Stability of Data Networks: Stationary and Bursty Models

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Abstract

This paper studies stability of network models that capture macroscopic features of data communication networks including the Internet. The network model consists of a set of links and a set of possible routes which are fixed subsets of links. A connection is dynamically established along one of the routes to transmit data as requested, and terminated after the transmission is over. The transmission bandwidth of a link is dynamically allocated, according to specific bandwidth allocation policy, to ongoing connections that traverse the link. A network model is said to be stable under a given bandwidth allocation policy if, roughly, the number of ongoing connections in the network will not blow up over time. We consider a stationary and a bursty network model; The former assumes stochastically stationary arrival processes of connections as did in many theoretical studies, while the latter allows more realistic bursty and correlated arrival processes. For both models under a necessary stability condition (i.e., the average offered transmission load on each link is within its bandwidth capacity), we show that the proportionally fair, the minimum potential delay, the max-min fair and a class of utility maximizing bandwidth allocation policies ensure network model stability, while some priority oriented and the maximum throughput policies do not. Interestingly, the bandwidth allocation policy that maximizes the $\arctan(\cdot)$ utility ensures the stability of the stationary model but *not* the bursty model. This raises a serious concern about the current practice in the Internet protocol design, since such a policy is thought as a good approximation of a most widely used TCP in the Internet.

Keywords: data network, Internet, rate control, bandwidth allocation, burstiness, stability, fluid network model, Lyapunov function.

1 Introduction

In modern data communication networks, digitized documents, like email and electronic files of texts, images, and sounds, are transmitted from one node to another traversing a series of transmission links. An abstract model of these data networks consists of a set of transmission *links*, and a set of possible *routes* with each route traversing a fixed subset of links. Each link, which may be traversed by several routes simultaneously, has a transmission *bandwidth* *capacity* that specifies the maximum data transmission rate through the link. When a request arrives for the transmission of a certain amount of data through a route, a *connection* (also called a *session* (Bertsekas and Gallager 1992) or a *flow* (Kelly 1997) with slight variations in meaning) is established along the route; and when the transmission is over, the connection is terminated. In this paper, we assume that the routing of connections in the network is determined according to some given protocol. The amount of data to be transmitted is referred to as the volume of connections (similarly referred to as session length or flow volume for the other terminologies). Several connections may be ongoing along the same route. All connections passing one link at the same time share the bandwidth of the link for their data transmissions. The bandwidth allocation follows a pre-determined *bandwidth allocation/sharing policy*, such as the maximum throughput, the proportionally fair, the minimum potential delay and the max-min fair policies, etc. (We will briefly review the bandwidth allocation policies studied analytically in the available literature later.) The dynamics of establishing and terminating connections is recorded by the ongoing connection process $N(t)$, which is a vector with each component being the number of ongoing connections on a route at time t. Given a bandwidth allocation policy, the network model is said to be stable if, roughly speaking, the ongoing connection process $N(t)$ will not blow up over time. (More precise definitions of stability will be given after we specify the network model.)

The network model stability is affected by the average load of the connection arrival processes and their variabilities. A necessary condition, called the *normal offered load condition*, is that the average offered transmission load on each link is within its bandwidth capacity. However, this is not sufficient for the stability as shown by Bonald and Massoulie (2001) with a counterexample. In the example, the connection arrival processes are Poisson and the bandwidth allocation policy gives priority to certain connections (and consequently the maximum throughput is achieved). Then, even under the normal offered load condition, the total number of ongoing connections in the network will grow unboundedly. This raises a fundamental question on the analysis and the design of the data transmission control for data networks: given a bandwidth allocation policy, is the network stable under the normal offered load condition?

