}$.
2. If $e > d$ and $f \leq d$, let
   $\widetilde{\mu} = \nu_1 \wedge \cdots \wedge \nu_{q-1} \wedge (x_{p_q} + d \mathbf{<} x_n) \wedge (x_n + f \mathbf{=} x_{p_{q+1}}) \wedge \nu_{q+1} \wedge \cdots \wedge \nu_{n-2}$.
3. If $e \leq d$ and $f > d$, let
   $\widetilde{\mu} = \nu_1 \wedge \cdots \wedge \nu_{q-1} \wedge (x_{p_q} + e \mathbf{=} x_n) \wedge (x_n + d \mathbf{<} x_{p_{q+1}}) \wedge \nu_{q+1} \wedge \cdots \wedge \nu_{n-2}$.
4. If $e \leq d$ and $f \leq d$, let
   $\widetilde{\mu} = \nu_1 \wedge \cdots \wedge \nu_{q-1} \wedge (x_{p_q} + e \mathbf{=} x_n) \wedge (x_n + f \mathbf{=} x_{p_{q+1}}) \wedge \nu_{q+1} \wedge \cdots \wedge \nu_{n-2}$.

It is readily checked that $(\widetilde{\mu} - \{n\}) \subseteq \mu$ and $(a_1, \ldots, a_n) \in \widetilde{\mu}^{\mathfrak{A}}$.

We now prove the converse: If $(a_1, \ldots, a_n) \in \widetilde{\mu}^{\mathfrak{A}}$ for some $\widetilde{\mu} \in (\mu + \{n\})$ then $(a_1, \ldots, a_n) \in (\mathbf{C}\mu)^{\mathfrak{A}}$. Let $a_{p_1} \leq \cdots \leq a_{p_n}$ where $\{p_1, \ldots, p_n\} = \{1, \ldots, n\}$. Hence, if $\widetilde{\mu} = \widetilde{\nu}_1 \wedge \cdots \wedge \widetilde{\nu}_{n-1}$ then for every $q \in \{1, \ldots, n-1\}$:

$$\mathit{left}(\widetilde{\nu}_q) \;=\; x_{p_q} \qquad \text{and} \qquad \mathit{right}(\widetilde{\nu}_q) \;=\; x_{p_{q+1}} \;.$$

Let now q be a fixed index in $\{1, \ldots, n\}$ such that $p_q = n$. Then:

$$\widetilde{\nu}_{q-1} = \; (x_{p_{q-1}} + d_{q-1} \;\#_{q-1}\; x_n) \qquad \text{and} \qquad \widetilde{\nu}_q = \; (x_n + d_q \;\#_q\; x_{p_{q+1}})$$

where $0 \leq d_{q-1}, d_q \leq d$ and $\#_{q-1}, \#_q \in \{\mathbf{=}, \mathbf{<}\}$. It is now easy to verify that because $\widetilde{\mu} \in (\mu + \{n\})$, it must be that $\mu = \widetilde{\nu}_1 \wedge \cdots \wedge \widetilde{\nu}_{q-1} \wedge \nu \wedge \widetilde{\nu}_{q+2} \wedge \cdots \wedge \widetilde{\nu}_{n-1}$ for some appropriate basic formula ν, and moreover $(a_1, \ldots, a_{n-1}) \in \mu^{\mathfrak{A}}$. It follows that $(a_1, \ldots, a_n) \in (\mathbf{C}\mu)^{\mathfrak{A}}$. ∎

Lemma 3.7 *If $n \geq 3$ and $d \in \omega$, then $\mathcal{M}_{n,d} \;=\; \bigcup \{\; (\mu + \{n\}) \mid \mu \in \mathcal{M}_{n-1,d} \;\}$.*

