NEWTON POLYGON RELATIVE TO AN ARC

TZEE-CHAR KUO AND ADAM PARUSIŃSKI

ABSTRACT. The notion of Newton polygon is well-known. We define a generalisation and apply it to study polar curves, Lojasiewicz exponents, singularities at infinity of complex polynomials (Ha Huy Vui's theorem), and μ-constant deformations. Philosophically speaking, the Newton polygon relative to an arc λ exposes f in a horn neighborhood of λ. The gradient of a function behaves erratically in the process of blowing up. Our method indicates how it can be handled without resort to blow-ups.

Throughout this paper let $f(x, y)$ denote a germ of holomorphic function with Taylor expansion:

$$f(x, y) = H_k(x, y) + H_{k+1}(x, y) + \cdots .$$

We shall assume $f(x, y)$ is *mini-regular in x of order k* in the sense that $H_k(1, 0) \neq 0$. (This can be achieved by a linear transformation $x' = x$, $y' = y + cx$, c a generic constant.)

By a fractional (convergent) power series we mean a series of the form

$$\lambda : x = \lambda(y) := c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots , \quad c_i \in \mathbb{C},$$

where $N \leq n_1 < n_2 < \cdots$ are positive integers, having no common divisor, such that $\lambda(t^N)$ has positve radius of convergence. We can identify λ with the analytic arc $\lambda : x = c_1 t^{n_1} + c_2 t^{n_2} + \cdots , y = t^N$, $|t|$ small, which is not tangent to the x-axis (since $n_1/N \geq 1$).

Let us apply the change of variables

$$X = x - \lambda(y), \quad Y = y,$$

to $f(x, y)$, yielding

$$F(X, Y) := f(X + \lambda(Y), Y) := \sum c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a *Newton dot*. The set of Newton dots is called *the Newton diagram*. They generate a convex hull, whose boundary is

---

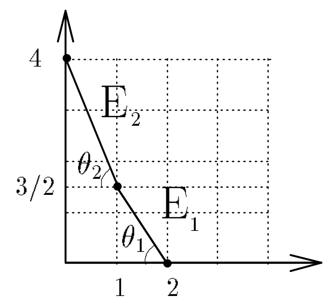
1991 *Mathematics Subject Classification.* 32S05, 32S15, 14H20.

The paper was prepared during the first author's stay in Angers. We would like to thank PRIA (Pôle de Recherche et d'Innovation à Angers) for the financial support which made this visit possible.

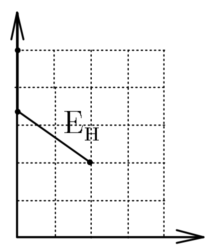
The first author was also supported by an ARC Institutional Grant.

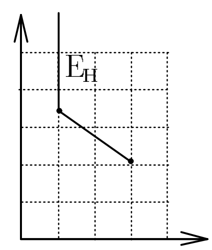
called *the Newton polygon of f relative to λ*, to be denoted by $\mathbb{P}(f,\lambda)$. Note that this is just the Newton polygon of F in the usual sense.

The Newton edges E_s and their associated Newton angles θ_s are defined in an obvious way as illustrated in the following example. Take $f(x,y) = x^2 - y^3 + y^4$, $\lambda : x = y^{3/2}$. Then $\mathbb{P}(f,\lambda)$ has compact edges E_1, E_2 with $\tan\theta_1 = 3/2$, $\tan\theta_2 = 5/2$.



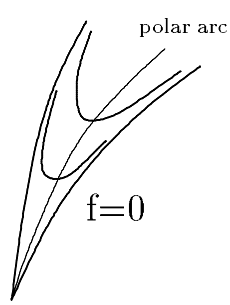
Convention: The *"highest Newton edge"*, often denoted by E_H, means the following: If the highest vertex is on the y-axis, E_H is the compact edge to its right; otherwise, E_H is the vertical edge sitting on this vertex, as illustrated below.





1. Polar Curves

The loci defined by $\partial f/\partial x = 0$ is called a *polar curve*. It consists of points where the level curves $f = const$ have horizontal tangents as illustrated below.



Observe that the sharper is the cusp $f = 0$, the larger is the contact order with its polar. Hence the contact order is a measurement of how singular the cusp is. The following theorem can be used to calculate the contact order, see [5].

Take any $q \in \mathbb{Q}^+$ (positive rationals). Take two fractional power series $\beta(y), \bar{\beta}(y)$. We say β and $\bar{\beta}$ are *congruent modulo* q if their difference has the form

$$\beta(y) - \bar{\beta}(y) = cy^q + \cdots, \quad c \in \mathbb{C}.$$

In this case we write $\beta \equiv \bar{\beta} \mod q$.

We say $\beta(y)$ is a *(Newton-Puiseux) root mod q of $f = 0$* if there exists $\bar{\beta}(y)$ such that

$$f(\bar{\beta}(y), y) \equiv 0 \quad \text{and} \quad \beta \equiv \bar{\beta} \mod q.$$

If, in the Newton-Puiseux factorization,

$$f(x, y) = unit \cdot \prod_{i=1}^{k} [x - \beta_i(y)],$$

there are exactly m roots β_i, $\beta_i \equiv \beta \mod q$, we call m the *multiplicity* of β.

Theorem 1.1. (Compare Lemma (3.3), [5]). *If $\beta(y)$ is a mod q root of $f = 0$ of multiplicity m, then it is a mod q root of $\partial f/\partial x = 0$ of multiplicity $m - 1$.*

There is also a version of Rolle's theorem. Take real numbers $a_1, \ldots, a_{s-1}, b_1, b_2$, with $b_1 < b_2$. Let $\varepsilon > 0$ be sufficiently small. Take $q_i \in \mathbb{Q}^+$, $q_1 < \cdots < q_s$.

Theorem 1.2. *Let $f(x, y)$ be real analytic. Suppose, for $i = 1, 2$,*

$$\beta_i(y) := a_1 y^{q_1} + \cdots + a_{s-1} y^{q_{s-1}} + b_i y^{q_s},$$

are mod $(q_s + \varepsilon)$ roots of $f = 0$. Then there is a real number a_s, $b_1 < a_s < b_2$, such that

$$\gamma(y) := a_1 y^{q_1} + \cdots + a_{s-1} y^{q_{s-1}} + a_s y^{q_s}$$

is a mod $(q_s + \varepsilon)$ root of $\partial f/\partial x = 0$.