For this question, some recent results can be found in Massoulie and Roberts (2000), de Veciana, *et al.* (2001), Fayolle, *et al.* (2001), Bonald and Massoulie (2001) and Ye (2003). In these works, it is assumed that connection arrivals to the network follow Poisson processes and connection volumes are exponentially distributed. In de Veciana, *et al.* (2001), it is proved that the network is stable under the normal offered load condition for the (weighted) max-min fair and the proportionally fair bandwidth allocation policies. Fayolle, *et al.* (2001) later provided a simplified proof for the max-min fair policy. Bonald and Massoulie (2001) proved the stability result for a class of (p, α)-proportionally fair bandwidth allocation policies that maximize some power utility functions. Included in the class are the proportionally fair and the minimum potential delay bandwidth allocations. This result is further extended by Ye (2003) to a broader class of bandwidth allocation policies that maximize some general utility functions, which with an additional property (i.e., the partial radial homogeneity property defined later) are equivalent to U-utility maximizing policies in this paper. This class of allocation policies include the (p, α)-proportionally fair allocation as a special case, as well as the allocation that maximizes the $\arctan(\cdot)$ utility function. The latter allocation is thought to be a good approximation to the bandwidth allocation imbedded in a type of widely used TCP (Transmission Control Protocol) in the Internet (see Kelly (2001) and Low (2002)).

The assumptions that connection arrivals follow Poisson processes and that connection volumes are exponentially distributed are convenient for theoretical analysis, but has been questioned for its realism (Paxson and Floyd (1995)). In the real data networks, such arrivals may be seriously correlated and bursty, which is often hard to be analyzed using a conventional stochastic model. To model the bursty phenomenon in data arrivals, instead of assuming stochastic stationary arrival processes, Cruz (1991a, b) introduced bursty models for data communication networks, which were later extended by Borodin, *et al.* (2001) to queueing networks. It is claimed that stability results and performance bounds derived for bursty models are more robust in real world applications. These studies on the modelling of the bursty phenomenon provide us another approach to study the stability of the data network using a bursty network model, which can be viewed as a compliment, rather than an opposition, to the stationary model. In fact, as shown by our results, the stability of the network is indeed affected by the burstiness of the arrival processes.

The aim of our work is to extend the stability results to more realistic data network models. As a first step, we relax the assumption of Poisson arrivals and consider a stationary stochastic model with general stationary renewal arrival processes. But *the connection volumes are still assumed exponentially distributed*. Then we consider a bursty model of the network which admits bursty arrivals of data transmission loads. As discussed above, this could be a useful step towards more realistic modelling and analysis of data networks. For both of the stationary and bursty network models, we investigate the impact of all the above mentioned bandwidth allocation policies on the network stability. In particular, we prove that the proportionally fair, minimum potential delay, max-min fair, (p, α)-proportionally fair, and U-utility maximizing bandwidth allocation policies ensure the stability for both stationary and bursty data network models provided that the normal offered load condition is satisfied. However, some priority based bandwidth allocation policies and the maximum throughput bandwidth allocation policy may lead to instability of the network models even when the normal offered load condition is satisfied. The most interesting case is concerned with the arctan-utility maximizing policy. It has been proven in Ye (2003) that, under the normal offered load condition, this policy ensures stability of the stationary network model with Poisson arrivals of connections. However, it is shown in this paper that the policy may cause instability of the bursty network model even under the normal offered load condition. Since the arctan-utility maximizing policy is thought to be a good approximation of a type of most widely used TCP in the Internet, this raises a serious concern about the the current practice in the Internet protocol design. We note that its stability for general stationary stochastic models is left open for further investigation.

