RATIONAL ROTATION NUMBERS FOR HOMEOMORPHISMS WITH TWO BREAK-TYPE SINGULARITIES*

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1. Introduction

This work deals with the family of homeomorphisms of the circle $T_a : S^1 \to S^1$, where $S^1 = [0,1)$, $a \in [0,1]$. Each T_a is defined by its *lift* $f_a : \mathbb{R} \to \mathbb{R}$ such that $\{f_a(x)\} = Tx$. We consider simpliest family $f_a = f + a$, where $0 \le f(0) < 1$.

Suppose that $T = \{f\}$ has two break-type singularities at points $x_{\rm cr}^{(1)} \ne x_{\rm cr}^{(2)}$, i.e., the following conditions hold:

1) $T \in C^{2+\varepsilon}(S^1 \setminus \{x_{\rm cr}^{(1)}, x_{\rm cr}^{(2)}\})$,

2) There exists $T'(x_{\rm cr}^{(i)}-)$ and $T'(x_{\rm cr}^{(i)}+)$, $i = 1,2$, and

$$c_i = \sqrt{\frac{T'(x_{\rm cr}^{(i)}-)}{T'(x_{\rm cr}^{(i)}+)}} \ne 1.$$

Denote by $\rho(T_a) = \rho(f_a) = \lim_{n\to\infty} f_a^n(x_0)/n$ the rotation number of the homeomorphism $\rho(T_a)$.

Theorem 1. *Suppose $c_1c_2 \ne 1$ and $c_1 \ne c_2$. Then the Lesbesgue measure of the set $\{a \mid \rho(T_a) \in [0,1] \setminus \mathbb{Q}\}$ is zero.*

This theorem for one break-type singualrity was proven in [1].

2. Notations and apriori estimates

Let us denote an interval with endpoints α and β as follows

$$I(\alpha,\beta) = \begin{cases} (\alpha,\beta), & \text{if } \alpha \le \beta, \\ (\beta,\alpha), & \text{if } \alpha > \beta. \end{cases}$$

We also put $I[\alpha,\beta] = \overline{I(\alpha,\beta)}$. A good description for Farey numbers is given in [1]. We shall need only the following properties of Farey interval $A = (p_1/q_1, p_2/q_2) \subset (0,1)$:

(1) $p_2q_1 - p_1q_2 = 1$.

(2) All rationals inside A have the form $(kp_1 + lp_2)/(kq_1 + lq_2)$. The rational number with smallest denominator is $(p_1+p_2)/(q_1+q_2)$.

*этот текст получен из файла 2.TEX

Consider homeomorphism T with two break-type singularities and corresponding lifting f. Choose arbitrary point x_0 on the circle and consider the trajectory $\{x_i = T^i x_0, 0 \le i \le q_1 + q_2\}$. Denote by $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$ the closed intervals $[x_0, x_{q_1}]$ and $[x_{q_2}, x_0]$, respectively. Denote by $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ the images of these intervals under the action of T:

$$\Delta_i^{(1)} = T^i \Delta_0^{(1)},\ \Delta_i^{(2)} = T^j \Delta_0^{(2)},$$

The following Preposition was proved in [1] and the proof works in our situation without any change.

Preposition 1. *Suppose $\rho(T) \in (p_1/q_1, p_2/q_2)$. The trajectory $\{x_i = T^i x_0, 0 \le i \le q_1 + q_2\}$ forms a partition of the circle consisting of intervals $\Delta_i^{(1)}$, $0 \le i \le q_2$ and $\Delta_j^{(2)}$, $0 \le j \le q_1$.*

Denote this partition by $\xi(A, x_0)$. Denote

$$v = |\ln c_1^2| + |\ln c_2^2| + \operatorname{Var}_{S^1} \ln f'.$$

Consider now trajectory $y_i = T^i y_0$ of point y_0, such that $y_i \ne x_{\mathrm{cr}}^{(1)}$ and $y_i \ne x_{\mathrm{cr}}^{(2)}$ for any $0 \le i \le q_2$. Denote $q = \max(q_1, q_2)$ and $p = \max(p_1, p_2)$.

Preposition 2. *Suppose $\rho(T) \in I((p_1 + p_2)/(q_1 + q_2), p/q)$ or $\rho(T) = p/q$. Then*

$$e^{-v} \le \prod_{i=0}^{q-1} f'(y_i) \le e^{v}. \tag{1}$$

Proof. We follow the lines of proof given in [1] with corrections due to the second singularity point. To be definite, assume that $q_2 > q_1$. Then $q = q_2$, $p = p_2$. Let us consider the case $\rho(T) \ne p_2/q_2$ (the same arguments apply to case $\rho(T) = p_2/q_2$ because there exists a periodic trajectory $x_0, \ldots, x_{q_2-1}$, such that $x_0, \ldots, x_{q_2-1}$ form a proper partition of the circle). Consider partition $\xi((p_1 + p_2)/(q_1 + q_2), p_2/q_2; x_0)$, where the trajectory $x_i = T^i x_0$ of x_0 do not touch critical points. Notice that $x_{q_1+q_2} \in \Delta_0^{(1)}$, since $\rho(T) \in ((p_1 + p_2)/(q_1 + q_2), p_2/q_2)$. Let us show that

$$\left| \sum_{i=0}^{q_2-1} \ln f'(y_i) - \sum_{i=0}^{q_2-1} \ln f'(x_i) \right| \le \operatorname{Var}_{S_1} \ln f'. \tag{2}$$

Indeed, assume that $y_0 \in \Delta_j^{(2)}$, $0 \le j \le q_1$. Let x_{k+j} correspond to y_k for $0 \le k < q_1 - j$ and let x_{k+j-q_1} correspond to y_k for $q_1 - j \le k < q_2$. We have $[x_{k+j}, y_k] \subset \Delta_{k+j}^{(2)}$, $0 \le k < q_1 - j$ for $0 \le k < q_1 - j$ and $[y_k, x_{k+j-q_1}] \subset \Delta_{k+j-q_1}^{(1)}$ for $q_1 - j \le k < q_2$, i.e., the intervals do not intersect each other. Hence, we get estimate (2). The same arguments apply to the case $y_0 \in \Delta_i^{(1)}$, $0 \le i \le q_2$.

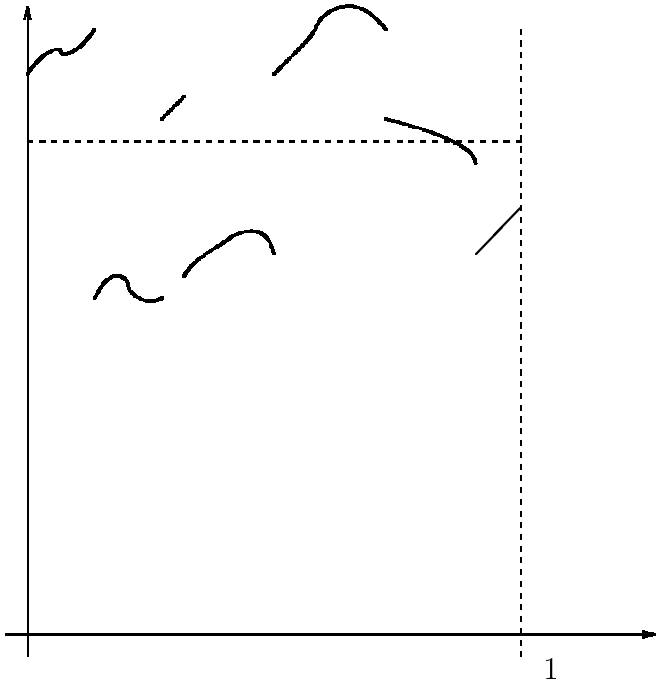

FIGURE 1

Notice now that

$$\int\limits_{S_1=[0,1)} \prod_{i=0}^{q_2-1} f'(y_i)dy_0 = \int\limits_0^1 \frac{\partial}{\partial y_0} f^{q_2}(y_0)dy_0 = f^{q_2}(1) - f^{q_2}(0) = 1.$$

Notice that $g = \partial f^{q_2}/\partial y_0$ is not equal 1. It follows that g takes values above and below 1. Without loss of generality suppose that $x_{\rm cr}^{(1)} = 0$ and $x_{\rm cr}^{(2)} \in (0,1)$. If g is arbitrary close to 1 in neighbourhood of some point y_0, then we are done (this case is shown on Figure 1).

Hence, assume that g never approaches 1, as shown on Figure 2. Notice that g is piecewise-continuous (and even of class $C^{1+\varepsilon}$ on each interval of continuity). All disconinuity points are formed by preimages $T^{-i}x_{\rm cr}^{(1)}$ and $T^{-j}x_{\rm cr}^{(2)}$ of singularities $x_{\rm cr}^{(1)}$ and $x_{\rm cr}^{(2)}$ for $0 \le i, j \le q_2$. For some i, j we can have $T^{-i}x_{\rm cr}^{(1)} = T^{-j}x_{\rm cr}^{(2)}$ and this imply that corresponding singularity of g has break c_1c_2. Other singularities have break equal c_1 or c_2. Notice that no other breaks can happen. Indeed, if for some $0 \le i_1$, j_1, i_2, $j_2 \le q_2$ we have

$$T^{-i_1}x_{\rm cr}^{(1)} = T^{-j_1}x_{\rm cr}^{(2)} \text{ and } T^{-i_2}x_{\rm cr}^{(1)} = T^{-j_2}x_{\rm cr}^{(2)},$$

then point $y^* = T^{-j_2}x_{\rm cr}^{(2)}$ is periodic of period $q^* = |(j_1 - j_2) + (i_1 - i_2)|$, since

$$T^{-j_2} \circ T^{j_1} \circ T^{-i_1} \circ T^{i_2}y^* = y^*$$

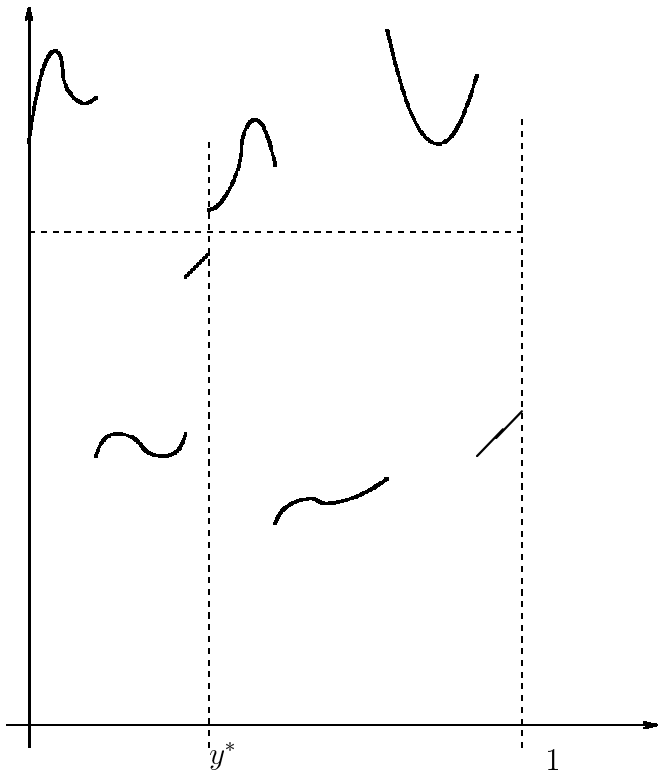

FIGURE 2

Hence $\rho(T) = p^*/q^*$ for some p^*. But $q^* = |(j_1 - j_2) + (i_1 - i_2)| \le 2q_2$, and this contradicts property (2) of Farey intervals, since

$$\frac{p^*}{q^*} \in (\frac{p_1 + p_2}{q_1 + q_2}, \frac{p_2}{q_2}) = B$$

and the rational with smallest possible denominator in B is $(p_1 + 2p_2)/(q_1 + 2q_2)$, but

$$q_1 + 2q_2 \ge 1 + 2q_2 > 2q_2 \ge q^*.$$

Using the same arguments we arrive at contradiction if $T^{-i_1} x_{\rm cr}^{(1)} = T^{-i_2} x_{\rm cr}^{(1)}$ or $T^{-i_1} x_{\rm cr}^{(1)} = T^{-i_2} x_{\rm cr}^{(1)}$ for some $0 \le i_1,\, i_2 \le q_2$.

The case $\rho(T) = p_2/q_2$ is even easier. If for some i, j we have $T^{-i} x_{\rm cr}^{(1)} = T^{-j} x_{\rm cr}^{(2)}$, then $x_{\rm cr}^{(1)} = T^{i-j} x_{\rm cr}^{(2)}$ and all q_2 singularities have break $c_1 c_2$. If $T^{-i_1} x_{\rm cr}^{(j)} = T^{-i_2} x_{\rm cr}^{(j)}$ for some $i_1 < i_2$, $i_2 - i_1 < q_2$, $j = 1, 2$, then we have $\rho(T) = p^*/q^*$, where $q^* = i_2 - i_1$. But $q^* < q_2$ and we arrive at contradiction with $\rho(T) = p_2/q_2$,

Hence the following inequality holds for every singular point y^*:

$$\left| \ln \sqrt{\frac{g(y_* -)}{g(y_* +)}} \right| \le |\ln c_1| + |\ln c_2| = C_{1,2}. \tag{3}$$

Notice now that since g take values above and below 1, one can choose a singularity point y^* such that one of the following conditions holds.