This means that by adding higher order terms, with possibly non-real coefficients, one can find a root of $\partial f/\partial x$.

For a proof of Theorem 1.1, let us consider the Newton polygon of f relative to

$$\beta^*(y) := \beta(y) + gy^q,$$

where g is a generic constant. Let $(\mathrm{E}_1, \theta_1), \ldots, (\mathrm{E}_H, \theta_H)$ denote the compact Newton edges and the corresponding angles, E_H being the highest edge. Note that $(k, 0)$ is a vertex of E_1. Since g is a generic number, β^* cannot be a root of f, E_H has a vertex on the Y-axis, say $(0, h_0)$. The other vertex of E_H ought to be (m, h_1), $h_1 < h_0$, where m is the multiplicity of β. The reason is as follows.

Take any edge E_s. The associated polynomial $\mathcal{E}_s(z)$ is defined to be $\mathcal{E}_s(z) := \mathcal{E}_s(z,1)$, where

$$\mathcal{E}_s(X,Y) := \sum c_{ij} X^i Y^{j/N}, \quad (i,j) \in \mathrm{E}_s \,.$$

As in [6], every root $c \neq 0$ of $\mathcal{E}_s(z) = 0$ leads to a root of f of the form

$$\beta^*(y) + [cy^{\tan\theta_s} + \cdots],$$

having contact order $\tan\theta_s$ with β^*. It follows that $\tan\theta_H = q$ and $\tan\theta_s < q$ for $s < H$. Since m is the multiplicity of β, $\mathcal{E}_H(z)$ must be of degree m.

The Newton diagram of $\partial F/\partial X$ can be easily found. Namely, move every Newton dot $(i, j/N)$ of F to $(i-1, j/N)$, if $i \geq 1$, and delete all Newton dots $(0, j/N)$. This is simply because $\frac{\partial}{\partial X}(X^i Y^{j/N}) = iX^{i-1}Y^{j/N}$. Therefore the highest Newton edge of $\partial F/\partial X$ has vertices at $(0, h_0 - \tan\theta_H)$ and $(m-1, h_1)$. The associated polynomial equation is $\frac{d}{dz}\mathcal{E}_H(z) = 0$. All $m-1$ roots are non-zero (since g is generic). Each root leads to a root of $\partial F/\partial X = 0$, which is congruent to β^* modulo q. The roots derived from the other edges are not congruent to β^* modulo q.

Note that the polar curve does not change when we move from (x,y) to (X,Y), because $\partial f/\partial x = \partial F/\partial X$. Hence, for $m \geq 2$, the proof is complete.

For the case $m = 1$, the above wording can be slightly modified to show that β is not a modulo q root of $\partial f/\partial x = 0$.

Rolle's Theorem is proved in the same way. Take a generic real number g and let

$$\beta^*(y) := a_1 y^{q_1} + \cdots + a_{s-1} y^{q_{s-1}} + g y^{q_s}.$$

Consider the highest Newton edge E_H of $\mathbb{P}(f, \beta^*)$. The associated equation $\mathcal{E}_H(z) = 0$ has b_1 and b_2 as real roots. Hence its derivative $\mathcal{E}_H'(z)$ has a real root, a_s, between b_1 and b_2.

2. Łojasiewicz exponent

In this section we assume $f(x,y)$ has an isolated singularity at 0, that is,

$$\|\operatorname{grad} f\|^2 = |\partial f/\partial x|^2 + |\partial f/\partial y|^2 > 0, \quad (x,y) \neq (0,0), \qquad \text{near } (0,0).$$

Take any analytic arc (convergent power series)

$$\lambda: \; x = a_1 t^{n_1} + a_2 t^{n_2} + \cdots, \quad y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots,$$

$|t|$ small. Let $N = \min\{n_1, m_1\}$. Let us define $\ell(\lambda) \in \mathbb{Q}^+$ by

$$\|\operatorname{grad} f(\lambda(t))\| \sim \|\lambda(t)\|^{\ell(\lambda)} \sim |t|^{N\ell(\lambda)}, \tag{2.1}$$

where $A \sim B$ means that A/B lies between two positive constants. We call

$$\mathrm{L(f)} := \sup_{\lambda}\{\ell(\lambda)\} \tag{2.2}$$

the Łojasiewicz exponent of f. The purpose of this section is to show the following

Theorem 2.1. *The Lojasiewicz exponent,* L(f)*, is attained along the polar curve. More precisely, there is a Newton-Puiseux root of* $\partial f/\partial x = 0$*,*

$$\gamma : \ x = c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots, \quad 1 \le N \le n_1 < n_2 < \cdots, \tag{2.3}$$

such that $\mathrm{L(f)} = \ell(\gamma)$.

Here $\ell(\gamma)$ is defined by identifying γ with the arc

$$\gamma : \ x = c_1 t^{n_1} + c_2 t^{n_2} + \cdots, \quad y = t^N. \tag{2.4}$$

Following [6], γ is called a *"branch"* of the polar curve, or simply a *"polar branch"*.

For real analytic $f(x, y)$ we have the following version of Theorem 2.1. Define the *real Lojasiewicz exponent* $\mathrm{L(f)} := \sup_\lambda\{\ell(\lambda)\}$ by taking the supremum over real analytic λ. Take a Newton-Puiseux root of $\partial f/\partial x = 0$,

$$\gamma : \ x = a_1 y^{n_1/N} + \cdots + a_{s-1} y^{n_{s-1}/N} + c_s y^{n_s/N} + \cdots, \tag{2.5}$$

where $a_i \in \mathbb{R}$, c_s is the first non-real coefficient, if there is one. Let us replace c_s by a generic real number g, and call

$$\gamma_{\mathbb{R}} : \ x = a_1 y^{n_1/N} + \cdots + a_{s-1} y^{n_{s-1}/N} + g y^{n_s/N} \tag{2.6}$$

a *real polar branch.* In case $s = \infty$, let $\gamma_{\mathbb{R}} = \gamma$.

Theorem 2.2. *The real Lojasiewicz exponent,* L(f)*, is attained along a real polar branch.*

We shall now prove Theorems 2.1 and 2.2. It is easy to see that if λ is tangent to the x-axis, then $\ell(\lambda) = k - 1$, k the multiplicity of f. We can therefore ignore these arcs.