Proof: Let $\mathfrak{A}$ be an arbitrary finite ordered structure, with universe $[0, u]$. By Lemma 3.4, $\bigcup\{ \mu^{\mathfrak{A}} \mid \mu \in \mathcal{M}_{n-1,d} \} = [0, u]^{(n-1)}$, so that $\bigcup\{ (\mathbf{C}\mu)^{\mathfrak{A}} \mid \mu \in \mathcal{M}_{n-1,d} \} = [0, u]^{(n)}$. Hence, by Lemma 3.6:

$$\bigcup\{ \tilde{\mu}^{\mathfrak{A}} \mid \tilde{\mu} \in (\mu + \{n\}) \text{ and } \mu \in \mathcal{M}_{n-1,d} \} = [0, u]^{(n)}$$

On the other hand,

$$\bigcup\{ \tilde{\mu} \mid \tilde{\mu} \in (\mu + \{n\}) \text{ and } \mu \in \mathcal{M}_{n-1,d} \} \subseteq \mathcal{M}_{n,d}$$

Hence, by Lemma 3.4 again:

$$\bigcup\{ \tilde{\mu} \mid \tilde{\mu} \in (\mu + \{n\}) \text{ and } \mu \in \mathcal{M}_{n-1,d} \} = \mathcal{M}_{n,d}$$

which is the desired conclusion. ∎

Lemma 3.8 *For every $Q \in \mathcal{Q} \boldsymbol{\times} \mathcal{Q}$ and every 2-arrangement μ there is a $Q' \in \mathcal{Q}$ such that $\mathsf{P}(Q \boldsymbol{\cap} \mu) \equiv Q'$.*

Proof: Let $Q = Q_1 \boldsymbol{\times} Q_2$, where $Q_1, Q_2 \in \mathcal{Q}$. Consider an arbitrary finite ordered structure $\mathfrak{A}$ and let $X_i = Q_i^{\mathfrak{A}}$ for $i \in \{1, 2\}$. We have the following equivalences:

$$\begin{aligned} & a_1 \in (\mathbf{P}((Q_1 \boldsymbol{\times} Q_2) \boldsymbol{\cap} \mu))^{\mathfrak{A}} \\ \iff & \exists a_2.\, (a_1, a_2) \in (X_1 \times X_2) \cap \mu^{\mathfrak{A}} \\ \iff & a_1 \in X_1 \wedge\ [\exists a_2 \in X_2.\, (a_1, a_2) \in \mu^{\mathfrak{A}}] \end{aligned} \tag{1}$$

Depending on what is μ we have different cases to consider. As μ is a 2-arrangement, μ is just a basic formula $(y + d \# z)$ where $\{y, z\} = \{x_1, x_2\}$, $d \in \omega$ and $\# \in \{\boldsymbol{=}, \boldsymbol{<}\}$. There are altogether 4 cases. Consider first the case when μ is $(x_1 + d \boldsymbol{=} x_2)$, for which we have the equivalence:

$$(1) \iff a_1 \in X_1 \cap (X_2 \ominus d) \tag{2}$$

For this case the desired Q' is $Q_1 \boldsymbol{\cap} (Q_2 \boldsymbol{\ominus} d)$. Consider next the case when μ is $(x_1 + d \boldsymbol{<} x_2)$, for which we have the equivalence:

$$(1) \iff a_1 \in X_1 \cap (X_2 \ominus d \ominus 1 \ominus \infty) \tag{3}$$

To see why (3) is correct, observe that $(X_2 \ominus d \ominus 1 \ominus \infty) = \{ a \mid a + d < max(X_2) \}$. For this case the desired Q' is $Q_1 \boldsymbol{\cap} (Q_2 \boldsymbol{\ominus} d \boldsymbol{\ominus} 1 \boldsymbol{\ominus} \infty)$. We omit the remaining two cases, when μ is $(x_2 + d \boldsymbol{=} x_1)$ or $(x_2 + d \boldsymbol{<} x_1)$, as they are similar to the previous two. ∎

We now state and prove a generalization of Lemma 3.8.