Now we briefly review some bandwidth allocation policies that are widely discussed in the literature and investigated in this paper. (The rigorous definitions of these policies in the context of our study are presented in Section 2.) The classical max-min allocation policy, discussed in Bertsekas and Gallager (1992), intends to give the greatest possible allocation to the most poorly treated connections. The proportional fairness allocation policy, proposed by Kelly (1997), seeks an allocation for which the aggregate of proportional changes with respect to any other connection is zero or negative. Such an allocation still intends to favor those poorly treated connections, but not as much as the max-min allocation. This policy is further studied and experimentally validated by Hurley, *et al.* (1999). It was generalized to (p, α)-proportional fairness allocation policies by Mo and Walrand (2000). Massoulie and Roberts (1999) discussed how to achieve the different types of bandwidth allocations by network flows and transmission control protocols. They also introduced an allocation that minimizes the potential delay which is defined as the least amount of expected delay as experienced by connections. The approach to represent bandwidth allocation policies as solutions to optimization problems that maximize certain utility functions was first taken up by Kelly (1997) and Kelly, *et al.* (1998). Kelly (2001) derived the $\arctan(\cdot)$ utility function that approximates the bandwidth allocation achieved by a type of TCP rate control protocol. Low (2002) provided a systematic procedure for deriving the utility functions corresponding to various TCP rate control protocols. A central problem in these papers is whether the bandwidth allocations under various rate control schemes (as well as active queue/buffer management) converge to some equilibria. For this problem, positive answers have been given for data networks with some commonly used TCP; That is, the bandwidth allocated to each connection converges to the solution of an optimization problem that maximizes some of the above mentioned utilities after a short transitional period, if the number of ongoing connections is fixed on all possible routes. These results provide a justification for the connection level network model in our study, where the ongoing connections are established dynamically, and the bandwidth allocation is implicitly assumed to be adapted accordingly and immediately. Such an equilibrium property can be regarded as a *microscopic* stability property of the transmission rate control and bandwidth allocation of data networks, as compared to the connection level *macroscopic* stability that we investigate in this work.

A technical novelty in our work is the use of the fluid model approach to obtain the stability results for both the stationary and bursty network models. The fluid model approach, first proposed by Rybko and Stolyar (1992) and then extended by Dai (1995), Chen (1995), Dai and Meyn (1995), Stolyar (1995) and Bramson (1998), was developed in the study of the stability of stochastic queueing networks in the past decade. An elegant result states that a queueing network is stable if its corresponding fluid model (a continuous analog of the queueing network) is stable. This approach is transplanted by Gamarnik (2000) to the study of stability of queueing systems with bursty arrivals. To apply these results, we first identify the corresponding fluid network model for the stationary or bursty data network model, then use Lyapunov function approach to prove the stability of the fluid network model, and finally prove the stability of the original data network model. Such a fluid model approach is also employed in Bonald and Massoulie (2001) to prove the stability of data networks with the (p, α)-proportionally fair bandwidth allocation under the normal offered load condition, where the existence of the corresponding fluid network model seems to be implicitly assumed. In fact, the rigorous justification of their use of the fluid model approach can be found in our paper (under the assumption that any connection has an exponentially distributed amount of data to be transmitted).

The paper is organized as follows. In next section, we describe the "infrastructure" of the data network model and various bandwidth allocation policies mentioned above. In Section 3 we focus on the stationary and the bursty data network models separately in two subsections. In each subsection, we first describe the dynamics of the (stationary/bursty) data network model, and then present the stability and instability results of the data network model with various bandwidth allocation policies. The proofs of stability results are given in Section 4, which may be skipped for readers who are only interested in the results of this paper. We conclude in Section 5.

Finally, we introduce some notation and convention that are used throughout this paper. The J-dimensional Euclidean space and its nonnegative orthant are denoted by $\mathcal{R}^J$ and $\mathcal{R}^J_+ := [0,\infty)^J$, respectively. Particularly, $\mathcal{R} = \mathcal{R}^1$ and $\mathcal{R}_+ = \mathcal{R}^1_+$. The sets of integers and nonnegative integers are denoted by $\mathcal{Z}$ and $\mathcal{Z}_+$, respectively. For set S, its cardinal is denoted as $|S|$, and it is the number of elements when S has finite elements. Vectors are understood to be column vectors. Let e be the vector with all its components being ones, and e_r be the vector with the component corresponding to index r being one and all the other components being zeros. The dimension of e and e_r should be recognized from the context. For any $x \in \mathcal{R}^J$, the norm is defined to be $||x|| = \sum_{j=1}^J |x_j|$. Let $C^J(\mathcal{R}_+)$ be the space of all continuous functions $f : \mathcal{R}_+ \to \mathcal{R}^J$. A sequence of functions, $\{f_n(t)\}$, in $C^J(\mathcal{R}_+)$, is said to converge uniformly on compact set (u.o.c.) to a function $f(t) \in C^J(\mathcal{R}_+)$, if for any $T > 0$,