Lemma 2.3. *In the Newton diagram of* f *relative to a given*

$$\beta : \ x = c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots, \quad 1 \le N \le n_1 < n_2 < \cdots, \tag{2.7}$$

let $(0, h_0)$ *and* $(1, h_1)$ *denote the lowest Newton dots on* $X = 0$ *and* $X = 1$ *respectively. Then*

$$\ell(\beta) = \min\{h_0 - 1, h_1\}. \tag{2.8}$$

This Lemma is important, but the proof is easy. Let us write

$$F(X, Y) := f(X + \beta(Y), Y) = unit \cdot Y^{h_0} + unit \cdot Y^{h_1} X + \text{terms divisible by } X^2.$$

By the Chain Rule, $\partial F/\partial X = \partial f/\partial x$, $\partial F/\partial Y = \beta'(Y)\partial f/\partial x + \partial f/\partial y$. Hence

$$|\partial F/\partial X| + |\partial F/\partial Y| \sim |\partial f/\partial x| + |\partial f/\partial y|.$$

Clearly, along $X = 0$, we have

$$|\partial F/\partial X| \sim |Y|^{h_1}, |\partial F/\partial Y| \sim |Y|^{h_0 - 1},$$

whence the result.

Next, we define the notion of *sliding*. Beginning with a given $\beta(y)$, we wish to find a root of $f=0$ which approximates β. In case β is already a root, we are done. Otherwise we can "slide β along f" to get a better approximation. The precise definition is as follows.

Suppose β is not a root. Consider $\mathbb{P}(f,\beta)$. Take any root c ($c\neq 0$) of $\mathcal{E}_H(z)=0$, the polynomial equation associated to the highest Newton edge E_H. We then call

$$\beta_1 : x = \beta(y) + c y^{\tan\theta_H}$$

a *sliding* of β *along* f. A recursive sliding $\beta \to \beta_1 \to \beta_2 \to \cdots$ produces a limit, β_∞, which is a root of $f=0$. This is the well-known procedure of finding a Newton-Puiseux root of $f=0$, as in [6].

Assertion 1. Let γ denote a sliding of β along $\partial f/\partial x$. Then $\ell(\beta)\le \ell(\gamma)$.

Theorem 2.1 clearly follows from this assertion.

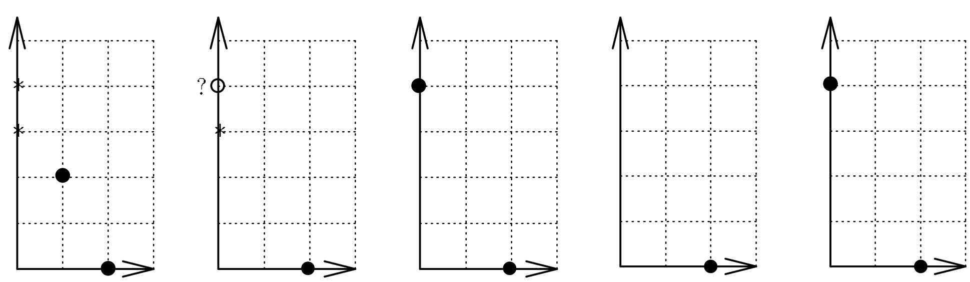
In the Newton diagram of $f(x,y)$ relative to β, let $(0,h_0)$, $(1,h_1)$ denote the lowest Newton dots on $X=0$ and $X=1$ respectively. In the Newton diagram of f relative to γ, let $(0,\eta_0)$, $(1,\eta_1)$ denote the lowest Newton dots on these lines. Consider also the Newton diagram of $\partial f/\partial x$ relative to β. Let $\mathrm{E}_{H'}$ denote the highest Newton edge; let $(0,h_0')$ denote its vertex on $X=0$.

Assertion 2. $\eta_1 > h_1$ and $\eta_0 \ge \min\{h_0, h_1+\tan\theta_{H'}\}$.

As an illustrative example, take

$$f(x,y) = x^2 + 2xy^2 + c_3y^3 + c_4y^4, \quad \beta(y)\equiv 0.$$

In case $c_3\neq 0$, we have $h_0=\eta_0=3$; in case $c_3=0\neq c_4\neq 1$, $h_0=\eta_0=4$; in case $c_3=0$ and $c_4=1$, $h_0=4$, $\eta_0=\infty$; and in case $c_3=c_4=0$, $h_0=\infty$, $\eta_0=4$. In all cases, $h_1=\tan\theta_{H'}=2$.



f(x,y) $\quad c_3\neq 0 \quad c_3=0\neq c_4\neq 1 \quad c_3=0, c_4=1 \quad c_3=c_4=0$

In the expansion $F(X,Y)=\sum c_{ij}X^iY^{j/N}$, let us collect all the terms whose derivatives lie on $\mathrm{E}_{H'}$:

$$\varphi_{H'}(X,Y) := \sum c_{ij}X^iY^{j/N}, \quad (i-1, j/N)\in \mathrm{E}_{H'}\,.$$

Note that $\frac{d}{dz}\varphi_{H'}(z,1) = \mathcal{E}_{H'}(z)$. In the expansion of $\varphi_{H'}(X + cY^{\tan\theta_{H'}}, Y)$, the term $XY^{h'_0}$ has coefficient 0, since $\mathcal{E}_{H'}(c) = 0$. (The coefficient of $Y^{h_1+\tan\theta_{H'}}$ may or may not vanish.) Thus Assertion 2 follows.

Assertion 1 now follows from Assertion 2, Lemma 2.3, and the fact that $\tan\theta_{H'} \geq 1$.

Remark 2.4. After the preparation of this paper we have learnt that Theorem 2.1 was also proved by a different argument by Bogusławska [2].

3. Singularities at infinity of complex polynomials

Let $P(x,y)$ be a polynomial over $\mathbb{C}$.

Theorem 3.1. (Ha Huy Vui, [3, 4]) *The following conditions are equivalent:*

(1) *For any sequence* $(x_n, y_n) \to \infty$ *with the property that* $P(x_n, y_n) \to 0$, *the sequence* $\|\text{grad } P(x_n, y_n)\|$ *does not tend to* 0.
(2) *For any sequence* $(x_n, y_n) \to \infty$ *with the property that* $P(x_n, y_n) \to 0$, *the sequence* $\|\text{grad } P(x_n, y_n)\| \, \|(x_n, y_n)\|$ *does not tend to* 0.