1) $g(y^*-) < 1$, $g(y^*+) > 1$,

2) $g(y^*-) > 1$, $g(y^*+) < 1$.

Consider case 1). Then, by (3)

$$\frac{g(y^*-)}{g(y^*+)} \geq \frac{1}{e^{2C_{1,2}}}. \tag{4}$$

Since $g(y^*+) > 1$,

$$g(y^*-) > \frac{1}{e^{2C_{1,2}}},$$

and since $g(y^*-) < 1$,

$$e^{2C_{1,2}} > g(y^*+).$$

The same argument works for case 2). Hence,

$$g(y^*-), g(y^*+) \in (e^{-2C_{1,2}}, e^{2C_{1,2}}).$$

Hence given $\varepsilon > 0$ we can choose x_0 such that

$$|\ln g(x_0)| < 2C_{1,2} + \varepsilon = |\ln c_1^2| + |\ln c_2^2| + \varepsilon. \tag{5}$$

Estimates (2) and (5) imply

$$\left|\sum_{i=0}^{q_2-1} \ln f'(y_i)\right| \leq \mathrm{Var}_{S_1} \ln f' + |\ln c_1^2| + |\ln c_2^2| + \varepsilon. \tag{6}$$

Since ε is arbitrary, inequality (6) holds for $\varepsilon = 0$ and taking exponents of both sides of (6) we obtain (1). □

Now denote

$$q = \max(q_1, q_2); \qquad q' = \min(q_1, q_2)$$
$$p = \max(p_1, p_2); \qquad p' = \min(p_1, p_2).$$

Preposition 2 gives a-priori estimate for derivative of f^q for $\rho(T) \in I((p_1 + p_2)/(q_1 + q_2), p/q)$ and $\rho(T) = p/q$. The following prepositions provide esitmates for derivatives of $T^{q'}$ and T^q on the whole interval $[p_1/q_1, p_2/q_2]$. Both prepositions are simple corollaries of Preposition 2. They are analogous to Prepositions 3.3 and 3.4 in [1] and are proven in the same way.

Preposition 3. *Suppose one of the following conditions hold:*

1) $\rho(T) \in (p_1/q_1, p_2/q_2)$;

2) $\rho(T) = p'/q'$;

3) $\rho(T) = p/q$, $q \geq 2q'$.

Then

$$e^{-v} \leq \prod_{i=0}^{q'-1} f'(y_i) \leq e^{v}.$$

If the following condition 3') holds

3') $\rho(T) = p/q$ and $q < 2q'$,
then

$$e^{-2v} \le \prod_{i=0}^{q'-1} f'(y_i) \le e^{2v}. \tag{7}$$

Proof. Take $0 < q_3 = q - lq' < q'$, $l \in \mathbb{N}$, and, respectevely, $p_3 = p - lp'$. Applying of Preposition 1 to Farey interval $I(p_3/q_3, p'/q')$ we obtain that the following estimate

$$e^{-v} \le \prod_{i=0}^{q'-1} f'(y_i) \le e^{v} \tag{8}$$

holds if $\rho(T) \in ((p_3 + p')/(q_3 + q'), p'/q')$ or $\rho(T) = p'/q'$. Hence, we immidiately obtain our statement in case 2). Case 1) is also obtained from (8), since

$$(\frac{p_1}{q_1}, \frac{p_2}{q_2}) = I(\frac{p'}{q'}, \frac{p}{q}) \subset I(\frac{p'}{q'}, \frac{p_3 + p'}{q_3 + q'}).$$

Consider now case 3). Notice that if $q = 2q'$, then by property (1) of Farey intervals we have

$$1 = |pq' - p'q| = |pq' - 2p'q'| = |p - 2p'|q',$$

and since q' is integer, we have $q' = 1$. Hence, our estimate is trivial. Therefore, assume that $q > 2q'$. Then $q_3 = q - lq'$ with $l \ge 2$ and hence

$$\frac{p}{q} \in I(\frac{p'}{q'}, \frac{p_3 + p'}{q_3 + q'}).$$

It stays to conisder case 3'). We have $q < 2q'$. Apply our statement to Farey interval $I((p - p')/(q - q'), p'/q')$. Since $q - q' < q'$ and $\rho(T) = p/q \in I((p - p')/(q - q'), p'/q')$, we fall in case 1) considered before. Hence, the following estimate hold:

$$e^{-v} \le \prod_{i=0}^{q-q'-1} f'(z_i) \le e^{v}.$$

Applying Preposition 1 to Farey interval $I(p'/q', p/q)$, and remembering that $\rho(T) = p/q$, we obtain

$$e^{-v} \le \prod_{i=0}^{q-1} f'(z_i) \le e^{v}.$$

Hence,

$$e^{-2v} \le \prod_{i=0}^{q-1} f'(z_i) \Big/ \prod_{i=0}^{q-q'-1} f'(z_i) \le e^{2v}$$

or

$$e^{-2v} \le \prod_{i=q-q'}^{q-1} f'(z_i) \le e^{2v}.$$

Choosing $z_{q-q'} = y_0$ we obtain (7). □

Preposition 4. *Choose $q_3 = q - lq'$ such that $0 < q_3 < q'$, $l \in \mathbb{N}$.*

1) If $\rho(T) \in I(p'/q', (p_1+p_2)/(q_1+q_2))$, then

$$e^{-(l+1)v} \le \prod_{i=0}^{q-1} f'(y_i) \le e^{(l+1)v}. \tag{9}$$

2) If $\rho(T) = p'/q'$, then

$$e^{-(l+2)v} \le \prod_{i=0}^{q-1} f'(y_i) \le e^{(l+2)v}. \tag{10}$$

3) Denote

$$Y_k = I(\frac{(k+1)p'+p}{(k+1)q'+q}, \frac{kp'+p}{kq'+q}).$$

If $\rho(T) \in Y_k$ or $\rho(T) = (kp'+p)/(kq'+q)$ for $k \ge 1$, then

$$e^{-(k+1)v} \le \prod_{i=0}^{q-1} f'(y_i) \le e^{(k+1)v} \tag{11}$$

Proof. Consider Farey interval $I(p_3/q_3, p'/q')$. If $\rho(T) \in I(p_3/q_3, p'/q')$, then, by case 1) of Preposition 3 we have estimate

$$e^{-v} \le \prod_{i=0}^{q_3-1} f'(z_i) \le e^{v}. \tag{12}$$

Notice also, that $q'+q \ge 2q'$. Hence Farey interval $I(p'/q', (p'+p)/(q'+q))$ and rotation number $\rho(T) \in I[p'/q', (p'+p)/(q'+q)]$ satisfy conditions 1)–3) of Preposition 3. Therefore,

$$e^{-v} \le \prod_{i=0}^{q'-1} f'(z_i) \le e^{v}. \tag{13}$$

Since $q = lq' + q_3$, combining (12) and (13) we obtain (9).

If $\rho(T) = p'/q'$, then for Farey interval $I(p_3/q_3, p'/q')$ conditions of cases 3) or 3') of Preposition 3 holds. In worst case 3') we have estimate

$$e^{-v} \le \prod_{i=0}^{q_3-1} f'(z_i) \le e^{v}.$$

Combining the last inequality with (13), we get (10).

Now consider case 3), i.e., $\rho(T) \in Y_k$ or $\rho(Tf) = (kp'+p)/(kq'+q)$. Apply Preposition 2 to Farey interval $I(p'/q', (kp'+p)/(kq'+q))$. It follows that

$$e^{-v} \le \prod_{i=0}^{kq'+q-1} f'(z_i) \le e^{v} \text{ for } \rho(T) \in Y_k \text{ or } \rho(Tf) = \frac{kp'+p}{kq'+q}. \tag{14}$$

Application of Preposition 3 to Farey interval $I(p'/q,(kp'+p)/(kq'+q))$ yields

$$e^{-v} \le \prod_{i=0}^{q'-1} f'(z_i) \le e^v \text{ for } \rho(T) \in Y_k \text{ or } \rho(Tf) = \frac{kp'+p}{kq'+q}, \tag{15}$$

since $kq'+q \ge q'+q \ge 2q'$ and estimates for cases 1)–3) of Preposition 3 hold. Combining (14), (15) and $q = lq' + q_3$ we get (11). □

As one can see, if $q < lq'$, then we have effective estimates for derivatives of T^q and $T^{q'}$. In fact, this is one of reasons for introducing so-called "good" Farey intervals used by G. Świątek [2] and K. Khanin [1] in similiar setting. A Farey interval $A = (p_1/q_1, p_2/q_2)$ is called "good" if

$$q = \max(q_1, q_2) < 2q' = \max(q_1, q_2).$$

It follows from statement 3) of Preposition 4 that if rotation number is not too close to the ends of Farey interval $(p_1/q_1, p_2/q_2)$, then we also have effective estimates for derivatives. In particular, the following corollary holds.

Corollary 1. *Consider Farey interval* $(p_1/q_1, p_2/q_2)$. *Suppose* $\rho(T) = (p_1 + p_2)/(q_1 + q_2)$. *Then*

$$e^{-v} \le \prod_{i=0}^{q_1+q_2-1} f'(y_i) \le e^v, \tag{16}$$

$$e^{-2v} \le \prod_{i=0}^{q_1-1} f'(y_i), \prod_{i=0}^{q_2-1} f'(y_i) \le e^{2v} \tag{17}$$

for any trajectory $y_i = T^i y_0$, $i = 0, \ldots, q_1 + q_2 - 1$ *such that* $y_i \ne x_{\mathrm{cr}}^{(j)}$, $j = 1, 2$.

Corollary 2. *Consider Farey interval* $(p_1/q_1, p_2/q_2)$. *Suppose*

$$\rho(T) \in [\frac{p_1+p_2}{q_1+q_2}, \frac{p_1+2q_2}{q_1+2q_2}].$$

Then

$$e^{-3v} \le \prod_{i=0}^{q_1-1} f'(y_i), \prod_{i=0}^{q_2-1} f'(y_i) \le e^{3v} \tag{18}$$

Proof. Let us prove an estimate for $\partial f^{q_1}/\partial x$. If $q_1 \le q_2$, then Preposition 7, gives an estiamate

$$e^{-2v} \le \prod_{i=0}^{q_1-1} f'(y_i) \le e^{2v}.$$

If $q_1 > q_2$, then $q' = \min(q_1, q_2) = q_2$, $q = \max(q_1, q_2) = q_1$ and we can apply Preposition 4 for

$$Y_1 = I(\frac{2p'+p}{2q'+q}, \frac{p'+p}{q'+q}) = (\frac{p_1+p_2}{q_1+q_2}, \frac{p_1+2p_2}{q_1+2q_2}),$$

and we obtain that

$$e^{-2v} \le \prod_{i=0}^{q_1-1} f'(y_i) \le e^{2v} \text{ for all } \rho(T) \in Y_1.$$

Application of case 3) of preposition 4 for $\rho(T) = (p_1 + 2p_2)(q_1 + 2q_2)$ yields estimate

$$e^{-3v} \le \prod_{i=0}^{q_1-1} f'(y_i) \le e^{3v} \text{ for all } \rho(T)\bar{Y}_k. \quad \square$$

Notice that estimate (2) imply the following important Lemma about decay of lengths of intervals $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$, that could be proven like Preposition 3.5 in [1].

Lemma 1. *Take $\lambda = (1 + e^{-v})^{-1/2} < 1$. Suppose that $\rho(T) \in [p_1/q_1, p_2/q_2]$ and the expansion of p_1/q_1 to the continuous fraction has length n:*

$$\frac{p_1}{q_1} = [k_1, \dots, k_n],\ k_n \ge 2.$$

Then

$$|\Delta_i^{(1)}|, |\Delta_j^{(2)}| \le \text{const}\, \lambda^n$$

for all $0 \le i \le q_2$, $0 \le j \le q_1$.

3. Periodic rotation numbers

Here and further we shall denote by const any *universal* constant that depends on homeomorphism f only and do not depend on a. In this section we consider f_a for some a such that $\rho(f_a) = p/q = [k_1, \dots, k_n]$, $k_n \ge 2$. For sake of shortness we shall omit index a till the end of the section.