$$\lim_{n\to\infty} \sup_{0\le t\le T} ||f_n(t) - f(t)|| = 0,$$

and we denote it by

$$f_n \to f, \quad u.o.c. \text{ as } n \to \infty.$$

Let operator $\partial_k f$ be a shorthand for the partial derivative of function $f(\cdot)$ with respect to its k-th variable, i.e.,

$$\partial_k f(x_1, ..., x_k, ..., x_n) = \frac{\partial f}{\partial x_k}(x_1, ..., x_k, ..., x_n).$$

For convenience, we adopt the convention

$$0 \cdot \infty \equiv 0 \text{ and } \frac{0}{0} \equiv 0. \tag{1}$$

2 Network Infrastructure and Bandwidth Allocations

Consider a data communication network with a set L of transmission links. Each link $l \in L$ has a bandwidth capacity $C_l > 0$, which is the maximum amount of data that can be transmitted through the link in unit time. Let $C = \{C_l : l \in L\}$. Define route r as a non-empty subset of L, and denote the set of all possible routes as R. A 0-1 incidence matrix $M = (M_{lr}, l \in L, r \in R)$ can be used to indicate which links are in a particular route, e.g., $M_{lr} = 1$ if link l is in route r (i.e., $l \in r$) and $M_{lr} = 0$ otherwise. We also denote $R(l) := \{r \in R : l \in r\}$ as the set of all routes that traverse the link $l \in L$. Thus, the quadruple (L, C, R, M) characterizes the "infrastructure" of the network.

In this paper,we consider those bandwidth allocation policies depending only on the number of ongoing connections in all the routes. Suppose n_r is the number of ongoing connections on route $r \in R$ and let $n = \{n_r : r \in R\}$. Then, we denote $a_r(n)$ as the bandwidth (the amount of data per unit time) allocated to each connection on route r, and $\Lambda_r(n) = n_r a_r(n)$ as the total bandwidth allocated to all connections on route r. Here we have assumed all connections on route r are treated equally, similar to the earlier literature. A bandwidth allocation $\Lambda(n) = \{\Lambda_r(n), r \in R\}$ is feasible if and only if the following conditions are satisfied (with Λ_r replaced by $\Lambda_r(n)$):

$$\sum_{r\in R(l)} \Lambda_r \le C_l, \quad \text{for } l \in L \tag{2}$$

$$\Lambda_r = 0 \text{ if } n_r = 0, \quad \text{for } r \in R \tag{3}$$

$$\Lambda_r \ge 0, \quad \text{for } r \in R. \tag{4}$$

We investigate the network model stability for the following bandwidth allocation policies. In the description of bandwidth allocation policies in the following subsections, although n_r ($r \in R$) refers to the number of ongoing connections on route r, it is straightforward to see that it can be regarded as a real number in the following definitions and thus $\Lambda(n)$ can be extended to a (vector) function in the nonnegative orthant of Euclidian space, i.e., in $\mathcal{R}_+^{|R|}$. Indeed, this is necessary in our proofs of main results later in the paper.