Lemma 3.9 *Let $n \geq 3$. For every $Q \in \mathcal{Q} \boldsymbol{\times} \cdots \boldsymbol{\times} \mathcal{Q}$ (n times) and every n-arrangement μ there is a $Q' \in \mathcal{Q} \boldsymbol{\times} \cdots \boldsymbol{\times} \mathcal{Q}$ ($n-1$ times) such that* $\mathsf{P}(Q \boldsymbol{\cap} \mu) \;\equiv\; Q' \boldsymbol{\cap} (\mu - \{n\})$.

Proof: Let $Q = Q_1 \boldsymbol{\times} \cdots \boldsymbol{\times} Q_n$, where $Q_1, \ldots, Q_n \in \mathcal{Q}$. Let $\mu = \nu_1 \wedge \cdots \wedge \nu_{n-1}$. We write $\nu \in \mu$ to mean that $\nu \in \{\nu_1, \ldots, \nu_{n-1}\}$. Consider an arbitrary finite ordered structure $\mathfrak{A}$ and let $X_i = Q_i^{\mathfrak{A}}$ for $i = 1, \ldots, n$. We have the following equivalences:

$$
\begin{aligned}
& (a_1, \ldots, a_{n-1}) \in (\mathsf{P}((Q_1 \boldsymbol{\times} \cdots \boldsymbol{\times} Q_n) \boldsymbol{\cap} \mu))^{\mathfrak{A}} \\
\Longleftrightarrow \;& \exists a_n .\ (a_1, \ldots, a_n) \in (X_1 \times \cdots \times X_n) \cap \mu^{\mathfrak{A}} \\
\Longleftrightarrow \;& a_1 \in X_1 \wedge \cdots \wedge a_{n-1} \in X_{n-1} \wedge \; [\exists a_n \in X_n .\ (a_1, \ldots, a_n) \in \mu^{\mathfrak{A}}] \\
\Longleftrightarrow \;& (a_1, \ldots, a_{n-1}) \in (X'_1 \times \cdots \times X'_{n-1}) \cap (\mu - \{n\})^{\mathfrak{A}}
\end{aligned}
$$

where for $i = 1, \ldots, n-1$:

$$
X'_i \;=\; \begin{cases}
X_i \cap (X_n \ominus d), & \text{if } (x_i + d \mathbin{\boldsymbol{=}} x_n) \in \mu \text{ for some } d \in \omega \ , \\
X_i \cap (X_n \ominus d \ominus 1 \ominus \infty), & \text{if } (x_i + d \mathbin{\boldsymbol{<}} x_n) \in \mu \text{ for some } d \in \omega \ , \\
X_i \cap (X_n \oplus d), & \text{if } (x_n + d \mathbin{\boldsymbol{=}} x_i) \in \mu \text{ for some } d \in \omega \ , \\
X_i \cap (X_n \oplus d \oplus 1 \oplus \infty), & \text{if } (x_n + d \mathbin{\boldsymbol{<}} x_i) \in \mu \text{ for some } d \in \omega \ , \\
X_i, & \text{otherwise} \ .
\end{cases}
$$

The correctness of this definition follows from the fact that $(X \ominus d \ominus 1 \ominus \infty) = \{\, a \mid a + d < max(X) \,\}$ and $(X \oplus d \oplus 1 \oplus \infty) = \{\, a \mid a + d > min(X) \,\}$. The desired $Q' = (Q'_1 \boldsymbol{\times} \cdots \boldsymbol{\times} Q'_{n-1})$ is therefore given by:

$$
Q'_i \;=\; \begin{cases}
Q_i \boldsymbol{\cap} (Q_n \boldsymbol{\ominus} d), & \text{if } (x_i + d \mathbin{\boldsymbol{=}} x_n) \in \mu \text{ for some } d \in \omega \ , \\
Q_i \boldsymbol{\cap} (Q_n \boldsymbol{\ominus} d \boldsymbol{\ominus} 1 \boldsymbol{\ominus} \infty), & \text{if } (x_i + d \mathbin{\boldsymbol{<}} x_n) \in \mu \text{ for some } d \in \omega \ , \\
Q_i \boldsymbol{\cap} (Q_n \boldsymbol{\oplus} d), & \text{if } (x_n + d \mathbin{\boldsymbol{=}} x_i) \in \mu \text{ for some } d \in \omega \ , \\
Q_i \boldsymbol{\cap} (Q_n \boldsymbol{\oplus} d \boldsymbol{\oplus} 1 \boldsymbol{\oplus} \infty), & \text{if } (x_n + d \mathbin{\boldsymbol{<}} x_i) \in \mu \text{ for some } d \in \omega \ , \\
Q_i, & \text{otherwise} \ ,
\end{cases}
$$