It is well-known that for $f = f_a$ there exists a periodic trajectory of period q. Denote by $\Delta_0 = [y_1, y_2]$ an interval of this trajectory, contaning point $x_{\text{cr}}^{(1)}$. Denote $\Delta_i = f^i\Delta_0$. For some r we have $\Delta_r \ni x_{\text{cr}}^{(2)}$. Let us introduce relative coordinates of $x_{\text{cr}}^{(1)}$, $x_{\text{cr}}^{(2)}$ and $f^{-r}x_{\text{cr}}^{(2)}$ on intervals Δ_0, Δ_r and Δ_0, resp. Clearly,

$$d_1 = \frac{x_{\text{cr}}^{(1)} - y_1}{y_2 - y_1} = \frac{x_{\text{cr}}^{(1)} - y_1}{|\Delta_0|},$$

$$d_2 = \frac{x_{\text{cr}}^{(2)} - f^r y_1}{f^r y_2 - f^r y_1} = \frac{x_{\text{cr}}^{(2)} - f^r y_1}{|\Delta_r|},$$

$$\tilde{d}_2 = \frac{f^{-r}x_{\text{cr}}^{(2)} - y_1}{y_2 - y_1} = \frac{f^{-r}x_{\text{cr}}^{(2)} - y_1}{|\Delta_0|}.$$

Let us introduce

$$\bar{f}(w) = \frac{1}{y_2 - y_1}(f^q(y_1 + w(y_2 - y_1)) - y_1),$$

where $w \in [0, 1]$. Clearly, $\bar{f}(w)$ is a rescaled return map $f^q : \Delta_0 \to \Delta_0$.

In order to study $\bar{f}(w)$ we need the following notations:

$$f^l_{c,d}(w) = \frac{c^2 w}{1 + d(c^2 - 1)},$$
$$f^r_{c,d}(w) = \frac{w + d(c^2 - 1)}{1 + d(c^2 - 1)},$$
$$F_M(w) = \frac{w}{M(1 - w) + w}.$$

One can see that function

$$f^{lr}_{c,d}(w) = \begin{cases} f^l_{c,d}(w), & w \in [0, d], \\ f^r_{c,d}(w), & w \in [d, 1], \end{cases}$$

is a continuous piecewise-linear map $[0, 1] \to [0, 1]$ such that

$$f^{lr}_{c,d}(0) = 0,\ f^{lr}_{c,d}(1) = 1,\ \frac{\partial}{\partial w} f^{lr}_{c,d}(w)\bigg|_{w=d-} \Big/ \frac{\partial}{\partial w} f^{lr}_{c,d}(w)\bigg|_{w=d+} = c^2.$$

This map will be used to approximate mapping f on intervals Δ_0 and Δ_r, since by Lemma 1 they are exponentially small.

As was shown in Preposition 4.3 in [1] the mapping $F_M(w)$ is a very good approximation for mapping $f^{r_2 - r_1} : \Delta_{r_1} \to \Delta_{r_2}$ if intervals Δ_i do not cover critical points $x^{(1)}_{\mathrm{cr}}$ and $x^{(2)}_{\mathrm{cr}}$ for $i = r_1, \ldots, r_2 - 1$. Denote

$$M(r_1, r_2) = \exp\left(\sum_{i=r_1}^{r_2 - 1} \int_{y \in \Delta_i} \frac{f''(y)}{2f'(y)} dy\right).$$

Also, define

$$f^{r_2, r_1}(w) = \frac{1}{|\Delta_{r_2}|}(f^{r_2 - r_1}((1 - w)f^{r_1} y_1 + w f^{r_1} y_2) - f^{r_2} y_1)$$

Lemma 2. *Suppose intervals Δ_i do not cover critical points $x^{(1)}_{\mathrm{cr}}$ and $x^{(2)}_{\mathrm{cr}}$ for $i = r_1, \ldots, r_2 - 1$. Then*

$$\|F_{M(r_1, r_2)} - f^{r_2, r_1}\|_{C^2([0,1])} \le \mathrm{const}\, \lambda^n,$$

where λ is a universal constant, given in Lemma 1.

This lemma is proved as Preposition 4.3 in [1]. The only prerequisite required in proof of Preposition 4.3 [1] is an apriori estimates for geometrical decay of size of intervals. In our setting, this estimate is provided with Lemma 1. Hence we omit the proof for sake of saving space.

Let us define

$$M_1 = M(1,r) = \exp\left(\sum_{i=1}^{r-1}\int_{y\in\Delta_i}\frac{f''(y)}{2f'(y)}dy\right),$$
$$M_2 = M(r+1,q) = \exp\left(\sum_{i=r+1}^{q-1}\int_{y\in\Delta_i}\frac{f''(y)}{2f'(y)}dy\right).$$

Functions $f^l_{c,d}$, $f^r_{c,d}$ and F_M have the following useful properties:

$$F_N \circ F_M = F_{MN}, \tag{19}$$
$$f^l_{c,d}(d) = f^r_{c,d}(d) = F_{1/c^2}(d) \tag{20}$$

for all M, N, c, d. Indeed,

$$F_M(w) = \frac{w}{M(1-w)+w},\ 1-F_M(w) = \frac{M(1-w)}{M(1-w)+w} \tag{21}$$

and

$$F_N\circ F_M(w) = \frac{\dfrac{w}{M(1-w)+w}}{N\dfrac{M(1-w)}{M(1-w)+w}+\dfrac{w}{M(1-w)+w}} = \frac{w}{MN(1-w)+w} = F_{MN}(w),$$

as required for (19). To see why (20) holds, one needs very simple algebra:

$$f^l_{c,d}(d) = f^r_{c,d}(d) = \frac{c^2d}{1+d(c^2-1)} = \frac{c^2d}{(1-d)+c^2d} = \frac{d}{\dfrac{1}{c^2}(1-d)+d} = F_{1/c^2}(d).$$

Now let us introduce function

$$G_{c_1,d_1,c_2,\tilde{d}_2,M_1,d_2}(w) : [0,1]\to[0,1]$$

by the following rules. If $0\le d_1\le\tilde{d}_2$, then

$$G_{c_1,d_1,c_2,\tilde{d}_2,M_1,d_2}(w) = \begin{cases} F_{\frac{c_1c_2}{M_1}}\circ f^l_{c_2,d_2}\circ F_{M_1}\circ f^l_{c_1,d_1}(w) & \text{for } 0\le w<d_1,\\ F_{\frac{c_1c_2}{M_1}}\circ f^l_{c_2,d_2}\circ F_{M_1}\circ f^r_{c_1,d_1}(w) & \text{for } d_1\le w<\tilde{d}_2,\\ F_{\frac{c_1c_2}{M_1}}\circ f^r_{c_2,d_2}\circ F_{M_1}\circ f^r_{c_1,d_1}(w) & \text{for } \tilde{d}_2\le w\le 1.\end{cases}$$

If $\tilde d_2 < d_1 \le 1$, we have

$$G_{c_1,d_1,c_2,\tilde d_2,M_1,d_2}(w) = \begin{cases} F_{\frac{c_1c_2}{M_1}} \circ f^l_{c_2,d_2} \circ F_{M_1} \circ f^l_{c_1,d_1}(w) & \text{for } 0 \le w < \tilde d_2, \\ F_{\frac{c_1c_2}{M_1}} \circ f^r_{c_2,d_2} \circ F_{M_1} \circ f^l_{c_1,d_1}(w) & \text{for } \tilde d_2 \le w < d_1, \\ F_{\frac{c_1c_2}{M_1}} \circ f^r_{c_2,d_2} \circ F_{M_1} \circ f^r_{c_1,d_1}(w) & \text{for } d_1 \le w \le 1. \end{cases}$$

Lemma 3.

$$\|\bar f(w) - G_{c_1,d_1,c_2,\tilde d_2,M_1,d_2}\|_{C^2([0,1]\setminus\{d_1,\tilde d_2\})} \le \text{const}\,\lambda^{n\varepsilon}. \tag{22}$$

Proof. Clearly,

$$\bar f(w) = f^{r+1,q} \circ f^{r,r+1} \circ f^{1,r} \circ f^{0,1}.$$

We can use Lemma 1 to approximate $f^{0,1}$ with $f^{lr}_{c_1,d_1}$ and $f^{r,r+1}$ with $f^{lr}_{c_2,d_2}$. Lemma 2 imply that $f^{1,r}$ is close to $F_{M_1}(w)$ and $f^{r+1,q}$ is close to $F_{M_2}(w)$. To obtain formula for G we only need to notice that $M_1M_2 \approx c_1c_2$. More precisely,

$$\begin{aligned} \ln M_1 + \ln M_2 &= \sum_{i=0}^{q-1} \int_{\Delta_i} \frac{f''(y)}{2f'(y)}dy - \int_{\Delta_0} \frac{f''(y)}{2f'(y)}dy - \int_{\Delta_r} \frac{f''(y)}{2f'(y)}dy \\ &= \ln c_1c_2 - \int_{\Delta_0} \frac{f''(y)}{2f'(y)}dy - \int_{\Delta_r} \frac{f''(y)}{2f'(y)}dy. \end{aligned}$$

By Lemma 1

$$\left| \int_{\Delta_0} \frac{f''(y)}{2f'(y)}dy + \int_{\Delta_r} \frac{f''(y)}{2f'(y)}dy \right| \le \text{const}\,\lambda^n,$$

as required. □

Let us introduce

$$\hat d_2(\tilde d_2) = \begin{cases} F_{M_1}(f^l_{c_1,d_1}(\tilde d_2)) & 0 \le \tilde d_2 \le d_1, \\ F_{M_1}(f^r_{c_1,d_1}(\tilde d_2)) & d_1 \le \tilde d_2 \le 1. \end{cases} \tag{23}$$

Lemma 4. *We have*

$$\|G_{c_1,d_1,c_2,\tilde d_2,M_1,d_2} - G_{c_1,d_1,c_2,\tilde d_2,M_1,\hat d_2(\tilde d_2)}\|_{C^2([0,1]\setminus\{d_1,\tilde d_2\})} \le \text{const}\,\lambda^{n\varepsilon} \tag{24}$$

Proof. Notice that

$$|\hat d_2(\tilde d_2) - d_2| \le C\lambda^{n\varepsilon}.$$

Both functions are continuous on the same intervals. On each of them they are just fractional-linear functions with exponentially close parameters. Moreover, all parameters are bounded away from 0 and infinity. Hence, these functions are close with both derivatives. □

Therefore, the following preposition holds

Preposition 5.

$$\|G_{c_1,d_1,c_2,\tilde d_2,M_1,\hat d_2(\tilde d_2)} - \bar f\|_{C^2([0,1]\setminus\{d_1,\tilde d_2\})} \le \text{const}\,\lambda^{n\varepsilon}$$

For any monotonous function $g : [0,1] \to [0,1]$, satisfying $g(0) = 0$ and $g(1) = 1$ let us introduce relation $R(g,d)$, given by formula

$$R(g,d) = \frac{1-g(d)}{g(d)} : \frac{1-d}{d}.$$

This relation satisfy the following important equality:

$$R(F_M, d) = M. \tag{25}$$

Indeed, by (21)

$$R(F_M, d) = \frac{\dfrac{M(1-d)}{M(1-d)+d}}{\dfrac{d}{M(1-d)+d}} : \frac{1-d}{d} = M.$$

Lemma 5. *We have*

$$\left|\frac{R(G_{c_1,d_1,c_2,\tilde{d}_2,M_1,\hat{d}_2(\tilde{d}_2)}, d_1)}{R(\bar{f}, d_1)} - 1\right| \to 0$$

as $n \to \infty$ for any fixed f (and hence fixed c_1, c_2).

Proof. For sake of shortness let us denote

$$G(w) = G_{c_1,d_1,c_2,\tilde{d}_2,M_1,\hat{d}_2(\tilde{d}_2)}(w).$$

Notice now, that

$$\frac{R(G,d_1)}{R(\bar{f},d_1)} = \frac{1-G(d_1)}{1-\bar{f}(d_1)}\frac{\bar{f}(d_1)}{G(d_1)}.$$

Fix some $\varepsilon > 0$. There exists $\delta > 0$ such that for all $d_1 > \varepsilon$ we have $G(d_1) > G(\varepsilon) > \delta > 0$. Let us consider fraction $\frac{\bar{f}(d_1)}{G(d_1)}$. Clearly,

$$\left|\frac{\bar{f}(d_1)}{G(d_1)} - 1\right| = \left|\frac{\bar{f}(d_1) - G(d_1)}{G(d_1)}\right|$$

Since $\|\bar{f}(d_1) - G(d_1)\| \le \lambda^n$, we have

$$\left|\frac{\bar{f}(d_1) - G(d_1)}{G(d_1)}\right| \le \frac{\lambda^n}{\delta} \le \varepsilon$$

for all n large enough. Hence, the only "dangerous" case is $d_1 < \varepsilon$.

The following three situation can occur (cf 3):

1) $d_1 < \tilde{d}_2$,

2) $d_1 > \tilde{d}_2$ and $\tilde{d}_2 < d_1 - \tilde{d}_2$,

3) $d_1 > \tilde{d}_2$ and $\tilde{d}_2 > d_1 - \tilde{d}_2$.