2.1 Maximum throughput allocation Λ^{mt}

The maximum throughput allocation, denoted by $\Lambda_r^{mt}(n)$, is characterized as *a solution* to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} \Lambda_r \\ \text{subject to} \quad & \text{feasibility conditions (2)-(4).} \end{aligned} \tag{5}$$

The solution to this optimization problem is not unique. Hence, in general, the maximum throughput fairness principle does not give a unique bandwidth allocation for a fixed set of n ongoing connections.

2.2 Arctan-utility maximizing allocation Λ^{arctan}: a TCP type allocation

In the Internet, many applications implement a type of data transmission rate control conforming to the TCP algorithm initiated by Jacobson (1988). Kelly (2001) and Low (2002) have argued that this type of allocation, denoted by $\Lambda^{arctan}(n)$, can be approximated by the unique optimal solution to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} w_r n_r \arctan\left(\frac{\Lambda_r}{w_r n_r}\right) \\ \text{subject to} \quad & \text{feasibility conditions (2)-(4),} \end{aligned} \tag{6}$$

where w_r is a positive constant.

Note that, when $n_r = 0$ and thus $\Lambda_r = 0$, the corresponding term in the objective function (6) should be dropped out. This is consistent with the convention (1), and we adopt the same convention when similar situations occur below.

2.3 Proportionally fair allocation Λ^{pp}

The proportional fair bandwidth allocation, $\Lambda^{pp}(n)$, firstly discussed in Kelly (1997), is determined by the unique optimal solution to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} n_r \log(\Lambda_r) \\ \text{subject to} \quad & \text{feasibility conditions (2)-(4).} \end{aligned} \tag{7}$$

2.4 Minimal potential delay allocation Λ^{pd}

The minimal potential delay allocation Λ^{pd} is proposed by Massoulie and Roberts (1999). It is characterized as the unique optimal solution to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} -\frac{n_r^2}{\Lambda_r} \\ \text{subject to} \quad & \text{feasibility conditions (2)-(4).} \end{aligned} \tag{8}$$

2.5 Max-min fair allocation Λ^{mm}

A bandwidth allocation is called max-min fair allocation if the bandwidth allocated to a connection cannot be increased without decreasing that of a connection having a smaller or equal allocation. As presented in Bertsekas and Gallager (1992), the max-min fair allocation $\Lambda^{mm}(n)$ assigns bandwidth to connections in a hierarchical procedure. We outline this procedure with slightly modifications in the context of our model. In the first hierarchy we increase the bandwidth of all the connections from 0 by an equal amount until some links' bandwidths are fully utilized. The amount of increment is given by $a^{(1)}(n) := \min_{l\in L} C_l / \sum_{r\in R(l)} n_r$. The set of the fully utilized links are denoted as $L_B^{(1)}$, and the set of the routes that pass through at least one of these links is denoted as $R_B^{(1)}$. We can see that for the connections on the routes in $R_B^{(1)}$, their maximum possible bandwidth allocation is $a^{(1)}(n)$, which is their allocation according to the max-min fair allocation policy. Then, we obtain a reduced network by deleting all the routes in $R_B^{(1)}$ and all the links in $L_B^{(1)}$, decreasing the capacity of the links in $L \backslash L_B^{(1)}$ by the amount $\sum_{r\in R_B^{(1)}\cap R(l)} n_r \cdot a^{(1)}(n)$ taken up by the connections on the routes in $R_B^{(1)}$, and releasing the bandwidth allocated to the connections on the remaining routes in $R \setminus R_B^{(1)}$. Repeat the above procedure in the reduced network until all the routes with ongoing connections have got their maximum possible allocation. We formalize this allocation procedure by the following algorithm.

Max-Min Algorithm:

Initialization. Let $L^{(1)} := L$, $R^{(1)} := R$, and $C_l^{(1)} := C_l$ and $R^{(1)}(l) := \{r \in R^{(1)} : l \in r\} = R(l)$ for $l \in L$; and iteration counter $k = 1$.