When $P(x,y)$ has degree d and the coefficient of x^d is non-zero, we say $P(x,y)$ is *regular in* x. Note that in this case, if $P(x_n, y_n) \to 0$, $(x_n, y_n) \to \infty$, then x_n/y_n is bounded.

Addendum. Suppose P is regular in x and not of the form $P(x,y) = g(x - cy)$, $c \in \mathbb{C}$, g a polynomial of one variable. Then the above conditions are also equivalent to

(3) *For any sequence* $(x_n, y_n) \to \infty$ *such that* $P(x_n, y_n) \to 0$, $\|\text{grad } P(x_n, y_n)\|$ *tends to infinity.*

In case $P(x,y) = g(x - cy)$, (1) and (2) hold if and only if g has distinct roots. This case is trivial.

For a proof of the theorem it suffices to show that (2) implies (1) since the implication (1)⇒(2) is trivial. We shall prove the addendum later on.

Assume condition (1) fails along a sequence (x_n, y_n). Using the Curve Selection Lemma, or Hironaka's Theorem, we can assume that (x_n, y_n) lies on an analytic curve

$$\lambda : x = c_1 s^{n_1} + c_2 s^{n_2} + \cdots, \quad y = s^{-N}, \tag{3.1}$$

where $s \to 0$, $N > 0$, $n_1 < n_2 < \cdots$, (n_1 need not be positive). We must have $n_1 + N \geq 0$, since x_n/y_n is bounded. We can rewrite λ as a fractional power series

$$\lambda : x = c_1 y^{-n_1/N} + c_2 y^{-n_2/N} + \cdots, \quad -N \leq n_1 < n_2 < \cdots. \tag{3.2}$$

Let us apply the change of variables

$$X = x - \lambda(y), \quad Y = y^{-1}, \tag{3.3}$$

and consider the Newton diagram of

$$M(X,Y) := P(X + \lambda(1/Y), 1/Y). \tag{3.4}$$

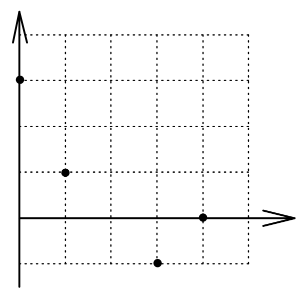
Clearly, it has at most finitely many dots lying on or below the X-axis. Moreover, there is one dot at $(d, 0)$, since $P(x, y)$ is regular in x.

Note that in the (X,Y)-plane, λ is just the Y-axis. The assumption $P(x_n, y_n) \to 0$ means that $M(0,Y) \to 0$ as $Y \to 0$. All Newton dots of $M(0,Y)$ lie above the X-axis.

Let $(1, h_1)$ denote the lowest Newton dot on $X = 1$. We must have $h_1 > 0$, since otherwise (1) would hold along λ. Therefore, we can use the Newton dots on or below the X-axis to "swallow" $(1, h_1)$. This means, more precisely, that we let λ slide along $\partial M/\partial X$, say to an arc γ_1.

Example 3.2. $M(X,Y) = Y^3 - 2XY + X^3Y^{-1} + X^4$.

The dot $(1,1)$ represents $-2XY$. Let us take a root $c \neq 0$ of $z^3 - 2z = 0$, say $c = \sqrt{2}$, and let γ_1 be $X = \sqrt{2}Y$. Then $-2XY$ is "swallowed" by X^3Y^{-1}.



The lowest Newton dot on $X = 1$ of $M(X + \gamma_1(Y), Y)$ is higher than $(1, h_1)$. On $X = 0$, all dots remain above the X-axis.

A recursive sliding $\lambda \to \gamma_1 \to \gamma_2 \to \cdots$, will then yield a root γ of the polar curve $\partial P/\partial x = \partial M/\partial X = 0$, for which

$$\widetilde{M}(X,Y) := M(X + \gamma(Y), Y)$$

has no dots on $X = 1$, and dots on $X = 0$ all lie above the X-axis.

An easy calculation, using the Chain Rule, yields

$$y(\partial P/\partial x) = Y^{-1}(\partial \widetilde{M}/\partial X), \quad y(\partial P/\partial y) = Y(\partial \widetilde{M}/\partial Y) - Y\gamma'(Y)(\partial \widetilde{M}/\partial X),$$

whence condition (2) fails along γ. Thus Theorem 3.1 is proven.

Remark 3.3. The above proof actually shows that Conditions (1) and (2) of Theorem 3.1 are equivalent to

(4) *For any root* $\gamma : x = \gamma(y)$ *of the polar curve* $\partial P/\partial x = 0$, $\lim_{y\to\infty} P(\gamma(y), y) \neq 0$ *(this limit can be infinite).*

Moreover the equivalent conditions (1), (2), or (4) admit the following geometric interpretation, see [3, 4]. We say that $P(x,y) = 0$ has *no singularities at infinity* if there is a "neighbourhood" $\mathcal{U}$ of infinity and a positive constant δ such that P induces a trivial fibration

$$P : \mathcal{U} \cap \{|P| < \delta\} \longrightarrow \{z \in \mathbb{C} \mid |z| < \delta\}. \tag{3.5}$$

The conditions (1), (2), or (4) are also equivalent to

(5) $P(x, y) = 0$ *has no singularities at infinity.*

Indeed, consider the analytic map

$$\Phi : \mathbb{C}^2 \to \mathbb{C}^2, \quad \Phi(x, y) = (P(x, y), y).$$

Since $P(x, y)$ is regular in x, Φ is proper. It is easy to see that Φ is an analytic covering branched along the polar curve $\partial P/\partial x = 0$. Suppose condition (4) holds. Then, for ε, δ positive and sufficiently small and $\mathcal{U} = \mathcal{U}_\varepsilon = \{(x, y)| \, |x| > \varepsilon^{-1}\}$ the map defined in (3.5) is a covering space, so topologically trivial.

On the other hand, suppose that (4) fails along a branch γ of $\partial P/\partial x = 0$. We may also suppose that $P(x, y) = 0$ does not have multiple components (otherwise P cannot be topologically trivial at generic points of such components). Consider, as before, $\mathcal{U} = \mathcal{U}_\varepsilon = \{(x, y)| \, |x| > \varepsilon^{-1}\}$ and the fibres $\mathcal{U}_{\varepsilon,z} = \mathcal{U}_\varepsilon \cap P^{-1}(z)$, $|z| < \delta$. Choose ε, δ small but positive. Then the projections

$$\mathcal{U}_{\varepsilon,z} \to \{x| \, |x| < \varepsilon\},$$

induced by $(x, y) \to x$, are analytic coverings branched along the points of the polar curve. We may assume that there are no such points for $z = 0$, but the existence of γ shows that the set of such points is non-empty for $z \neq 0$ and small. This means, in particular, that the Euler characteristic $\chi(\mathcal{U}_{\varepsilon,z})$ changes at $z = 0$ and (5) fails.