In case 1) we have

$$\frac{\bar{f}(d_1) - G(d_1)}{G(d_1)} = \frac{\bar{f}(d_1)/d_1 - G(d_1)/d_1}{G(d_1)/d_1} = \frac{\bar{f}'(\xi) - G'(\zeta)}{G'(\zeta)},$$

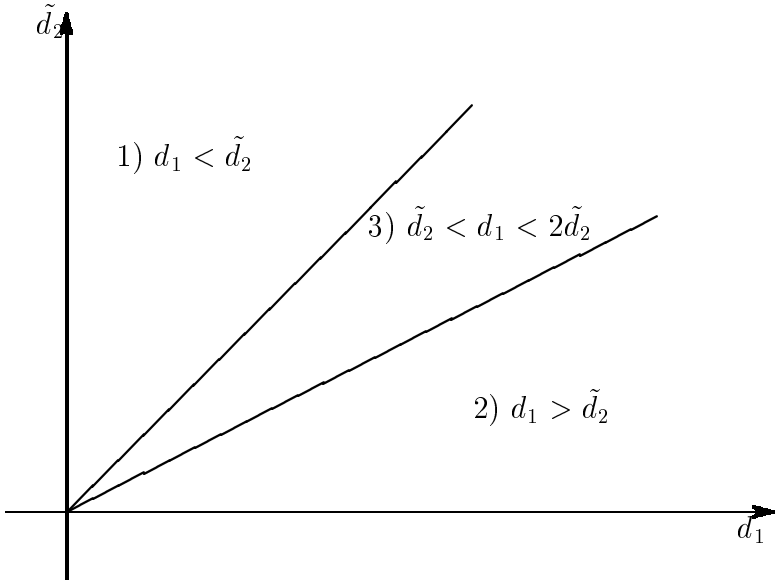

FIGURE 3

where ξ, $\zeta \in (0, d_1)$. Notice that second derivatives of $\bar{f}$ and G are bounded by C_2 and first derivatives are bounded away from 0 and ∞. Hence, $G'(\zeta) > C_1 > 0$ and

$$|\bar{f}'(\xi) - G'(\zeta)| = |\bar{f}'(0) - G'(0) + \bar{f}''(\xi_1)\xi - G''(\zeta_1)\zeta| \le |\bar{f}'(0) - G'(0)| + 2C_2\varepsilon.$$

Hence, for all n large enough we have

$$\frac{\bar{f}(d_1) - G(d_1)}{G(d_1)} \le \frac{\bar{f}'(0) - G'(0)}{C_1} + \frac{2C_2\varepsilon}{C_1} \le C_4\varepsilon,$$

where $C_4 > 1$ is independent of ε (here we used that $|f'(0) - G'(0)| < \text{const}\,\lambda^n$). Hence if $d_1 < \tilde{d}_2$, then for any fixed ε for all n large enough we have

$$|\frac{\bar{f}(d_1) - G(d_1)}{G(d_1)}| \le C_4\varepsilon.$$

Therefore, this fraction converges to 0 as $n \to \infty$.

If case 2) or 3) holds we use Lagrange theorem twice for $\bar{f}$ and for G:

$$\bar{f}(d_1) = \bar{f}(\tilde{d}_2) + \bar{f}'(\xi_2)(d_1 - \tilde{d}_2) = \bar{f}'(\xi_1)\tilde{d}_2 + \bar{f}(\xi_2)(d_1 - \tilde{d}_2),$$
$$G(d_1) = G'(\zeta_1)\tilde{d}_2 + G'(\zeta_2)(d_1 - \tilde{d}_2),$$

where $\xi_1, \zeta_1 \in (0, \tilde{d}_2)$, $\xi_2, \zeta_2 \in (\tilde{d}_2, d_1)$. Again, it is enough to consider case $d_1 < \varepsilon$ and show that the following inequality holds:

$$|\frac{\bar{f}(d_1) - G(d_1)}{G(d_1)}| = |\frac{\bar{f}'(\xi_1)\tilde{d}_2 + \bar{f}(\xi_2)(d_1 - \tilde{d}_2)}{G'(\zeta_1)\tilde{d}_2 + G'(\zeta_2)(d_1 - \tilde{d}_2)}| = |\frac{P}{Q}| \le \text{const}\,\varepsilon$$

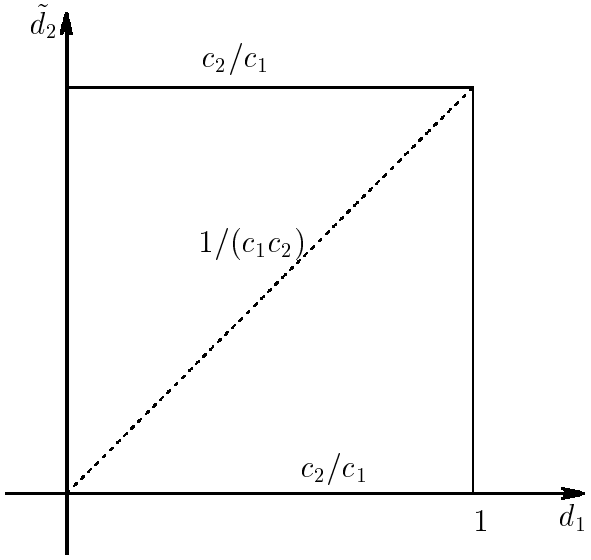

FIGURE 4

Suppose $\tilde{d}_2 \le d_1 - \tilde{d}_2$, i.e., case 2) holds. Let us multiply P and Q by $1/(d_1 - \tilde{d}_2)$. In this case

$$\begin{aligned}
|\frac{P}{d_1 - \tilde{d}_2}| &= |(\bar{f}'(\xi_1) - G'(\xi_1))\frac{\tilde{d}_2}{d_1 - \tilde{d}_2} + \bar{f}'(\xi_2) - G'(\zeta_2)| \\
&\le |\bar{f}'(\xi_1) - G'(\xi_1)| + |\bar{f}'(\xi_2) - G'(\zeta_2)| \\
&= |\bar{f}'(0) - G'(0)| + |f''(\xi_{11})\xi_1 - G''(\zeta_{11})\zeta_1| \\
&\qquad + |\bar{f}'(\tilde{d}_2) - G'(\tilde{d}_2)| + |f''(\xi_{21})\xi_2 - G''(\zeta_{21})\zeta_2| \\
&\le 2\lambda^n + 4C_2\varepsilon,
\end{aligned}$$

where $\xi_{11} \in (0, \xi_1)$, $\zeta_{11} \in (0, \zeta_1)$, $\xi_{21} \in (\tilde{d}_2, \xi_2)$ and $\zeta_{21} \in (\tilde{d}_2, \zeta_2)$. We also have

$$|\frac{Q}{d_1 - \tilde{d}_2}| = G'(\zeta_1)\frac{\tilde{d}_2}{d_1 - \tilde{d}_2} + G'(\zeta_2) \ge G'(\zeta_2) > C_1.$$

Hence,

$$|\frac{P}{Q}| \le \frac{2\lambda^n + 4C_2\varepsilon}{C_1} \le 4C_4\varepsilon$$

for all n large enough.

Case 3) is considered in the same way with multiplying P and Q by $1/\tilde{d}_2$. □

Let us denote

$$R(d_1, \tilde{d}_2) = R(G_{c_1, d_1, c_2, \tilde{d}_2, M_1, \hat{d}_2(\tilde{d}_2)}, d_1).$$

Lemma 6. *We have $R(d_1, d_1) = 1/(c_1c_2)$ and $R(d_1, 0) = R(d_1, 1) = c_2/c_1$. For any fixed d_1 the function $g(\tilde{d}_2) = R(d_1, \tilde{d}_2)$ is monotonous for $\tilde{d}_2 \in [0, d_1]$ and for*

$\tilde d_2 \in [d_1, 1]$. *Hence the ratio* $R(d_1, \tilde d_2)$ *takes values between* $1/(c_1c_2)$ *and* c_2/c_1 *(cf 4).*

Proof. Notice that by (19)–(20) and (23),

$$\begin{aligned} G_{c_1,d_1,c_2,d_1,M_1,\hat d_2(d_1)}(d_1) &= F_{\frac{c_1c_2}{M_1}} \circ f^l_{c_2,\hat d_2(d_1)} \circ F_{M_1} \circ f^l_{c_1,d_1}(d_1) \\ &= F_{\frac{c_1c_2}{M_1}} \circ F_{1/c_2^2} \circ F_{M_1} \circ F_{1/c_1^2}(d_1) = F_{1/(c_1c_2)}(d_1). \end{aligned}$$

Using (25), we obtain

$$R(d_1, d_1) = R(G_{c_1,d_1,c_2,d_1,M_1,\hat d_2(d_1)}, d_1) = R(F_{1/(c_1c_2)}, d_1) = \frac{1}{c_1c_2}$$

as required.

If $\tilde d_2 > d_1$, then

$$G_{c_1,d_1,c_2,\tilde d_2,M_1,\hat d_2(\tilde d_2)}(d_1) = F_{\frac{c_1c_2}{M_1}} \circ f^l_{c_2,\hat d_2(\tilde d_2)} \circ F_{M_1} \circ f^l_{c_1,d_1}(d_1).$$

Suppose $0 < d_1 < 1$. Notice that $\hat d_2(\tilde d_2) = 1$ if $\tilde d_2 = 1$ and hence

$$G_{c_1,d_1,c_2,\tilde d_2,M_1,\hat d_2(\tilde d_2)} = G_{c_1,d_1,c_2,1,M_1,1}$$

when $\tilde d_2 = 1$. Using (19) and (20), we obtain

$$\begin{aligned} G_{c_1,d_1,c_2,1,M_1,1}(d_1) &= F_{\frac{c_1c_2}{M_1}} \circ f^l_{c_2,1} \circ F_{M_1} \circ f^l_{c_1,d_1}(d_1) \\ &= F_{\frac{c_1c_2}{M_1}} \circ F_{M_1} \circ f^l_{c_1,d_1}(d_1) \\ &= F_{c_1c_2} \circ f^l_{c_1,d_1}(d_1) = F_{c_1c_2} \circ F_{1/c_1^2}(d_1) = \\ &= F_{c_2/c_1}(d_1). \end{aligned}$$

Using (25) we obtain

$$R(d_1, 1) = R(F_{c_2/c_1}, d_1) = \frac{c_2}{c_1}.$$

If $\tilde d_2 < d_1$, then

$$G_{c_1,d_1,c_2,\tilde d_2,M_1,\hat d_2(\tilde d_2)}(d_1) = F_{\frac{c_1c_2}{M_1}} \circ f^r_{c_2,\hat d_2(\tilde d_2)} \circ F_{M_1} \circ f^l_{c_1,d_1}(d_1).$$

For $\tilde d_2 = 0$ we have

$$\begin{aligned} G_{c_1,d_1,c_2,0,M_1,0}(d_1) &= F_{\frac{c_1c_2}{M_1}} \circ f^r_{c_2,0} \circ F_{M_1} \circ f^l_{c_1,d_1}(d_1) \\ &= F_{\frac{c_1c_2}{M_1}} \circ F_{M_1} \circ f^l_{c_1,d_1}(d_1) = F_{c_1c_2} f^l_{c_1,d_1}(d_1) = F_{c_2/c_1}(d_1), \end{aligned}$$

and hence,

$$R(d_1, 0) = R(d_1, 1) = \frac{c_2}{c_1}$$

Finally, let us study the derivative of $R(d_1, \tilde d_2)$ w.r.t. $\tilde d_2$. Using chain rule we obtain

$$\frac{\partial}{\partial \tilde d_2} R(d_1, \tilde d_2) = \frac{d_1}{1 - d_1} \frac{1}{(G(d_1))^2} \frac{\partial}{\partial \tilde d_2} G(d_1).$$

Now notice that if $d_1 < \tilde{d}_2$,

$$\frac{\partial}{\partial \tilde{d}_2} G(d_1) = \frac{\partial}{\partial \tilde{d}_2} F_{c_1 c_2/M_1} \circ f^l_{c_2,\hat{d}_2(\tilde{d}_2)}(F_{M_1} \circ f^l_{c_1,d_1}(d_1)) =$$
$$= F'_{c_1 c_2/M_1}(f^l_{c_2,\hat{d}_2(\tilde{d}_2)}(w)) \frac{\partial}{\partial \tilde{d}_2} f^l_{c_2,\hat{d}_2(\tilde{d}_2)}(w),$$

where $w = F_{M_1}(f^l_{c_1,d_1}(d_1))$. Notice now that

$$\frac{\partial}{\partial \tilde{d}_2} f^l_{c_2,\hat{d}_2(\tilde{d}_2)}(w) = \frac{\partial}{\partial \tilde{d}_2}\left(\frac{c_2^2 w}{1+\hat{d}_2(\tilde{d}_2)(c_2^2-1)}\right) = -\frac{c_2^2 w(c_2^2-1)}{(1+\hat{d}_2(\tilde{d}_2)(c_2^2-1))^2}\frac{\partial}{\partial \tilde{d}_2}\hat{d}_2(\tilde{d}_2).$$

Hence the sign of the derivative $\partial R(d_1,\tilde{d}_2)/\partial \tilde{d}_2$ conicide with the sign of $-(c_2^2-1)$ when $\tilde{d}_2 > d_1$. Proceeding in the same way we obtain that for $\tilde{d}_2 < d_1$ the sign of the derivative $\partial R(d_1,\tilde{d}_2)/\partial \tilde{d}_2$ is the same as the sign of derivative

$$\frac{\partial}{\partial \tilde{d}_2} f^r_{c_2,\hat{d}_2(\tilde{d}_2)} = \frac{(c_2^2-1)(1-w)}{(1+\hat{d}_2(\tilde{d}_2)(c_2^2-1))^2}\frac{\partial}{\partial \tilde{d}_2}\hat{d}_2(\tilde{d}_2),$$

i.e., the sign of $\partial R(d_1,\tilde{d}_2)/\partial \tilde{d}_2$ is the same as the sign of (c_2^2-1).