Step 1. Let

$$
\begin{aligned}
& a^{(k)}(n) = \min_{l\in L^{(k)}} \frac{C_l^{(k)}}{\sum_{r\in R^{(k)}(l)} n_r}; && (9)\\
& L_B^{(k)} = \{l \in L^{(k)} : \frac{C_l^{(k)}}{\sum_{r\in R^{(k)}(l)} n_r} = a^{(k)}(n)\}; \\
& R_B^{(k)} = \{r : r \in R^{(k)}(l) \text{ for some } l \in L_B^{(k)}\}; \\
& \Lambda_r^{mm}(n) = n_r \cdot a^{(k)}(n), \qquad \text{for } r \in R_B^{(k)}; && (10)\\
& L^{(k+1)} := L^{(k)} \setminus L_B^{(k)}; \\
& R^{(k+1)} := R^{(k)} \setminus R_B^{(k)}; \\
& C_l^{(k+1)} := C_l^{(k)} - \sum_{r\in R_B^{(k)}\cap R^{(k)}(l)} \Lambda_r^{mm}(n), \qquad \text{for } l \in L^{(k+1)}; \\
& R^{(k+1)}(l) := \{r \in R^{(k+1)} : l \in r\}, \qquad \text{for } l \in L^{(k+1)}.
\end{aligned}
$$

Step 2. If

$$n_r = 0 \;\text{ for any } r \in R^{(k+1)}, \text{ or } R^{(k+1)} = \phi;$$

then, let

$$\Lambda_r^{mm}(n) = 0 \;\text{ for all } r \in R^{(k+1)},$$

set

$$K(n) = k,$$

and stop; else, let the iteration counter $k := k + 1$ and goto *step 1.*

Remarks 1. In the max-min algorithm, $a^{(k)}(n)$ is the bandwidth assigned to the connections passing through the kth level bottleneck links, $\Lambda_r^{mm}(n)$ the bandwidth assigned to the routes passing through the kth level bottleneck links, $L_B^{(k)}$ the set of the kth level bottleneck links, $R_B^{(k)}$ the set of the routes passing through the kth level bottleneck links, $L^{(k)}$ the reduced set of the network links with all the higher level (from the 1st to $(k-1)$st level) bottleneck links deleted, $R^{(k)}$ the set of the routes with all the routes passing through higher level bottleneck links deleted, $C_l^{(k)}$ the remaining bandwidth of link $l \in L^{(k+1)}$ after deleting all the higher level bottleneck links and routes, and $R^{(k)}(l)$ the set of all the routes in $R^{(k)}$ that traverse link $l \in L^{(k)}$.

2. Since at least one bottleneck link is found after each iteration, the max-min algorithm stops after finite iterations. The number of the iterations, $K(n)$, is equal to the number of the hierarchical levels of bottleneck links. □

2.6 (p, α)-proportionally fair allocation Λ^α

First introduced by Mo and Walrand (2000), the class of (p, α)-proportionally fair bandwidth allocations $\Lambda^\alpha(n)$, is characterized as the unique optimal solution to the following optimization problem for any given number α $(\alpha > 0, \alpha \neq 1)$:

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} p_r n_r^\alpha \frac{\Lambda_r^{1-\alpha}}{1-\alpha}, \\ \text{subject to} \quad & \text{feasibility conditions (2)-(4)}, \end{aligned} \tag{11}$$

where p_r, $r \in R$, are fixed parameters. It is pointed out by Bonald and Massoulie (2001) that, assuming $p_r = 1$ for all $r \in R$ and given a fixed n, the bandwidth allocation $\Lambda^\alpha(n)$ approaches (or is equal to) a maximum throughput allocation $\Lambda^{mt}(n)$, the proportionally fair allocation $\Lambda^{pp}(n)$, the minimal potential delay allocation $\Lambda^{pd}(n)$ and the max-min fair allocation $\Lambda^{mm}(n)$ when $\alpha \to 0$, $\alpha \to 1$, $\alpha = 2$ and $\alpha \to \infty$, respectively.