Now we show the claim of Addendum. For this it suffices to show that (2) implies (3). We may use exactly the same argument as in the proof (2)⇒(1) if we know that for any analytic curve λ, as in (3.2),

$$M(X, Y) := P(X + \lambda(1/Y), 1/Y)$$

must have Newton dots strictly below the X-axis. This follows from the following two lemmas.

Lemma 3.4. *Let $P(x, y)$ be a polynomial regular in x, of degree d, and suppose that there exists an analytic curve (3.2) such that the Newton diagram of $M(X, Y) := P(X + \lambda(1/Y), 1/Y)$ has no dots below the X-axis. Then there exists a polynomial of one variable g and $c \in \mathbb{C}$ such that*

$$P(x, y) = g(x - cy). \tag{3.6}$$

Proof. Write

$$\lambda(Y^{-1}) = c_1 Y^{n_1/N} + c_2 Y^{n_2/N} + \cdots = \varphi(Y) + \psi(Y),$$

where $\varphi(Y)$ contains the negative powers of Y and $\psi(Y)$ the non-negative ones.

Develop $P(x,y)$ as a polynomial of x:

$$P(x,y) = ax^d + (by+e)x^{d-1} + \cdots . \tag{3.7}$$

Note that by our assumption $a \neq 0$. Then

$$P(X+\lambda(1/Y), 1/Y) = aX^d + [ad\lambda(Y^{-1}) + bY^{-1} + e]X^{d-1} + \cdots . \tag{3.8}$$

If $M(X,Y) := P(X+\lambda(1/Y),1/Y)$ has no Newton dots below the X-axis then, of course, $[ad\lambda(Y^{-1}) + bY^{-1} + e]$ has no terms with negative exponents, that is

$$\varphi(Y) = -\frac{b}{ad}Y^{-1}.$$

Set $c = -\frac{b}{ad}$. Then $\lambda(Y^{-1}) = cY^{-1} + \psi(Y)$. Now we go back to the original variables x, y. As we have just proved, $Q(x,y) = P(x+cy+\psi(y^{-1}), y)$ does not have Newton dots above the x-axis. Here $\psi(y)$ is a convergent power series.

Lemma 3.5. *If, for a polynomial $P(x,y)$ and a convergent power seris ψ, $Q(x,y) = P(x+cy+\psi(y^{-1}),y)$ does not have Newton dots above the x-axis, then $P(x+cy,y)$ does not depend on y. In other words, there is a polynomial of one variable g such that $P(x+cy,y) = g(x)$ (or equivalently $P(x,y) = g(x-cy)$).*

The proof is reduced to the case $c = 0$, when we change the variable x to $x + cy$.

Suppose $P(x,y)$ is not independent of y. Amongst all non-zero terms divisible by y, let $a_{ij}x^iy^j, j > 0$, be the one for which (i,j) is largest lexicographically. Then, clearly, $P(x+\psi(y^{-1}),y)$ still has $a_{ij}x^iy^j$ as a term. This is a contradiction. □

4. μ-constant families of type $F(x,y,t) = f(x,y) + tg(x,y)$

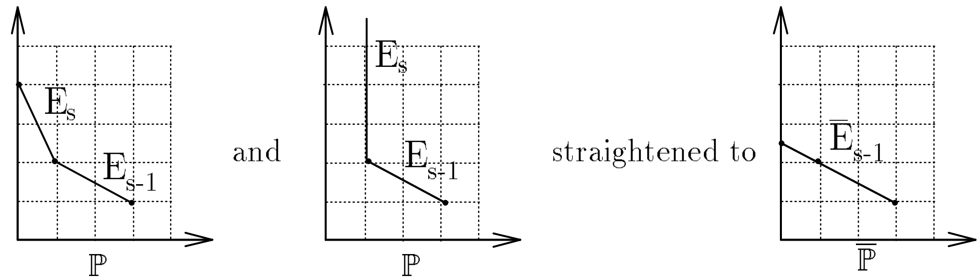
Let $f(x,y), g(x,y)$ be two germs of holomorphic funtions. Consider a one parameter deformation $F(x,y,t) = f(x,y) + tg(x,y)$, $t \in (\mathbb{C}, 0)$. Recall that F is μ constant if, and only if

$$|g(x,y)| \ll |F_x| + |F_y|. \tag{4.1}$$

We shall show in Theorem(4.1) below that F is a μ constant deformation if, and only if, the Newton polygon of f relative to any analytic arc $x = \lambda(y)$ is not disturbed by g (in a precise meaning defined below). This, in particular, implies that such μ-constant deformations are Whitney regular, that is

$$|g(x,y)| \leq C(|x|+|y|)(|F_x|+|F_y|), \quad C \text{ a constant}. \tag{4.2}$$

Consider a given Newton polygon $\mathbb{P}$. In case it has a vertex of the form $V_s = (1, h_s)$ on the line $x = 1$, we define the *straightened* polygon, denoted by $\overline{\mathbb{P}}$, as follows: Erase the edge E_s and extend E_{s-1} to the y-axis, as indicated below.



In case there is no vertex on $x = 1$, we set $\overline{\mathbb{P}} = \mathbb{P}$.

The Newton polygon relative to a polar branch does not have dots on $x = 1$, hence is already straightened.

Theorem 4.1. *Consider $F_t(x, y) := F(x, y, t) = f(x, y) + tg(x, y)$. The following conditions are equivalent:*

(a) $|g(x, y)| \ll \|\operatorname{grad}_{(x,y)} F(x, y, t)\|$, *as* $(x, y) \to 0$.