for $i = 1, \ldots, n-1$. ■

3.3 $\mathcal{S}$ no more expressive than $\mathcal{Q}$ over finite ordered structures

Over finite ordered structures where all underlying relations (other than $<$) are monadic, $\mathcal{S}$ is no more powerful than $\mathcal{Q}$ (Lemma 3.11), as far as expressing unary properties. The proof is by induction on $\mathcal{S}$. We first deal with a special case.

Lemma 3.10 *Let $S \in \mathcal{S}$ be an arbitrary FODS not mentioning the constructor* P. *(Such a FODS is necessarily of arity ≥ 1.)*

1. *If* arity$(S) = 1$ *then there is $P \in \mathcal{P}$ such that $S \equiv P$.*

2. *If* arity$(S) = n \geq 2$ *then there are* $P_1, \ldots, P_m \in \mathcal{P}_n$ *such that:* $S \equiv (P_1 \mathbf{\cap} \mu_1) \mathbf{\cup} \cdots \mathbf{\cup} (P_m \mathbf{\cap} \mu_m)$ *where* $\{\mu_1, \ldots, \mu_m\} = \mathcal{M}_{n,0}$.

(Part 1 is a special case of part 2; we separate them for clarity.)

Proof: By induction on FODS's not mentioning $\mathbf{P}$. If $S = \varnothing$ (resp. $S = R \in \{R_1 \ldots, R_\alpha\}$), then let $P = \varnothing$ (resp. $P = R$) in order to satisfy part 1 of the lemma.

If $S = \Delta$ or $S = \mathbf{<}$ then $arity(S) = n = 2$. There are 3 simple arrangements of degree 0 over $\{x_1, x_2\}$. Let $\mathcal{M}_{2,0} = \{\mu_1, \mu_2, \mu_3\}$, where $\mu_1 = (x_1 \mathbf{<} x_2)$, $\mu_2 = (x_2 \mathbf{<} x_1)$, and $\mu_3 = (x_1 \mathbf{=} x_2)$. If $S = \Delta$ then choose $P_1 = P_2 = \varnothing \mathbf{\times} \varnothing$ and $P_3 = (\mathbf{\sim}\varnothing)\mathbf{\times}(\mathbf{\sim}\varnothing)$ to satisfy part 2 of the lemma. If $S = \mathbf{<}$ then choose $P_2 = P_3 = \varnothing \mathbf{\times} \varnothing$ and $P_1 = (\mathbf{\sim}\varnothing)\mathbf{\times}(\mathbf{\sim}\varnothing)$ to satisfy part 2 of the lemma.

Proceeding inductively, let $S = (\mathbf{\sim} S_0)$ and $arity(S) = n$. Omitting the case $n = 1$, which is simpler, let $n \geq 2$. By the induction hypothesis, we assume that S_0 satisfies part 2 of the lemma. By part 3 of Lemma 3.2 ($\mathcal{P}_n$ is semantically closed under $\mathbf{\sim}$) and by Lemma 3.4 ($\{\mu^{\mathfrak{A}} \mid \mu \in \mathcal{M}_{n,0}\}$ is a partition of $[0, u]^{(n)}$), we conclude that S also satisfies part 2 here.