Hence, if $c_2 < 1$, then $g(\tilde{d}_2) = R(d_1,\tilde{d}_2)$ is an increasing function for $\tilde{d}_2 \in [0,d_1]$ and $g(\tilde{d}_2)$ is a decreasing function for $\tilde{d}_2 \in [d_1,1]$. Finally, $g(0) = c_2/c_1$, $g(d_1) = 1/c_1c_2$, $g(1) = c_2/c_1$.

If $c_2 > 1$, then $g(\tilde{d}_2) = R(d_1,\tilde{d}_2)$ is a decreasing function for $\tilde{d}_2 \in [0,d_1]$ and $g(\tilde{d}_2)$ is an increasing function for $\tilde{d}_2 \in [d_1,1]$. Again, $g(0) = c_2/c_1$, $g(d_1) = 1/c_1c_2$, $g(1) = c_2/c_1$. □

4. Critical periodic trajectory

Suppose now that $d_1 = 0$ or $d_1 = 1$. One can easily check that

$$G_{c_1,0,c_2,\tilde{d}_2,M_1,\hat{d}_2(\tilde{d}_2)} = G_{c_1,1,c_2,\tilde{d}_2,M_1,\hat{d}_2(\tilde{d}_2)} = G$$

and

$$G(w) = \begin{cases} F_{c_1c_2/M_1} \circ f^l_{c_2,d_2} \circ F_{M_1}(w), & 0 \le w < \tilde{d}_2, \\ F_{c_1c_2/M_1} \circ f^r_{c_2,d_2} \circ F_{M_1}(w), & \tilde{d}_2 \le w < 1, \end{cases}$$

Notice that

$$(F_a(w))'_{w=0} = \frac{1}{a} \text{ and } (F_a(w))'_{w=1} = a.$$

Let us introduce $m_- = G'(1)$ and $m_+ = G'(0)$. Since

$$F_{\frac{c_1c_2}{M_1}}(0) = f^l_{c_2,\hat{d}_2}(0) = F_{M_1}(0) = 0 \text{ and } F_{\frac{c_1c_2}{M_1}}(1) = f^l_{c_2,\hat{d}_2}(1) = F_{M_1}(1) = 1,$$

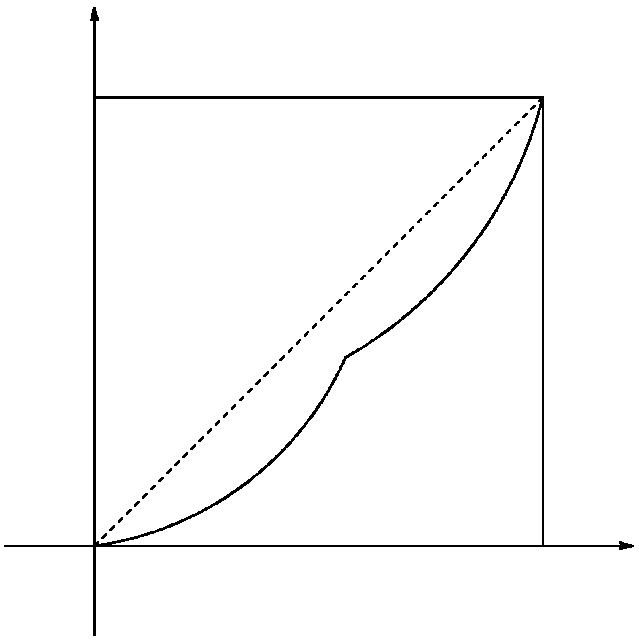

FIGURE 5

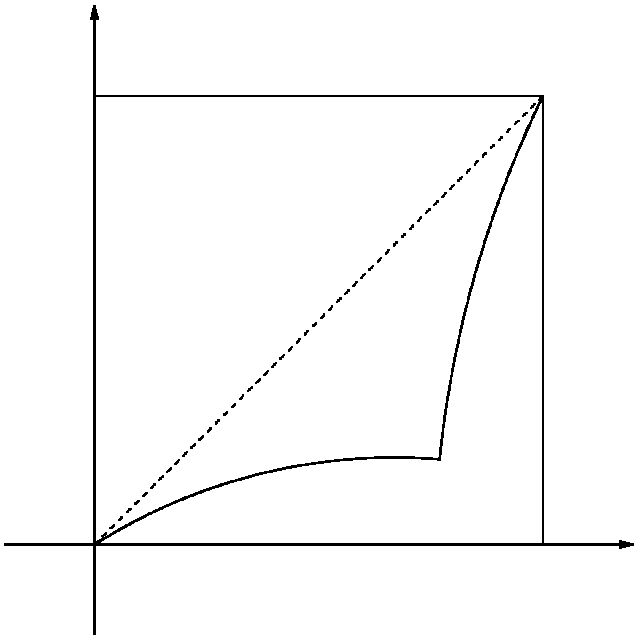

FIGURE 6

we have

$$m_- = G'(1) = \frac{c_1c_2}{M_1}\frac{1}{1+\hat{d}_2(\tilde{d}_2)(c_2^2-1)}M_1 = c_1c_2\frac{1}{1+\hat{d}_2(\tilde{d}_2)(c_2^2-1)}, \tag{26}$$

$$m_+ = G'(0) = \frac{M_1}{c_1c_2}\frac{c_2^2}{1+\hat{d}_2(\tilde{d}_2)(c_2^2-1)}\frac{1}{M_1} = \frac{c_2}{c_1}\frac{1}{1+\hat{d}_2(\tilde{d}_2)(c_2^2-1)}. \tag{27}$$

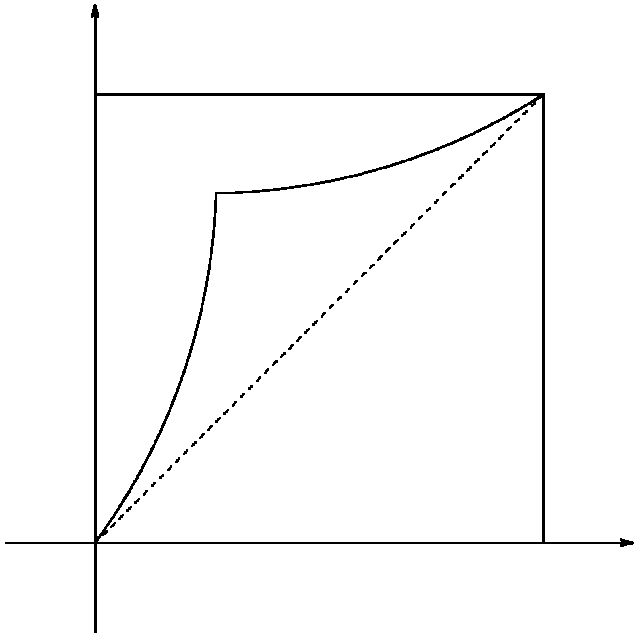

FIGURE 7

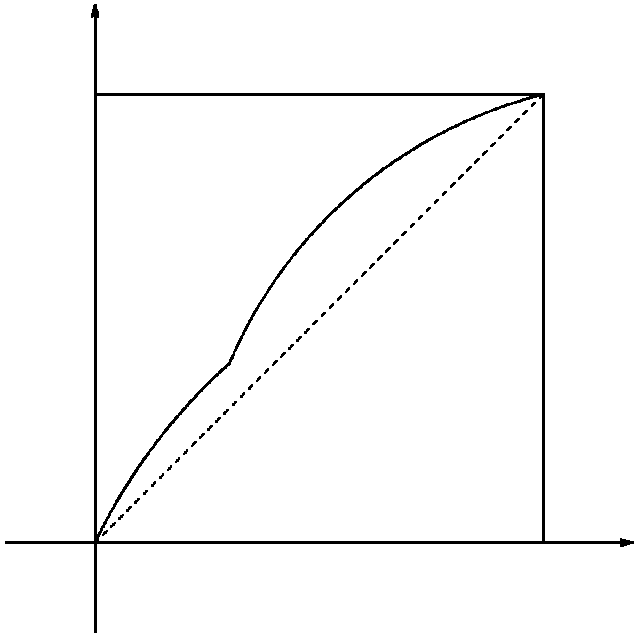

FIGURE 8

Notice that, as one could expect

$$\frac{m_-}{m_+} = c_1^2.$$

Let us also consider $G(\tilde d_2)$. Using (19) and (20), we obtain

$$\begin{aligned} G(\tilde d_2) &= F_{\frac{c_1c_2}{M_1}} \circ f^l_{c_2,\hat d_2(\tilde d_2)} \circ F_{M_1}(\tilde d_2) = F_{\frac{c_1c_2}{M_1}} \circ f^l_{c_2,\hat d_2(\tilde d_2)}(\hat d_2(\tilde d_2)) \\ &= F_{\frac{c_1c_2}{M_1}} \circ F_{\frac{1}{c_2^2}}(\hat d_2(\tilde d_2)) = F_{\frac{c_1c_2}{M_1}} \circ F_{\frac{1}{c_2^2}} \circ F_{M_1}(\tilde d_2) = F_{c_1/c_2}(\tilde d_2), \end{aligned}$$

i.e.,

$$G(\tilde d_2) = \frac{\tilde d_2}{(c_1/c_2)(1-\tilde d_2)+\tilde d_2}. \tag{28}$$

It follows from Lemma 6 and (26),(27) that

$$\begin{aligned} R(d_1,\tilde d_2) &\in I[\frac{1}{c_1c_2}, \frac{c_2}{c_1}], \\ m_+ &\in I[\frac{1}{c_1c_2}, \frac{c_2}{c_1}], \\ m_- &\in I[c_1c_2, \frac{c_1}{c_2}]. \end{aligned}$$

Now we would like to enumerate critical points in such a way, that segment $I[\frac{1}{c_1c_2}, \frac{c_2}{c_1}]$ have not touched point 1. As one can see, only the following two cases are possible.

A) If $c_1c_2 > 1$, then we have to enumerate $x^{(i)}_{\rm cr}$ in such a way, that $c_2/c_1 < 1$.

B) If $c_1c_2 < 1$, then we have to enumerate $x^{(i)}_{\rm cr}$ in such a way, that $c_2/c_1 > 1$.

Let us study the graph of $G(w)$. In both cases it would be a graph of piecwise-fractional-linear function.

In case A) $m_- = G'(1) > \text{const} > 1$ and $m_+ = G'(0) < \text{const} < 1$. Using (28), we obtain that for $\tilde d \in [\delta, 1-\delta]$ we have

$$G(\tilde d_2) \le \frac{\tilde d_2}{1+\varepsilon}$$

for some ε, since

$$\frac{c_1}{c_2}(1-\tilde d_2)+\tilde d_2 \in [1, \frac{c_1}{c_2}].$$

Hence, the graph of $G(w)$ is completely below the diagonal (cf Figures 5, 6).

In case B) $m_- = G'(1) < \text{const} < 1$ and $m_+ = G'(1) > \text{const} > 1$. Using (28), we obtain that for $\tilde d \in [\delta, 1-\delta]$ we have

$$G(\tilde d_2) \ge \frac{\tilde d_2}{1-\varepsilon}$$

for some ε, since

$$\frac{c_1}{c_2}(1-\tilde d_2)+\tilde d_2 \in [\frac{c_1}{c_2}, 1].$$

Hence, the graph of $G(w)$ is completly above the diagonal (cf Figures 7, 8).

Let us denote by $\bar I(p/q) = [a_0(p/q), a_1(p/q)]$ a closed interval of values a such that $\rho(f_a) = p/q$.

Theorem 2. *Suppose that critical points $x_{\rm cr}^{(i)}$ are enumerated in such a way that*
A) for $c_1c_2 < 1$ we have $c_2 < c_1$ and
B) for $c_1c_2 > 1$ we have $c_2 > c_1$.
In case A) for all $n >$ const there exists a unique critical periodic trajectory for $a = a_0(p/q)$. This is a trajectory of the critical point $x_{\rm cr}^{(1)}$.
In case B) for all $n >$ const there exists a unique critical periodic trajectory for $a = a_1(p/q)$. This is a trajectory of the critical point $x_{\rm cr}^{(1)}$.