2.7 U-utility maximizing allocation Λ^U

One motivation of our work is to study the stability and performance of the TCP Internet. As far as we understand, to some extents, any utility maximizing bandwidth allocation policy is an approximation of the actual bandwidth allocation in the TCP Internet. Thus, it is worth to study a more generic class of utility maximizing allocation policies so that the stability results would be more robust with respect to uncertainties in approximating the TCP bandwidth allocation. Technically, the generic utility maximizing policy, called U-utility maximizing policy, also leads to a unified treatment for the stability problem of the proportionally fair, minimum potential delay and (p, α)-proportionally fair allocation policies.

The U-utility maximizing allocation, denoted by $\Lambda^U(n)$, is characterized as the unique optimal solution to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{r \in R} U_r(n_r, \Lambda_r), \\ \text{subject to} \quad & \text{feasibility conditions (2)-(4)}, \end{aligned} \tag{12}$$

where the utility functions $U_r(n_r, \Lambda_r), r \in R$, defined on $\mathcal{R}_+^2$, are second order differentiable on $\mathcal{R}_+ \times (0, \infty)$ and satisfy the following assumptions

$$U_r(0, \Lambda_r) = 0, \quad \text{for } \Lambda_r > 0, \tag{13}$$

$$\partial_2 U_r(0, \Lambda_r) = 0, \quad \text{for } \Lambda_r > 0, \tag{14}$$

$$\partial_2 U_r(n_r, \Lambda_r) > 0, \quad \text{for } n_r, \Lambda_r > 0, \tag{15}$$

$$\partial_1 \partial_2 U_r(n_r, \Lambda_r) > 0, \quad \text{for } n_r, \Lambda_r > 0, \text{ and} \tag{16}$$

$$U_r(n_r, \cdot) \text{ is strictly concave for fixed } n_r > 0. \tag{17}$$

In addition, we need a partial radial homogeneity property on allocation Λ^U to help us prove the main results. That is,

$$\Lambda_r^U(cn) = \Lambda_r^U(n), \quad \text{for any } r \in R \text{ with } n_r > 0 \text{ and any } c > 0. \tag{18}$$

Remarks. 1. Assumptions (13)-(15) are intuitively appealing. The assumption (16) can be explained as follows. Consider pairs (x_r^1, Λ_r) and (x_r^2, Λ_r) with $x_r^1 < x_r^2$, i.e., more connections share the bandwidth Λ_r in the latter case. Then, $\partial_2 U_r(x_r^1, \Lambda_r) \leq \partial_2 U_r(x_r^2, \Lambda_r)$ (by assumption (16)) merely says that increasing the bandwidth is more rewarding in the latter case. The assumption (17) on concavity simply says that adding an extra bandwidth is more beneficial when the allocated bandwidth is small than when it is large.

2. The assumption (18) on partial radial homogeneity is a technical requirement, which is not as restrictive as it appears at the first sight. It can be verified that the utility functions for bandwidth allocations Λ^{pp}, Λ^{pd} and Λ^{α} satisfy all assumptions (13)-(18). Hence, all these bandwidth allocations are the special cases of Λ^U.

3. The partial radial homogeneity property is not satisfied for the *arctan*-utility maximizing bandwidth allocation Λ^{arctan}. Therefore, Λ^{arctan} is not a special case of U-utility maximizing policy Λ^U. However, the class of bandwidth allocation policies studied in Ye (2003) is in fact a class of U-utility maximizing policies without satisfying the partial radial homogeneity property and thus includes the *arctan*-utility maximizing bandwidth allocation as a special case.