(b) *Take any arc $\lambda : x = a_1 y^{n_1/N} + a_2 y^{n_2/N} + \cdots$, $1 \leq N \leq n_1 < n_2 < \cdots$. The straightened Newton polygon of F_t relative to λ is independent of t:*

$$\overline{\mathbb{P}}(F_t, \lambda) = \overline{\mathbb{P}}(f, \lambda), \quad (t \textit{ small }).$$

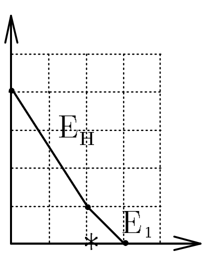
(c) *Take any polar branch, γ, of $f(x, y)$. The highest Newton edge of $\mathbb{P}(f, \gamma)$ is not disturbed by $g(x, y)$ in the sense defined below.*

Take r, $0 < r < 1$. Let $\mathcal{M}_r$ denote the contraction

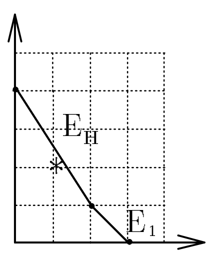
$$\mathcal{M}_r : (i, j) \mapsto (ri, rj), \quad (i, j) \in \mathbb{R}^2.$$

Take an edge, E_s, of a Newton polygon $\mathbb{P}$. We say E_s is *disturbed* by g if for some $r < 1$, $\mathcal{M}_r(\mathrm{E}_s)$ contains at least one Newton dot of g. We say $\mathbb{P}$ is *disturbed* if some edge of $\mathbb{P}$ is.

For example, in $x^3 + x^2 y + y^4 + tx^2$, $\mathrm{E}_H = \mathrm{E}_2$ is not disturbed, but E_1 is. In $x^3 + x^2 y + y^4 + txy^2$, E_H is disturbed, E_1 is not.



E_H not disturbed



E_H disturbed

A Newton dot of g can disturb at most two adjacent edges of $\mathbb{P}$.

We begin the proof of Theorem 4.1. Of course (b)$\Rightarrow$(c). By the Curve Selection Lemma, (b)$\Rightarrow$(a) is also clear. Assume (b) is not true, we shall show (a) and (c) are both false.

Take any arc λ for which $\overline{\mathbb{P}} := \overline{\mathbb{P}}(f, \lambda)$ is disturbed by $g(x, y)$. We define a number $r(\lambda)$, $0 < r(\lambda) < 1$, and a positive integer, $k(\lambda)$, as follows.

The number $r(\lambda)$ is just the smallest rational such that $\mathcal{M}_{r(\lambda)}(\overline{\mathbb{P}})$ contains at least one Newton dot of $g(X + \lambda(Y), Y)$. All Newton dots of g lie on or above $\mathcal{M}_{r(\lambda)}(\overline{\mathbb{P}})$.

Amongst these dots on $\mathcal{M}_{r(\lambda)}(\overline{\mathbb{P}})$, let $(i(\lambda), j(\lambda)/N)$ be the one for which the X-coordinate $i(\lambda)$ is minimal (i.e. closest to the Y-axis). Let $\mathrm{E}_{s(\lambda)}$ be the last edge of $\overline{\mathbb{P}}$ for which $\mathcal{M}_{r(\lambda)}(\mathrm{E}_{s(\lambda)})$ contains $(i(\lambda), j(\lambda)/N)$. Denote the initial vertex of $\mathrm{E}_{s(\lambda)}$ by $V_{s(\lambda)} = (k_{s(\lambda)}, h_{s(\lambda)})$. We then define $k(\lambda) := k_{s(\lambda)}$.

In order to simplify the notations we shall write $\mathrm{E}_s := \mathrm{E}_{s(\lambda)}$, $s := s(\lambda)$, $r := r(\lambda)$.

Among all λ which fail condition (b), let us choose one, still denoted by λ (abusing notation), such that $k(\lambda)$ is *minimal.*

Now collect terms of $f(X + \lambda(Y), Y)$:

$$\mathcal{E}_s(X, Y) := \sum a_{ij} X^i Y^{j/N}, \quad (i, j/N) \in \mathrm{E}_s,$$

and collect terms of $g(X + \lambda(Y), Y)$:

$$\widetilde{\mathcal{E}}_s(X, Y) := \sum c_{ij} X^i Y^{j/N}, \quad (i, j/N) \in \mathcal{M}_r(\mathrm{E}_s),$$

then consider the associated polynomials:

$$\mathcal{E}_s(z) := \mathcal{E}_s(z, 1), \qquad \widetilde{\mathcal{E}}_s(z) := \widetilde{\mathcal{E}}_s(z, 1).$$

Their degrees will be denoted by d and $\tilde{d}$ respectively.

We now begin to construct a polar branch, γ, which fails condition (c). Note that $d = k(\lambda)$ and $rd \geq \tilde{d}$. It follows immediately that there is a root c of $\frac{d}{dz}\mathcal{E}_s(z) = 0$, say of multiplicity m, such that $r(m + 1) \geq \tilde{m}$, where $\tilde{m} \geq 0$ is the multiplicity of c as a root of $\widetilde{\mathcal{E}}_s(z) = 0$.

Let us also write $e := e(\lambda) := \tan \theta_s$, and set

$$X_1 = X + cY^e, \quad Y_1 = Y, \quad \lambda_1(Y) = \lambda(y) + cy^e,$$

to transform $\mathbb{P}(f, \lambda)$ to $\mathbb{P}(f, \lambda_1)$ by sliding. Consider the faces $\mathrm{E}_i^{(1)}$, and vertices, $V_i^{(1)}$, of $\mathbb{P}(f, \lambda_1)$. The following hold:

(i) $\mathrm{E}_1^{(1)} = \mathrm{E}_1, \dots, \mathrm{E}_{s-1}^{(1)} = \mathrm{E}_{s-1}$; $\quad V_1^{(1)} = V_1, \dots, V_{s-1}^{(1)} = V_{s-1}, V_s^{(1)} = V_s$.

(ii) $g(X + \lambda_1(Y), Y)$ has a Newton dot of the form $(\tilde{m}, \tilde{h})$.

There are three cases to consider:

Case 1. The above number c is not a root of $\mathcal{E}_s(z) = 0$.

The vertex $V_{s+1}^{(1)}$ lies on the Y_1-axis; $\mathrm{E}_s^{(1)}$ is the highest edge; it has no Newton dot on the line $X_1 = 1$, and is disturbed by $(\tilde{m}, \tilde{h})$.

We can slide λ_1 to a polar branch, γ, as in §2. During the sliding, $\mathrm{E}_s^{(1)}$ is always the highest edge, being disturbed by $(\tilde{m}, \tilde{h})$. Finally, the highest edge of $\mathbb{P}(f, \gamma)$ is disturbed, condition (c) is false.