Proof. Without loss of generality, consider case A). If d_2 is bounded away from 0 and 1, then the result follows from Preposition 5, since the whole graph of $\bar f$ would lie below the diagonal. If d_2 is close to 0, then $\bar f(d_1) = G'(0)d_1 + o(d_1) < d_1$, since $G'(0) = m_+ <$ const < 1. The same argument works for d_2 close to 1. Using estimates of Preposition 5, we obtain that the whole graph of $\bar f$ lie below the diagonal. □

5. Non-linearity of f^q

Notice that it is enough to consider the case $c_1c_2 < 1$, $c_2/c_1 < 1$. Consider interval $I(p/q) = (a_0, a_1)$. Suppose that p/q is obtained from Farey interval $(p_1/q_1, p_2/q_2)$. It follows from Theorem 2 that $T_{a_0}^q x_{\rm cr}^{(1)} = x_{\rm cr}^{(1)}$ and

$$T_{a_0}^q \tilde x < \tilde x \text{ for any point } \tilde x \in (T_{a_0}^{q_2} x_{\rm cr}^{(1)}, x_{\rm cr}^{(1)}). \tag{29}$$

Denote now by R the ratio

$$\frac{x_{\rm cr}^{(1)} - \tilde x}{\tilde x - T_{a_0}^{q_1} x_{\rm cr}^{(1)}} = R.$$

It follows from the following lemma that there exists a unique $\tilde a \in I(p/q)$ such that $T_{\tilde a}^q \tilde x = \tilde x$.

Lemma 7. *Suppose $\rho(f_{a_1}) < p/q < \rho(f_{a_2})$. Then for any $x \in [0,1]$ there exists a unique $a(x) \in [a_1, a_2]$ such that $\rho(f_{a(x)}) = p/q$ and $f^q(x) = x + p$.*

Proof. Note that there exists K such that for all $k > K$ we have

$$f_{a_1}^{kq}(x) - x - kp < 0 \text{ and } f_{a_2}^{kq}(x) - x - kp > 0.$$

Since $f_a^{kq}(x)$ is a strictly monotonous function of a, there exists a unique $a = a(x)$, $a(x) \in [a_1, a_2]$ such that we have $f_a^{kq}(x) - x - kp = 0$.

Now note that if $f_a^q(x) - p > x$ for all $x \in \mathbb{R}$ then, by $f_a(x) - 1 = f_a(x-1)$, we have

$$f_a^{2q}(x) - 2p = f_a^q(f_a^q(x) - p) - p > f_a^q(x) - p > x$$

and by induction we obtain that $f_a^{lq}(x) - lp > x$ for all $l \geq 1$. Therefore, we arrive to contradiction with $f_a^{kq}(x) - kp = x$. The same arguments works for $f_a^q(x) - p < x$. Therefore, $f_a^q(x) - p = x$. □

Let us show that at moment $\tilde{a}$ the relative coordinate of $x_{\mathrm{cr}}^{(1)}$ on critical trajectory passing through $\tilde{x}$ is bounded away from 0 and 1 by constant.

Preposition 6.

$$\frac{1}{2}\frac{e^{-2v}}{1+e^{4v}}\frac{R}{R+1} \le \frac{x_{\mathrm{cr}}^{(1)}-\tilde{x}}{T_{\tilde{a}}^{q_1}\tilde{x}-\tilde{x}} \le \frac{R}{R+e^{-2v}}. \tag{30}$$

Proof. Notice that p/q is obtained from some Farey interval $A=(p_1/q_1, p_2/q_2)$, i.e., $p/q=(p_1+p_2)/(q_1+q_2)$. Preposition 1, monotonicity of T_a and (29) imply that the following order of points takes place:

$$T_{a_0}^{q_2}x_{\mathrm{cr}}^{(1)} < T_{a_0}^{q}\tilde{x} < \tilde{x} < x_{\mathrm{cr}}^{(1)} < T_{a_0}^{q_1}\tilde{x} < T_{a_0}^{q_1}x_{\mathrm{cr}}^{(1)}$$

and

$$T_{a_0}^{q_1}\tilde{x} < T_{\tilde{a}}^{q_1}\tilde{x}.$$

Since $\rho(T_{a_0})=p/q$, it follows from (17) that

$$e^{-2v} \le (\frac{\partial}{\partial x}T_a^{q_1})(x_0), (\frac{\partial}{\partial x}T_a^{q_2})(x_0) \le e^{2v} \tag{31}$$

for each $a\in[a_0,a_1]$ and for each point x_0, such that the derivative of T_a^q w.r.t. x_0 is well-defined.

Let us prove upper bound in (30). Notice that

$$\frac{x_{\mathrm{cr}}^{(1)}-\tilde{x}}{T_{\tilde{a}}^{q_1}\tilde{x}-\tilde{x}} \le \frac{x_{\mathrm{cr}}^{(1)}-\tilde{x}}{T_{a_0}^{q_1}\tilde{x}-\tilde{x}} = \frac{x_{\mathrm{cr}}^{(1)}-\tilde{x}}{x_{\mathrm{cr}}^{(1)}-\tilde{x}+T_{a_0}^{q_1}\tilde{x}-x_{\mathrm{cr}}^{(1)}}.$$

Using (31), we obtain

$$\begin{aligned} T_{a_0}^{q_1}\tilde{x}-x_{\mathrm{cr}}^{(1)} &= \int_{T_{a_0}^{q_2}x_{\mathrm{cr}}^{(1)}}^{\tilde{x}} \frac{\partial}{\partial x}T_{a_0}^{q_1}(x)dx \\ &\ge e^{-2v}(\tilde{x}-T_{a_0}^{q_2}x_{\mathrm{cr}}^{(1)}). \end{aligned}$$

But

$$\tilde{x}-T_{a_0}^{q_2} = \frac{1}{R}(x_{\mathrm{cr}}^{(1)}-\tilde{x}).$$

Hence

$$\begin{aligned} \frac{x_{\mathrm{cr}}^{(1)}-\tilde{x}}{x_{\mathrm{cr}}^{(1)}-\tilde{x}+T_{a_0}^{q_1}\tilde{x}-x_{\mathrm{cr}}^{(1)}} &\le \frac{x_{\mathrm{cr}}^{(1)}-\tilde{x}}{x_{\mathrm{cr}}^{(1)}-\tilde{x}+\dfrac{e^{-2v}}{R}(x_{\mathrm{cr}}^{(1)}-\tilde{x})} \\ &= \frac{R}{R+e^{-2v}}, \end{aligned}$$

as required.

In order to prove the lower bound in (30) we need an estimate

$$T_{\tilde{a}}^{q_1}\tilde{x}-T_{a_0}^{q_1}\tilde{x} \le e^{2v}(T_{\tilde{a}}^{q}\tilde{x}-T_{a_0}^{q}\tilde{x}). \tag{32}$$

To see why it is true, notice that by (49) (proven in Appendix),

$$\frac{\partial}{\partial a}T_a^q\tilde{x} = \frac{\partial}{\partial a}T_a^{q-q_1}\circ T_a^{q_1}\tilde{x}\prod_{j=q_1}^{q-1} f'(\tilde{x}_j(a))\frac{\partial}{\partial a}T_a^{q_1}\tilde{x} + (\frac{\partial}{\partial a}T_a^{q-q_1})(T_a^{q_1}\tilde{x}),$$

where $\tilde{x}_j(a) = T_a^j\tilde{x}$. Since the second summund in r.h.s. is non-negative,

$$\frac{\partial}{\partial a}T_a^q\tilde{x} \geq \prod_{j=q_1}^{q-1} f'(\tilde{x}_j(a))\frac{\partial}{\partial a}T_a^{q_1}\tilde{x}$$

or

$$\frac{\partial}{\partial a}T_a^q\tilde{x} \geq \prod_{j=0}^{q_2-1} f'(z_j(a))\frac{\partial}{\partial a}T_a^{q_1}\tilde{x},$$

where $z_j(a) = T_a^j z_0(a)$, $z_0(a) = \tilde{x}_{q_1}(a)$. But (31) imply that

$$e^{-2v} \leq \prod_{j=0}^{q_2-1} f'(z_j(a)) \leq e^{2v}$$

for all $a \in I(p/q)$ such that $\tilde{x}_i(a) \neq x_{\rm cr}^{(j)}$ for all $0 \leq i \leq q-1$, $j = 0, 1$. Therefore

$$\frac{\partial}{\partial a}T_a^{q_1}\tilde{x} \leq e^{2v}\frac{\partial}{\partial a}T_a^q\tilde{x} \tag{33}$$

for almost all $a \in I[p/q]$. To prove (32) it stays to notice that

$$\begin{aligned} T_{\tilde{a}}^{q_1}\tilde{x} - T_{a_0}^{q_1}\tilde{x} &= \int_{a_0}^{\tilde{a}}\frac{\partial}{\partial a}T_a^{q_1}(\tilde{x})da \\ &\leq e^{2v}\int_{a_0}^{\tilde{a}}\frac{\partial}{\partial a}T_a^q(\tilde{x})da = T_{\tilde{a}}^q\tilde{x} - T_{a_0}^q. \end{aligned}$$

Notice that

$$T_{\tilde{a}}^{q_1}\tilde{x} - \tilde{x} = T_{\tilde{a}}^{q_1}\tilde{x} - T_{a_0}^{q_1}\tilde{x} + T_{a_0}^{q_1}\tilde{x} - \tilde{x} \tag{34}$$

It follows from (32) that

$$T_{\tilde{a}}^{q_1}\tilde{x} - T_{a_0}^{q_1}\tilde{x} \leq e^{2v}(T_{\tilde{a}}^q\tilde{x} - T_{a_0}^q\tilde{x}) = e^{2v}(\tilde{x} - T_{a_0}^q\tilde{x}).$$

Notice that

$$T_{a_0}^{q_1}(T_{a_0}^q\tilde{x}, \tilde{x}) \subset (x_{\rm cr}^{(1)}, T_{a_0}^{q_1}\tilde{x}) \subset (\tilde{x}, T_{a_0}^{q_1}\tilde{x}).$$

Using (31), we obtain

$$e^{-2v}|(T_{a_0}^q\tilde{x}, \tilde{x})| \leq |(x_{\rm cr}^{(1)}, T_{a_0}^{q_1}\tilde{x})| \leq |(\tilde{x}, T_{a_0}^{q_1}\tilde{x})|.$$

To sum up

$$T_{\tilde{a}}^{q_1}\tilde{x} - T_{a_0}^{q_1}\tilde{x} \leq e^{4v}(T_{a_0}^{q_1}\tilde{x} - \tilde{x}).$$

Hence, (34) yields

$$T_{\tilde{a}}^{q_1}\tilde{x} - \tilde{x} \leq (1 + e^{4v})(T_{a_0}^{q_1}\tilde{x} - \tilde{x})$$

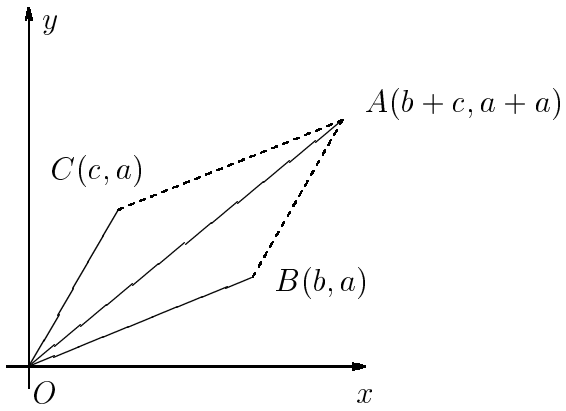

FIGURE 9

Therefore,

$$\frac{x_{\rm cr}^{(1)}-\tilde{x}}{T_{\tilde{a}}^{q_1}\tilde{x}-\tilde{x}} \geq \frac{1}{1+e^{4v}}\frac{x_{\rm cr}^{(1)}-\tilde{x}}{T_{a_0}^{q_1}\tilde{x}-\tilde{x}} \tag{35}$$

Now notice that

$$\frac{x_{\rm cr}^{(1)}-\tilde{x}}{T_{a_0}^{q_1}\tilde{x}-\tilde{x}} \geq \frac{x_{\rm cr}^{(1)}-\tilde{x}}{T_{a_0}^{q_1}x_{\rm cr}^{(1)}-T_{a_0}^{q_2}x_{\rm cr}^{(1)}}.$$

The r.h.s. fraction could be estimated by the following lemma.