Let $S = (S_1 \mathbf{\cup} S_2)$ and $arity(S) = n$. Omit the simpler case $n = 1$ and let $n \geq 2$. Assume both S_1 and S_2 satisfy part 2 of the lemma, by the induction hypothesis. By part 1 of Lemma 3.2, ($\mathcal{P}_n$ is closed under $\mathbf{\cup}$) and by Lemma 3.4 ($\{\mu^{\mathfrak{A}} \mid \mu \in \mathcal{M}_{n,0}\}$ is a partition of $[0, u]^{(n)}$), we conclude that S also satisfies part 2 here.

The case for $S = (S_1 \mathbf{\cap} S_2)$ is identical to the case for $S = (S_1 \mathbf{\cup} S_2)$. (It also follows by duality, in the presence of $\mathbf{\cup}$ and $\mathbf{\sim}$.)

Let $S = (\mathbf{C} S_0)$ and $arity(S_0) = n \geq 1$. Consider the case $n = 1$ first. By the induction hypothesis, $S_0 \equiv P$ for some $P \in \mathcal{P}$. Let $\mathcal{M}_{2,0} = \{\mu_1, \mu_2, \mu_3\}$ as defined in the second paragraph of this proof. If we define $P_1 = P_2 = P_3 = (P \mathbf{\times} (\mathbf{\sim}\varnothing))$ then $S \equiv (P_1 \mathbf{\cap} \mu_1) \mathbf{\cup} (P_2 \mathbf{\cap} \mu_2) \mathbf{\cup} (P_3 \mathbf{\cap} \mu_3)$, as desired. For the case $n \geq 2$, we have by the induction hypothesis:

$$S_0 \equiv (P_1 \mathbf{\cap} \mu_1) \ \mathbf{\cup} \cdots \mathbf{\cup} \ (P_m \mathbf{\cap} \mu_m)$$

for some $P_1, \ldots, P_m \in \mathcal{P}_n$ where $\mathcal{M}_{n,0} = \{\mu_1, \ldots, \mu_m\}$. From this, it readily follows that:

$$S = (\mathbf{C} S_0) \equiv ((\mathbf{C} P_1) \mathbf{\cap} (\mathbf{C} \mu_1)) \ \mathbf{\cup} \cdots \mathbf{\cup} \ ((\mathbf{C} P_m) \mathbf{\cap} (\mathbf{C} \mu_m))$$

By Lemma 3.6, $(\mathbf{C} \mu_i) \equiv \mu_{i,1} \vee \cdots \vee \mu_{i,k_i}$ where $\{\mu_{i,1}, \ldots, \mu_{i,k_i}\} = (\mu_i + \{n+1\})$ for $i = 1, \ldots, m$. Hence,

$$\begin{aligned} S \quad \equiv \quad & ((P_1 \mathbf{\times} (\mathbf{\sim}\varnothing)) \mathbf{\cap} \mu_{1,1}) \ \mathbf{\cup} \cdots \mathbf{\cup} \ ((P_1 \mathbf{\times} (\mathbf{\sim}\varnothing)) \mathbf{\cap} \mu_{1,k_1}) \ \mathbf{\cup} \\ & ((P_2 \mathbf{\times} (\mathbf{\sim}\varnothing)) \mathbf{\cap} \mu_{2,1}) \ \mathbf{\cup} \cdots \mathbf{\cup} \ ((P_2 \mathbf{\times} (\mathbf{\sim}\varnothing)) \mathbf{\cap} \mu_{2,k_2}) \ \mathbf{\cup} \end{aligned}$$

$$\vdots$$

$$((P_m \mathbf{\times} (\mathbf{\sim}\varnothing)) \mathbf{\cap} \mu_{m,1}) \mathbf{\cup} \cdots \mathbf{\cup} ((P_m \mathbf{\times} (\mathbf{\sim}\varnothing)) \mathbf{\cap} \mu_{m,k_m})$$

By part 4 of Lemma 3.2 (for every $P \in \mathcal{P}_n$ there is $P' \in \mathcal{P}_{n+1}$ such that $(P \mathbf{\times} \mathbf{\sim}\varnothing) \equiv P'$) and by Lemma 3.7, we conclude that S satisfies part 2 of the lemma here.