Case 2. The number c is a root of $\mathcal{E}_s(z) = 0$ with multiplicity $\leq d - 1$.

Of course, the multiplicity equals $m+1$. This case cannot happen, for we would find $k(\lambda_1)=m+1\le d-1<k(\lambda)$, a contradiction to the assumption that $k(\lambda)$ is minimal.

Case 3. $\mathcal{E}_s(z)=(z-c)^d$.

This case can be excluded from the offset, as follows.

First, we claim that λ can be chosen to have the additional property that $\mathbb{P}(f,\lambda)$ has no Newton dots on the line $X=d-1$.

This can be achieved by using the following idea of Bierstone-Milman [1], which generalizes the Tschirnhausen transformation. Take a Newton-Puiseux root, $\lambda^*(Y)$, of

$$\frac{\partial^{d-1}}{\partial X^{d-1}}f(X+\lambda(Y),Y)=0,\quad O_Y(\lambda^*)\ge e.$$

It then follows that $f(X+\lambda(Y)+\lambda^*(Y),Y)$ has no Newton dot on $X=d-1$. So we can replace λ by $\lambda+\lambda^*$.

With this additional property of λ, $\mathcal{E}_s(z)$ is a Tschirnhausen polynomial in the sense that it has the form

$$a_0z^d+a_2z^{d-2}+\cdots+a_d,\quad a_0\ne 0.$$

The case $a_2=\cdots=a_d=0$ can happen only if λ is already a polar branch, and E_s is vertical, disturbed by g. There is nothing more to prove.

In case not all $a_2,\ldots,a_d$ are zero, there are at least two distinct roots, Case 3 cannot happen.

We now begin to show Condition (a), too, is false. For this we need first the following algebraic lemma.

Take a pair of constants, w and $\tilde{w}$. Take a pair of polynomials, $\varphi(z)$ and $\tilde{\varphi}(z)$, say of degree d and $\tilde{d}$ respectively, written as

$$\varphi(z)=a_0(z-z_1)^{m_1}\cdots(z-z_q)^{m_q},\quad \tilde{\varphi}(z)=b_0(z-z_1)^{n_1}\cdots(z-z_q)^{n_q},$$

where $m_i\ge 0,n_i\ge 0,(m_i,n_i)\ne(0,0)$, and $z_i\ne z_j$ if $i\ne j$.

Recall that the standard symmetric function of roots are

$$s_k=\sum m_iz_i^k,\quad \tilde{s}_k=\sum n_iz_i^k,\quad k=0,1,2,\ldots.$$

Lemma 4.2. *Suppose* $\left|\begin{smallmatrix} w & m_i\\ \tilde{w} & n_i\end{smallmatrix}\right|\ne 0,\ 1\le i\le q$. *Suppose one of the following* $q-1$ *numbers*

$$\begin{vmatrix} w & s_i\\ \tilde{w} & \tilde{s}_i\end{vmatrix},\quad i=0,\ldots,q-2, \tag{4.3}$$

is not zero. Then there exists c *such that*

$$\begin{vmatrix} w\varphi(c) & \varphi'(c)\\ \tilde{w}\tilde{\varphi}(c) & \tilde{\varphi}'(c)\end{vmatrix}=0,\ \text{but } \varphi(c)\tilde{\varphi}(c)\ne 0. \tag{4.4}$$

Proof. Take a polynomial

$$\Psi(z) = \prod (z - z_i)^{e_i}.$$

Recall that $\dfrac{\Psi'(z)}{\Psi(z)} = \sum \dfrac{e_i}{z - z_i}$. Hence, we have

$$\begin{vmatrix} w\varphi(z) & \varphi'(z) \\ \tilde{w}\tilde{\varphi}(z) & \tilde{\varphi}'(z) \end{vmatrix} = \varphi(z)\tilde{\varphi}(z)R(z),$$

where

$$R(z) := \sum \left| \begin{smallmatrix} w & m_i \\ \tilde{w} & n_i \end{smallmatrix} \right| (z - z_i)^{-1}.$$

In particular, $R(z)$ is of the form

$$R(z) = \frac{q(z)}{(z - z_1)\cdots(z - z_q)}, \quad q(z_i) \neq 0.$$

To show the lemma it suffices to show that $\deg q(z) \geq 1$. The expansions

$$\frac{1}{z - z_i} = \frac{1}{z} + \frac{z_i}{z^2} + \cdots + \frac{z_i^{n-1}}{z^n} + \cdots, \ 1 \leq i \leq q,$$

lead to

$$R(z) = \sum \begin{vmatrix} w & s_k \\ \tilde{w} & \tilde{s}_k \end{vmatrix} z^{-(k+1)}.$$

By (4.3), there is an integer p, $1 \leq p \leq q - 1$, such that

$$\lim_{z \to \infty} z^p R(z) = \begin{vmatrix} w & s_{p-1} \\ \tilde{w} & \tilde{s}_{p-1} \end{vmatrix} \neq 0.$$

This implies that the degree of $q(z)$ is $q - p \geq 1$. This shows the lemma. □

Let us return to λ, which minimizes $k(\lambda)$. Let us take

$$\varphi(z) = \mathcal{E}_s(z), \quad \tilde{\varphi}(z) = \tilde{\mathcal{E}}_s(z).$$

Let w be the rational number such that $\mathcal{E}_s(cY^e, Y) \sim Y^w$, c a generic number, and define $\tilde{w}$ similarly using $\widetilde{\mathcal{E}}_s(cY^e, Y)$ instead.

Assertion 4.3. *For all i,* $\left| \begin{smallmatrix} w & m_i \\ \tilde{w} & n_i \end{smallmatrix} \right| \neq 0$, $1 \leq i \leq q$.

Take any i, and consider the root z_i. Let us slide λ to λ_1:

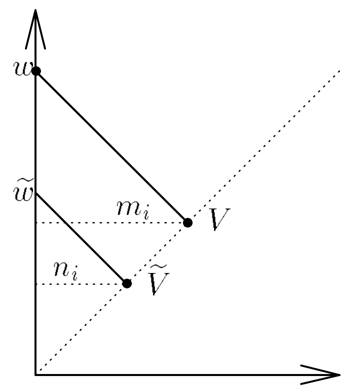
$$\lambda_1(y) = \lambda(y) + z_i y^e, \quad X_1 = X + z_i Y^e, \quad Y_1 = Y.$$

This substitution turns E_s into an edge $\mathrm{E}_s^{(1)}$ of $\mathbb{P}(f, \lambda_1)$, having the same initial vertex as E_s, but its terminal vertex, denoted by V (for simplicity of notation), has X_1-coordinate m_i. The disturbing terms $\widetilde{\mathrm{E}}_s$ are likewise transformed, having its leftest Newton dot, denoted by $\tilde{V}$, on the line $X_i = n_i$.