Lemma 8. *Given a, b, $c>0$ such that*

$$\frac{a}{b}\geq\delta \ \textit{and}\ \frac{a}{c}\geq\gamma,$$

the following inequality holds:

$$\frac{a+a}{b+c}\geq\min(\delta,\gamma),\ \textit{i.e.,}\ \frac{a}{b+c}\geq\frac{1}{2}\min(\delta,\gamma). \tag{36}$$

Proof. Consider points $B=(b,a)$, $C=(c,a)$, and $A=(b+c,a+a)$ on the plain xOy (cf Figure 9). The slope of ray OA lies between slopes of rays OB and OC. Hence,

$$\frac{a+a}{b+c}\geq\min(\frac{a}{b},\frac{a}{c})\geq\min(\delta,\gamma)$$

as required. □

Let us use (36) with $a=x_{\rm cr}^{(1)}-\tilde{x}$, $b=T_{a_0}^{q_1}x_{\rm cr}^{(1)}-x_{\rm cr}^{(1)}$ and $c=x_{\rm cr}^{(1)}-T_{a_0}^{q_2}x_{\rm cr}^{(1)}$. We have

$$\frac{a}{c}=\frac{x_{\rm cr}^{(1)}-\tilde{x}}{x_{\rm cr}^{(1)}-T_{a_0}^{q_2}x_{\rm cr}^{(1)}}=\frac{R}{R+1}$$

and

$$\frac{a}{b}=\frac{x_{\rm cr}^{(1)}-\tilde{x}}{T_{a_0}^{q_1}x_{\rm cr}^{(1)}-x_{\rm cr}^{(1)}}.$$

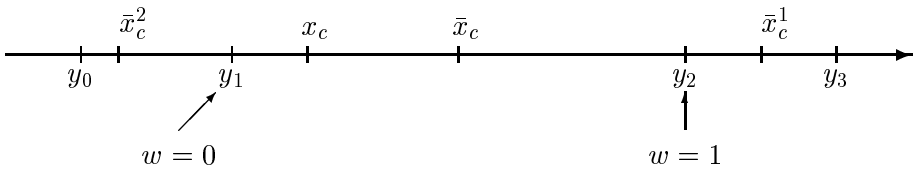

FIGURE 10. Intervals $\Delta_c^{(1)} = [\bar{x}_c^2, x_c]$, $\Delta_c^{(2)} = [x_c, \bar{x}_c]$, $\Delta_c^{(3)} = [\bar{x}_c, \bar{x}_c^1]$, where $\bar{x}_c^2 = Tf_{a_1}^{q_2} x_c$, $\bar{x}_c^1 = Tf_{a_1}^{q_1} x_c$, and $\bar{x}_c = Tf_{a_1}^{q} x_c$

Since

$$T_{a_0}^{q_1}(T_{a_0}^{q_2} x_{\rm cr}^{(1)}, x_{\rm cr}^{(1)}) = (x_{\rm cr}^{(1)}, T_{a_0}^{q_1} x_{\rm cr}^{(1)}),$$

we can apply (31) to obtain

$$e^{-2v}\frac{R}{R+1} \le \frac{x_{\rm cr}^{(1)} - \tilde{x}}{T_{a_0}^{q_1} x_{\rm cr}^{(1)} - x_{\rm cr}^{(1)}} \le e^{2v}\frac{R}{R+1}$$

Therefore,

$$\frac{x_{\rm cr}^{(1)} - \tilde{x}}{T_{a_0}^{q_1} x_{\rm cr}^{(1)} - T_{a_0}^{q_2} x_{\rm cr}^{(1)}} \ge \frac{e^{-2v}}{2}\frac{R}{R+1}$$

Combining our estimates, we obtain

$$\frac{x_{\rm cr}^{(1)} - \tilde{x}}{T_{\tilde{a}}^{q_1} - \tilde{x}} \ge \frac{1}{1+e^{4v}}\frac{e^{-2v}}{2}\frac{R}{R+1},$$

as required. □

Now take

$$\tilde{x} = \frac{T_{a_0}^{q_2} x_{\rm cr}^{(1)} + x_{\rm cr}^{(1)}}{2}.$$

By Lemma 7 there exists a unique $\tilde{a}$, corresponding to passing of periodic trajectory through $\tilde{x}$:

$$\rho(T_{\tilde{a}}) = \frac{p}{q},\ T_{\tilde{a}}^{q}\tilde{x} = \tilde{x}.$$

Denote $y_0 = T_{\tilde{a}}^{q_2} y_1$, $y_1 = \tilde{x}$, $y_2 = T_{\tilde{a}}^{q_1} y_1$, $y_3 = T_{\tilde{a}}^{q_1} y_2$. Clearly, $x_{\rm cr}^{(1)} \in (y_0, y_1)$.

It follows from (30), that relative coordinate $d_1(\tilde{a})$ of $x_{\rm cr}^{(1)}$ is bounded away from 0 and 1 for all n large enough:

$$\frac{1}{2}\frac{e^{-2v}}{1+e^{4v}}\frac{1}{2} \le d_a(\tilde{a}) = \frac{x_{\rm cr}^{(1)} - y_1}{y_2 - y_1} \le \frac{1}{1+e^{-2v}}. \tag{37}$$

Denote $\bar{x}_c^1 = Tf_{\tilde{a}}^{q_1} x_c)$, $\bar{x}_c^2 = Tf_{\tilde{a}}^{q_2}(x_c)$, $\bar{x}_c = Tf_{\tilde{a}}^{q=q_1+q_2}(x_c)$. Evidently, $\bar{x}_c^2 \in (y_0, y_1)$, $\bar{x}_c \in (y_2, y_1)$, $\bar{x}_c^1 \in (y_1, y_0)$. Notice that $\bar{x}_c > x_{\rm cr}^{(1)}$. Indeed, $T_{\tilde{a}}^{q} x_{\rm cr}^{(1)} > T_{a_0}^{q} x_{\rm cr}^{(1)} = x_{\rm cr}^{(1)}$.

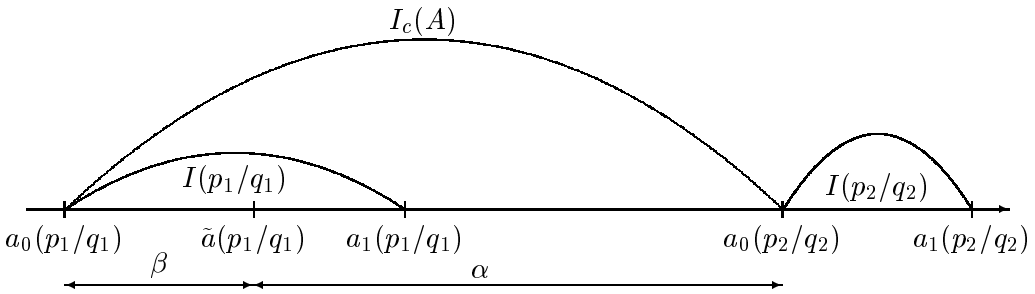

FIGURE 11. Intervals $I(p_1/q_1)$, $I(p_2/q_2)$ and $I_c(A)$

Points $\bar{x}_c^2$, $x_{\rm cr}^{(1)}$, $\bar{x}_c$, $\bar{x}_c^1$ form three successive intervals. Denote this intervals by $\Delta_c^{(1)}$, $\Delta_c^{(2)}$, $\Delta_c^{(3)}$, respectively. We have $\Delta_c^{(1)} = [\bar{x}_c^2, x_{\rm cr}^{(1)}]$, $\Delta_c^{(2)} = [x_{\rm cr}^{(1)}, \bar{x}_c]$, $\Delta_c^{(3)} = [\bar{x}_c, x_c^1]$ (see Figure 10). We shall show that the length of each of these intervals is of the same order.

Preposition 7. *There exists a constant C_4 such that for all n large enough*

$$\frac{\max(l(\Delta_c^{(1)}), l(\Delta_c^{(2)}), l(\Delta_c^{(3)}))}{\min(l(\Delta_c^{(1)}), l(\Delta_c^{(2)}), l(\Delta_c^{(3)}))} \le C_4.$$

Proof. Notice that (37) and Lemmas 5 and 6 imply that $\Delta_c^{(2)}/(x_{\rm cr}^{(1)} - y_0)$ and $\Delta_c^{(2)}/(y_1 - \bar{x}_c)$ are bounded away from 0 and 1 by universal constants. Hence it is sufficient to show that

$$(y_2 - y_1) \le \text{const}\,(y_1 - y_2),\ (y_0 - y_1) \le (y_1 - y_2).$$

But $[y_3, y_2] = Tf^{q_2}[y_2, y_1]$, $[y_1, y_0] = Tf^{q_1}[y_2, y_1]$ and these estimates follow from Corollary 1. □

6. Proof of the theorem 1

Let $A = (p_1/q_1, p_2/q_2)$ be a "good" Farey interval of rank n , i.e. a Farey interval such that $q = \max(q_1, q_2) < 2q' = 2\min(q_1, q_2)$. Denote by $a^{(1)} = a_0(p_1/q_1)$, $a^{(2)} = a_0(p_2/q_2)$ those ends of the intervals $I(p_1/q_1)$, $I(p_2/q_2)$ which correspond to the passage of the periodic trajectory through the singular point. By Theorem 2, these ends are left. Let $I_c(A)$ be the interval $(a^{(1)}, a^{(2)})$ and let $\bar{I}_c(A) = [a^{(1)}, a^{(2)}]$ be its closure (see Figure 11). Denote $x^m(a) = Tf_a^m(x_c)$ where $x_c = 0$ is a critical point and $m \ge 0$.

Preposition 8. *There exists $0 < \gamma < 1$ such that for all n large enough*

$$\frac{l(I(p_1/q_1))}{l(I_c(A))} \ge 1 - \gamma \tag{38}$$

This statement implies theorem 1.

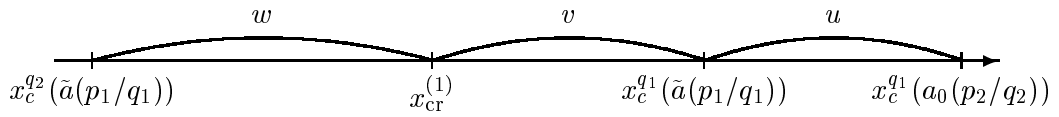

FIGURE 12. Intevals w, v, u

Proof of Theorem 1. Suppose that $\rho = \rho(a)$ is irrational and a is a condensation point for set $\{a \mid \rho(a) \in [0,1) \setminus \mathbb{Q}\}$. Then there exists an infinite sequence of "good" Farey intervals A_i, containing ρ. Since $a \in I_c(A_i)$ and $l(I_c(A_i)) \to 0$ as $i \to \infty$, using (38) we arrive at contradiction with assumption that a is a condensation point. □

Let us denote

$$M_{1,2} = \max_{a \in \bar{I}_c(A)} \left(\frac{\partial}{\partial a} x^{q_{1,2}}(a) \right)$$
$$m_{1,2} = \min_{a \in \bar{I}_c(A)} \left(\frac{\partial}{\partial a} x^{q_{1,2}}(a) \right)$$

Proof of Preposition 8. First notice, that the we have the following ordering:

$$a_0\left(\frac{p_1}{q_1}\right) < \tilde{a}\left(\frac{p_1}{q_1}\right) < a_1\left(\frac{p_1}{q_1}\right) < a_0\left(\frac{p_2}{q_2}\right) < a_1\left(\frac{p_2}{q_2}\right) \text{ (see Figure 11).}$$

Let us denote

$$\alpha = a_0\left(\frac{p_2}{q_2}\right) - \tilde{a}\left(\frac{p_1}{q_1}\right),\ \beta = \tilde{a}\left(\frac{p_1}{q_1}\right) - a_0\left(\frac{p_1}{q_1}\right).$$

For any function $f(z)$ such that $f'(z) > 0$ we have the following estimates for any $z_1 < z_2$:

$$\frac{f(z_2) - f(z_1)}{\max_{z \in [z_1, z_2]} f'(z)} \le z_2 - z_1 \le \frac{f(z_2) - f(z_1)}{\min_{z \in [z_1, z_2]} f'(z)}.$$

It follows that

$$\beta \ge \frac{x^{q_1}(\tilde{a}\left(\frac{p_1}{q_1}\right)) - x^{q_1}(a_0\left(\frac{p_1}{q_1}\right))}{M_1}. \tag{39}$$

Using the equality

$$x^{q_1}(a_0\left(\frac{p_1}{q_1}\right)) = x_c,$$

and notation

$$v = x^{q_1}(\tilde{a}\left(\frac{p_1}{q_1}\right)) - x_c \text{ (see Figure 6)}$$

we can rewrite (39) as follows:

$$\beta \geq \frac{x^{q_1}(\tilde{a}\left(\frac{p_1}{q_1}\right)) - x_c}{m_1} = \frac{v}{M_1}. \tag{40}$$

Also, we have

$$\frac{x^{q_1}(a_0\left(\frac{p_2}{q_2}\right)) - x^{q_1}(\tilde{a}\left(\frac{p_1}{q_1}\right))}{M_1} \leq \alpha \leq \frac{x^{q_1}(a_0\left(\frac{p_2}{q_2}\right)) - x^{q_1}(\tilde{a}\left(\frac{p_1}{q_1}\right))}{m_1},$$

and after notation

$$u = x^{q_1}(a_0\left(\frac{p_2}{q_2}\right)) - x^{q_1}(\tilde{a}\left(\frac{p_1}{q_1}\right)) \text{ (see Figure 6)},$$

we obtain

$$\frac{u}{M_1} \leq \alpha \leq \frac{u}{m_1}. \tag{41}$$

Now, let us write

$$\alpha \leq \frac{x^{q_2}(a_0\left(\frac{p_2}{q_2}\right)) - x^{q_2}(\tilde{a}\left(\frac{p_1}{q_1}\right))}{m_2}.$$