Finally, let $S = (\pi S_0)$ where π is a permutation of $\{1, \ldots, n\}$ and $arity(S_0) = n \geq 2$. It is easily seen that part 2 of the lemma is invariant under such a permutation. This concludes the induction and the proof. ∎

Lemma 3.11 *Let $S \in \mathcal{S}$ be an arbitrary FODS.*

1. *If* arity$(S) = 0$ *then there is no condition on S.*
2. *If* arity$(S) = 1$ *then there is $Q \in \mathcal{Q}$ such that $S \equiv Q$.*
3. *If* arity$(S) = n \geq 2$ *then there are $Q_1, \ldots, Q_m \in \mathcal{Q}_n$ such that: $S \equiv (Q_1 \mathbf{\cap} \mu_1) \mathbf{\cup} \cdots \mathbf{\cup} (Q_m \mathbf{\cap} \mu_m)$ where $\{\mu_1, \ldots, \mu_m\} = \mathcal{M}_{n,d}$ for some $d \in \omega$ depending on S.*

(As in Lemma 3.10, parts 1 and 2 are special cases of part 3; we separate them for clarity. Only the case $n = 1$ is used in Theorem 3.13; the cases $n = 0$ and $n \geq 2$ are included here to push the induction through.)

Proof: By induction on $\mathcal{S}$. By Corollary 2.5 we can restrict attention to FODS's in prenex form. Hence, by Lemma 3.10, it suffices to consider 3 cases to complete the induction; these are the cases for $S = (\mathbf{\sim} S_0)$, $S = (\pi S_0)$ and $S = (\mathbf{P} S_0)$. The cases for $S = (\mathbf{\sim} S_0)$ and $S = (\pi S_0)$ are treated just as in the proof of Lemma 3.10, as all parts of Lemma 3.2 invoked in that proof relative to $\mathcal{P}_n$ are also true relative to $\mathcal{Q}_n$. To conclude the induction, let $S = (\mathbf{P} S_0)$ and $arity(S_0) = n \geq 2$. By the induction hypothesis, there are $Q_1, \ldots, Q_m \in \mathcal{Q}_n$ such that:

$$S_0 \equiv (Q_1 \mathbf{\cap} \mu_1) \mathbf{\cup} \cdots \mathbf{\cup} (Q_m \mathbf{\cap} \mu_m)$$

where $\{\mu_1, \ldots, \mu_m\} = \mathcal{M}_{n,d}$ for some $d \in \omega$ depending on S_0. From which we can write:

$$S = (\mathbf{P} S_0) \equiv (\mathbf{P}(Q_1 \mathbf{\cap} \mu_1)) \mathbf{\cup} \cdots \mathbf{\cup} (\mathbf{P}(Q_m \mathbf{\cap} \mu_m))$$

If $n = 2$, then Lemma 3.8 implies that $S \equiv Q$ for some $Q \in \mathcal{Q}$, which is the desired conclusion. If $n \geq 3$ then, by Lemma 3.9:

$$S \equiv (Q'_1 \mathbf{\cap} (\mu_1 - \{n\})) \mathbf{\cup} \cdots \mathbf{\cup} (Q'_m \mathbf{\cap} (\mu_m - \{n\}))$$

for some $Q'_1, \ldots, Q'_m \in \mathcal{Q}_{n-1}$. Let $d' = max\{degree(\mu_1 - \{n\}), \ldots, degree(\mu_m - \{n\})\}$. As observed when we defined the operation $(\mu - \{n\})$ after Lemma 3.5, $d' = 1 + 2 \cdot d$. By Lemma 3.5, for every $i = 1, \ldots, m$, we have $(\mu_i - \{n\}) \equiv \mu_{i,1} \vee \cdots \vee \mu_{i,k_i}$ for some $\mu_{i,1}, \ldots, \mu_{i,k_i} \in \mathcal{M}_{n-1,d'}$. Hence,

$$\begin{aligned} S \;\equiv\;\; & (Q'_1 \cap \mu_{1,1}) \cup \cdots \cup (Q'_1 \cap \mu_{1,k_1}) \cup \\ & (Q'_2 \cap \mu_{2,1}) \cup \cdots \cup (Q'_2 \cap \mu_{2,k_2}) \cup \\ & \vdots \\ & (Q'_m \cap \mu_{m,1}) \cup \cdots \cup (Q'_m \cap \mu_{m,k_m}) \end{aligned}$$