Suppose $\left|\begin{smallmatrix} w & m_i \\ \tilde{w} & n_i \end{smallmatrix}\right| = 0$. Then by a simple Plane Geometry argument, we would have

$$\tilde{V} = \mathcal{M}_{r(\lambda)}(V) \in \mathcal{M}_{r(\lambda)}(\mathbb{P}(f, \lambda_1)).$$

It would then follow that $k(\lambda_1) < k(\lambda)$, a contradiction.



Now, take an indeterminate, Γ, and let $\lambda_\Gamma(y) := \lambda(y) + \Gamma y^e$. Then consider the Taylor expansions

$$f(X + \lambda_\Gamma(Y), Y) = Y^w \Phi_0(\Gamma, Y) + Y^{w-e}\Phi_1(\Gamma, Y)X + \cdots,$$

$$g(X + \lambda_\Gamma(Y), Y) = Y^{\tilde{w}} \tilde{\Phi}_0(\Gamma, Y) + Y^{\tilde{w}-e}\tilde{\Phi}_1(\Gamma, Y)X + \cdots.$$

Note that

$$\Phi_0(\Gamma, 0) = \varphi(\Gamma), \;\; \Phi_1(\Gamma, 0) = \varphi'(\Gamma);$$

$$\tilde{\Phi}_0(\Gamma, 0) = \tilde{\varphi}(\Gamma), \;\; \tilde{\Phi}_1(\Gamma, 0) = \tilde{\varphi}'(\Gamma).$$

We are now in a position to apply Lemma 4.2.

Case 1. Suppose $\left|\begin{smallmatrix} w & d \\ \tilde{w} & \tilde{d} \end{smallmatrix}\right| \neq 0$.

The first number in (4.3) is not zero, since $s_0 = d$, $\tilde{s}_0 = \tilde{d}$. Let us take a constant c satisfying (4.4).

We shall construct a *relative polar curve* $\mathcal{P}$, that is a curve along which

$$\partial F/\partial x = \partial F/\partial y = 0,$$

such that g is not identically zero on $\mathcal{P}$.

Let us write, as shorthands,

$$A(\Gamma, Y) := w\Phi_0 + Y\partial\Phi_0/\partial Y, \quad \tilde{A}(\Gamma, Y) := \tilde{w}\tilde{\Phi}_0 + Y\partial\tilde{\Phi}_0/\partial Y.$$

To construct $\mathcal{P}$ consider the following system of equations

$$\begin{cases} t\tilde{A}(\Gamma, Y) + Y^{w-\tilde{w}}A(\Gamma, Y) = 0 \\ t\tilde{\Phi}_1(\Gamma, Y) + Y^{w-\tilde{w}}\Phi_1(\Gamma, Y) = 0. \end{cases} \tag{4.5}$$

The coefficient determinant of (4.5)

$$\Delta(\Gamma, Y) := \begin{vmatrix} A(\Gamma, Y) & \Phi_1(\Gamma, Y) \\ \tilde{A}(\Gamma, Y) & \tilde{\Phi}_1(\Gamma, Y) \end{vmatrix}$$

is zero when $\Gamma = c$, $Y = 0$. Hence we can use the Newton-Puiseux Theorem to find a root $\Gamma(Y)$ of Δ:

$$\Delta(\Gamma(Y), Y) = 0, \quad \Gamma(0) = c.$$

Since $\tilde{A}(\Gamma(0), 0) = \tilde{\varphi}(c) \neq 0$, $t = t(Y) := Y^{w-\tilde{w}} A(\Gamma(Y), Y)/\tilde{A}(\Gamma(Y), Y)$ is a solution of (4.5), for $\Gamma = \Gamma(Y)$.

Consequently

$$\mathcal{P} : Y \longrightarrow (\Gamma(Y), Y, t(Y))$$

is a relative polar curve, $\partial F/\partial x = \partial F/\partial y = 0$ along $\mathcal{P}$, yet $|g(x,y)| \sim y^{\tilde{w}}$. This shows that Condition (a) is false in Case 1.

Case 2. Suppose $\left|\begin{smallmatrix} w & d \\ \tilde{w} & \tilde{d} \end{smallmatrix}\right| = 0$.

This case is actually easy. In the first place, we have $w - \tilde{w} \geq e$. This is proved by a simple argument of Plane Geometry. (Indeed, equality holds only in a trivial case.)

Take a generic constant Γ. Consider the curve $\lambda_\Gamma(y) := \lambda(y) + \Gamma y^e$. Along this curve

$$\partial f/\partial X \sim Y^{w-e}, \quad \partial f(X + \lambda_\Gamma(Y), Y)/\partial Y \sim Y^{w-1}, \quad g(X,Y) \sim Y^{\tilde{w}}.$$

This shows (a) is again false, thus completing the proof.

References

[1] E. Bierstone and P.D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. 128 (1997), 207-302.

[2] M. Bogusławska, *On the Lojasiewicz exponent of the gradient of holomorphic functions*, preprint University of Łódź.

[3] H.V. Hà, *Nombres de Lojasiewicz et singularité à l'infini des polynômes de deux variables complexes*, C.R. Acad. Sci. Paris Sér. I Math. 311 (1990), 429–432.

[4] H.V. Hà, *On the Irregularity at Infinity of Algebraic Plane Curves*, University of Hanoi preprint 91/4, 1991.

[5] T.-C. Kuo and Y.C. Lu, *On analytic function germs of two complex variables*, Topology, 16 (1977), 299-310.

[6] R. Walker, *Algebraic Curves*, Princeton University Press (1950).

School of Mathematics and Statistics, University of Sydney, Sydney, NSW, 2006, Australia

E-mail address: tck@maths.usyd.edu.au

Département de Mathématiques, Université d'Angers, 2, bd Lavoisier, 49045 Angers cedex, France

E-mail address: parus@tonton.univ-angers.fr