Since

$$x^{q_2}(a_0\left(\frac{p_2}{q_2}\right)) = x_c,$$

this estimate with notation

$$w = x^{(1)}_{\mathrm{cr}} - x^{q_2}(a_0\left(\frac{p_2}{q_2}\right)) \text{ (see Figure 6)}$$

yields

$$\alpha \leq \frac{w}{m_2} \tag{42}$$

Now let us notice that the following ordering takes place:

$$x^{q_2}(\tilde{a}\left(\frac{p_1}{q_1}\right)) < x_c < x^{q_1}(\tilde{a}\left(\frac{p_1}{q_1}\right)) < x^{q_1}(a_0\left(\frac{p_2}{q_2}\right)).$$

Suppose that p_1/q_1 is obtained from some Farey interval $(p_2/q_2, p_0/q_0)$, i.e. $q_1 = q_0 + q_1$, $p_1 = p_0 + p_2$. In this case we can use Lemma 7, where

$$\tilde{a} = \tilde{a}\left(\frac{p_1}{q_1}\right),\ p/q = p_1/q_1,\ \Delta^{(1)}_c = w,\ \Delta^{(2)}_c = v.$$

It follows from Lemma 7, that for some C we have

$$C \leq \frac{w}{v} \leq 1/C. \tag{43}$$

Combining estimates (40), (41), (42) and (43), we get the following inequalities

$$\begin{aligned}\frac{l(I(p_1/q_1))}{l(I_c(A))} &\geq \frac{\beta}{\beta+\alpha} = \frac{1}{1+\alpha/\beta} \\ &\geq \frac{1}{1+\dfrac{u}{m_1}\dfrac{M_1}{v}} = \frac{1}{1+\dfrac{M_1}{m_1}\dfrac{u}{w}\dfrac{w}{v}} \\ &\geq \frac{1}{1+\dfrac{M_1}{m_1}\dfrac{\alpha M_1}{\alpha m_2}\dfrac{w}{v}} = \frac{1}{1+\dfrac{M_1}{m_1}\dfrac{M_1}{m_2}\dfrac{w}{v}} \geq \frac{1}{1+C_1} < 1-\gamma,\end{aligned}$$

where we assumed that

$$\frac{M_1}{m_1}, \frac{M_1}{m_2} \leq \text{const } < \infty.$$

These inequalities are proven in the following Preposition 9. □

Preposition 9. *Let A be a "good " Farey interval. Then*

$$\frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{m_1}{M_2}, \frac{m_2}{M_1} \geq \text{const } > 0.$$

Proof. We use the following preposition (see [2] and [3, Preposition 6.3]). Denote

$$\Delta_i = \left(Tf^i_{a^{(1)}}(x_c), Tf^i_{a^{(2)}}(x_c)\right).$$

Preposition 10. *The system of intervals Δ_i, $1 \leq i \leq q = \max(q_1, q_2)$, covers any point of the circle at most twice.*

Corollary 3. $\sum_{i=1}^{q} l(\Delta_i) \leq 2$.

Let $\bar{q} \leq q = \max(q_1, q_2)$, $a \in I(A)$. It follows from (52) that

$$\frac{\partial}{\partial a} x^{\bar{q}}(a) = \sum_{i=0}^{\bar{q}-1} \prod_{j=i+1}^{\bar{q}-1} f'_a(x^j(a)), \tag{44}$$

where

$$x^j(a) = Tf^j_a(x_c).$$

Consider one of the summands in (44),

$$B^q_i(a) = \prod_{j=i+1}^{\bar{q}-1} f'_a(x^j(a))$$

Lemma 9. *There exists constant C such that for any a_1, $a_2 \in \bar{I}(A)$ we have*

$$\frac{1}{C} \leq \frac{B^{\bar{q}}_i(a_1)}{B^{\bar{q}}_i(a_2)} \leq C$$

Proof. It is easy to see that $x_c \notin \Delta_j$, $1 \le j \le q$. Hence,

$$\left|\ln B_i^{\bar q}(a_1) - \ln B_i^{\bar q}(a_2)\right| \le \left|\sum_{j=i}^{\bar q-1}(\ln f'(x_j(a_1)) - \ln f'(x_j(a_2)))\right|$$
$$\le \text{const} \max_{y\in S^1}\left|\frac{f''(y)}{f'(y)}\right| \sum_{j=i}^{\bar q} l(\Delta_j) \le 2\max_{y\in S^1}\left|\frac{f''(y)}{f'(y)}\right| = \ln C.$$

Here we used Corollary 3 and $(\ln f_a'(y))' = f_a''(y)/f_a'(y)$. □

Therefore, Lemma 9 implies that m_1/M_1, $m_2/M_2 \ge 1/C$. Now let us compare $\dfrac{\partial x^{q_1}(a)}{\partial a}$ with $\dfrac{\partial x^{q_2}(a)}{\partial a}$. Assume that $q = \max(q_1, q_2) = q_2$. Using (49) for $f(a) = f_a^{q_2-q_1}(x_c)$, $g(a) = f_a^{q_1}(x_c)$, we obtain

$$\frac{\partial}{\partial a}x^{q_2}(a) = \left(\prod_{j=q_1}^{q_2-1} f_a'(x^j(a))\right)\frac{\partial}{\partial a}x^{q_1}(a) + \frac{\partial}{\partial a}T f_a^{q_2-q_1}(x_0)\Big|_{x_0=x^{q_1}(a)}. \tag{45}$$

Applying Corollary 1 to the Farey interval $(p_1/q_1, (p_2-p_1)/(q_2-q_1))$, we get

$$e^{-2v} \le \prod_{j=q_1}^{q_2-1} f_a'(x^j(a)) \le e^{2v},\ a \in I(A).$$

It follows from (45), that

$$\frac{\partial x^{q_2}(a)/\partial a}{\partial x^{q_1}(a)/\partial a} \ge \left(\prod_{j=q_1}^{q_2-1} f_a'(x^j(a))\right) \ge e^{-2v}$$

and $m_2/M_1 \ge e^{-2v}/C$. The following estimates could be obtained like Lemma 9.

$$\frac{1}{C} \le \frac{(\partial/\partial a)T f_a^{q_2-q_1}(x_0)|_{x_0=x^{q_1}(a)}}{(\partial/\partial a)x^{q_2-q_1}(a)} \le C,\ a \in I(A). \tag{46}$$

As one can see, in order to prove (46), one have to consider

$$\bar B_i^{\bar q}(a) = \prod_{j=i+1}^{\bar q-1} f'(x_0^j(a)),$$

where $x_0^j(a) = T_a^{q_1}x^j(a)$.

Lemma 10. *There exists constant C such that for any a_1,$a_2 \in \bar I(A)$ we have*

$$\frac{1}{C} \le \frac{\bar B_i^{\bar q}(a_1)}{\bar B_i^{\bar q}(a_2)} \le C.$$

Proof.

$$|\ln \bar{B}_i^{\bar{q}}(a_1) - \ln \bar{B}_i^{\bar{q}}(a_2)| \le |\sum_{j=i}^{\bar{q}} (\ln f'(x_0^j(a_1)) - \ln f'(x^j(a_2)))|$$
$$\le |\sum_{j=i}^{\bar{q}} (\ln f'(T_{a_1}^{q_1} x^j(a_1)) - \ln f'(x^j(a_1))|$$
$$+ |\sum_{j=i}^{\bar{q}} (\ln f'(x^j(a_1)) - \ln f'(x^j(a_2)))|.$$

The second sum in r.h.s is estimated as in Lemma 9, and the first sum is bounded by constant, since the total length of intervals $[x^j(a_1), T_{a_1}^{q_1} x^j(a_1)]$, $j = 0, \dots \bar{q}$, do not exceed 1, i.e., the finite length of the circle. □

Using (49) for $f(a, \cdot) = f_a^{2q_1 - q_2}(\cdot)$, $g(a) = f_a^{q_2 - q_1}(x_c)$, we obtain

$$\frac{\partial}{\partial a} x^{q_1}(a) = \left(\prod_{j=q_2-q_1}^{q_1-1} f_a'(x^j(a)) \right) \frac{\partial}{\partial a} x^{q_2-q_1}(a) + \left. \frac{\partial}{\partial a} T f_a^{2q_2-q_1}(x_0) \right|_{x_0 = x^{q_2-q_1}(a)}. \quad (47)$$

We have

$$\frac{(\partial/\partial a) x^{q_1}(a)}{(\partial/\partial a) x^{q_2-q_1}(a)} \ge \prod_{j=q_2-q_1}^{q_1-1} f_a'(x^j(a)) \ge e^{-3v},\ a \in I(A), \quad (48)$$

where we applied [3, Preposition 3.4, p. 65] to the Farey interval $((2p_1 - p_2)/(2q_1 - q_2), (p_2 - p_1)/(q_2 - q_1))$. Using (46) and (48) we obtain

$$\left. \frac{\partial}{\partial a} T f_a^{q_2-q_1}(x_0) \right|_{x_0 = x^{q_1}(a)} \le C \frac{\partial}{\partial a} x^{q_2-q_1}(a) \le C e^{3v} \frac{\partial}{\partial a} x^{q_1}(a).$$

Substituting last estimate to (45) we obtain

$$\frac{\partial}{\partial a} x^{q_2}(a) \le (e^{2v} + e^{3v} C) \frac{\partial}{\partial a} x^{q_1}(a).$$

Let $\bar{a}$ be the value such that $\left. \dfrac{\partial x^{q_1}(a)}{\partial a} \right|_{a=\bar{a}}$ is minimal. Combining our estimates, we obtain

$$\frac{m_1}{M_2} = \frac{(\partial/\partial a) x^{q_1}(\bar{a})}{(\partial/\partial a) x^{q_2}(\bar{a})} \cdot \frac{(\partial/\partial a) x^{q_2}(\bar{a})}{M_2} \ge \left(\frac{1}{e^{2v} + C e^{3v}} \right) \frac{1}{C} \quad \square$$

Appendix

Consider function $f(x,y)\in C^1(\mathbb{R}\times\mathbb{R})$ and $g(y)\in C^1(\mathbb{R})$. Denote

$$f_x(x,y)=\frac{\partial}{\partial z}f(z,y)\bigg|_{z=x}\text{ and }f_y(x,y)=\frac{\partial}{\partial a}f(x,a)\bigg|_{a=y}.$$

Note that,

$$\frac{\partial}{\partial y}f(g(y),y)=f_x(g(y),y)g'(y)+f_y(g(y),y). \tag{49}$$

Denote by $f^n(x,y)$ the nth iteration of function f with respect to variable x: $f^n(x,y)=f(f(\dots f(f(x,y),y),\dots,y),y))$.

Lemma 11. *For any $n\in\mathbb{N}$*

$$\frac{\partial}{\partial y}f^n(g(y),y)=\sum_{i=1}^{n}f_y(x_i(y),y)\prod_{j=i+1}^{n}f_x(x_j(y),y)+g'(y)\prod_{j=1}^{n}f_x(x_j(y),y), \tag{50}$$

where $x_j(y)=f^{j-1}(g(y),y)$, $x_1(y)=g(y)$.

Here and later the product with low limit higher than upper limit is equal to 1.

Proof. It follows from (49) that the statement is true for $n=1$. By induction, suppose that (50) is true for $n=k-1$. To obtain this formula for $n=k$ we substitute $g(y)\mapsto f(g(y),y)$, $x_j(y)\mapsto x_{j+1}(y)$ in (50) for $n=k-1$.

$$\begin{aligned}\frac{\partial}{\partial y}f^k(g(y),y)&=\frac{\partial}{\partial y}f^{k-1}(f(g(y),y),y)\\
&=\sum_{i=1}^{k-1}f_y(x_{i+1}(y),y)\prod_{j=i+1}^{k-1}f_x(x_{j+1}(y),y)+\frac{\partial}{\partial y}f(g(y),y)\bigg(\prod_{j=1}^{k-1}f_x(x_{j+1}(y),y)\bigg)\\
&=\sum_{i=2}^{k}f_y(x_i(y),y)\prod_{j=i+1}^{k}f_x(x_j(y),y)\\
&\qquad+\big(f_x(g(y),y)g'(y)+f_y(g(y),y)\big)\bigg(\prod_{j=2}^{k}f_x(x_j(y),y)\bigg)\\
&=\sum_{i=1}^{k}f_y(x_i(y),y)\prod_{j=i+1}^{k}f_x(x_j(y),y)+g'(y)\prod_{j=1}^{k}f_x(x_j(y),y).\end{aligned}$$

Here we use (49). □

Lemma 12. *For any $k\in\mathbb{N}$*

$$\frac{\partial}{\partial y}f^k(x,y)=\sum_{i=0}^{k-1}f_y(x_i(y),y)\prod_{j=i+1}^{k-1}f_x(x_j(y),y). \tag{51}$$

where $x_j(y)=f^j(x,y)$, $x_0(y)=x$.

Proof. Fix x and put $g(y) = f(x, y)$, $n = k - 1$ in (50). □

It follows from (51) that

$$\frac{\partial}{\partial a} f_a^n(x) = \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} f_a'(x_j(a)), \tag{52}$$

where

$$x_j(a) = f_a^j(x).$$

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