Let $Q_0 = \varnothing \times \cdots \times \varnothing$ ($n-1$ times). Clearly $Q_0 \in \mathcal{Q}_{n-1}$. We add a conjunction $(Q_0 \cap \mu)$ to the above disjunction for every $(n-1)$-arrangement $\mu \in (\mathcal{M}_{n-1,d'} - \{\mu_{i,j} | 1 \leq i \leq m, 1 \leq j \leq k_i\})$ in order to make S satisfies part 3 of the lemma. ∎

3.4 $\mathcal{Q}$ no more expressive than $\mathcal{S}$ over finite ordered structures

As $\mathcal{S}$ has exactly the expressive power of first-order logic, by Propositions 2.2 and 2.3, the next result is the converse of Lemma 3.11.

Lemma 3.12 *Every $Q \in \mathcal{Q}$ defines a (unary) first-order property over a class of finite ordered structures.*

Proof: By induction on $\mathcal{Q}$ we prove that, over an arbitrary class of finite ordered structures, for every $Q \in \mathcal{Q}$ there is an equivalent FO formula $\varphi(x)$. For the basis case, if $Q = \varnothing$ (respectively, $Q = R \in \{R_1, \ldots, R_\alpha\}$) then define $\varphi(x) = \neg(x = x)$ (respectively, $\varphi(x) = R(x)$).

Let $Q = Q_0 \oplus n$ where $n \in \omega$. By the induction hypothesis, there is a FO formula $\varphi_0(x)$ equivalent to Q_0. If $n = 0$, define $\varphi(x) = \varphi_0(x)$. If $n \geq 1$, the desired $\varphi(x)$ is:

$$\varphi(x) \;=\; (\exists y_0) \cdots (\exists y_{n-1}) \, [\varphi_0(y_0) \wedge succ(y_0, y_1) \wedge \cdots \wedge succ(y_{n-2}, y_{n-1}) \wedge succ(y_{n-1}, x)]$$

where $succ(x, y) \;=\; (x < y) \;\wedge\; (\forall z)[x < z \rightarrow (y = z \;\vee\; y < z)]$.

Let $Q = Q_0 \oplus \infty$. By the induction hypothesis, there is a FO formula $\varphi_0(x)$ equivalent to Q_0. The desired $\varphi(x)$ is:

$$\varphi(x) \;=\; (\exists y) \, [\varphi_0(y) \;\wedge\; (y = x \;\vee\; y < x)] \, .$$

The cases for $Q = Q_0 \ominus n$ and $Q = Q_0 \ominus \infty$ are treated just as $Q = Q_0 \oplus n$ and $Q = Q_0 \oplus \infty$, respectively. Details omitted.

The cases for $Q = \sim Q_0$, $Q = Q_1 \cap Q_2$ and $Q = Q_1 \cup Q_2$ are also straightforward, using the induction hypothesis, and therefore omitted. ∎

We are ready to state the main result of this section.

Theorem 3.13 *Let $\mathcal{C} = \{\mathfrak{A}_i\}_{i\in\omega}$ be a class of finite ordered structures, and $\mathcal{X} = \{X_i\}_{i\in\omega}$ where $X_i \subseteq |\mathfrak{A}_i|$ for every $i \in \omega$. $\mathcal{X}$ is a first-order property over $\mathcal{C}$ iff there is a $Q \in \mathcal{Q}$ such that $X_i = Q^{\mathfrak{A}_i}$ for every $i \in \omega$.*

Proof: Immediate consequence of Lemmas 3.11 and 3.12. ■

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