4. For max-min fair policy Λ^{mm}, the partial radial homogeneity is implied by the max-min algorithm. However, this policy does not satisfy the assumption (17) on concavity, and thus is not a special case of U-utility maximizing policy Λ^U. □

3 Network Models and Their Stability

In real data networks, connections for data transmissions are established and terminated dynamically. Traditionally, establishment or arrival processes of connections are modelled as independent stationary renewal processes, for example, independent Poisson processes. Such stationary models ease the theoretical analysis, and provide acceptable approximations to real world situations. However, arrival processes are often correlated and bursty, which can affect the network performance significantly, and often cannot be ignored. Another approach to model arrival processes is to use the bursty model introduced in Cruz (1991a, b). We adopt the two approaches in the paper and propose two complementary models for the data network, namely the stationary network model and the bursty network model.

In the following two subsections, we describe the two network models in detail and present their instability examples and stability results under various bandwidth allocation policies. The proofs are left to Section 4. The concept of stability is also defined precisely for both models.

Roughly speaking, the stability of the stationary model is defined as positive Harris recurrence of the underlying Markov process that captures the dynamics of the model, while the stability of the bursty model is define as the boundedness of the total unfinished transmission workload remaining in the network. Such definitions are consistent with the definitions of stability of stochastic queueing networks (e.g., Dai 1995) and bursty queueing networks (e.g., Gamarnik 2000) in the literature.

3.1 Stationary Network Model

In the stationary network model, the connection arrivals to route $r \in R$ form independent stationary renewal processes with mean arrival rate λ_r. For route $r \in R$, denote the interarrival time between the $(i-1)$st and the ith connections as $u_r(i)$, and the amount of data to be transmitted by the ith connection as $v_r(i)$ $(i = 1, 2, ...)$. We assume that $u_r(i)$ $(i \geq 2)$ are i.i.d. random variables with mean $1/\lambda_r$ while the first arrival time $u_r(1)$ is the residual arrival time and follows the equilibrium distribution; and that $v_r(i)$ are i.i.d. exponentials with mean ν_r. We also need two technical conditions on $u_r(i)$, an unbounded condition and a spread out condition, which are, respectively,

$$P\{u_r(1) \geq x\} > 0, \quad \text{for any } x > 0, \tag{19}$$

and there exist some integer j_r and some function $p_r(x) \geq 0$ for $x \geq 0$ with $\int_0^\infty p_r(x)dx > 0$ such that

$$P\left\{a \leq \sum_{i=1}^{j_r} u_r(i) \leq b\right\} \geq \int_a^b p_r(x)dx, \quad \text{for any } 0 \leq a < b. \tag{20}$$

[These two conditions are used when we apply the fluid model approach to prove the stability result for the stationary network model in Section 4.2 and can be relaxed by introducing the concept called the petite set; see Bramson (1998).] Let $\lambda = \{\lambda_r : r \in R\}$ and $\nu = \{\nu_r : r \in R\}$, then the stationary stochastic model of the data network is represented by the sixtuple $(L, C, R, M, \lambda, \nu)$. In this model, the average offered transmission load to route $r \in R$ in terms of the amount of data per unit time is $\rho = (\rho_r)$ with

$$\rho_r = \lambda_r \nu_r. \tag{21}$$

Given a state dependent bandwidth allocation policy $\Lambda(\cdot)$, the dynamics of the stationary model can be captured by a Markov process. To describe the Markov process, we first define an $|R|$-dimensional *ongoing connection process* $N(t) = \{N_r(t) : r \in R\}$ with $N_r(t)$ being the number of on-going connections on route $r \in R$ at time t. If the connection arrival processes are Poisson processes, $N(t)$ is a continuous time Markov chain with transition rates given by

$$q(n, n') = \begin{cases} \lambda_r & \text{for } n' = n + e_r, \\ \nu_r^{-1}\Lambda_r(n) & \text{for } n' = n - e_r, n_r \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

for $n, n' \in \mathcal{Z}_+^{|R|}$. However, for general stationary renewal arrival processes, it is necessary to introduce a finer structure in order to capture the network dynamics by a Markov process. Let $U_r(t)$ $(r \in R, t \geq 0)$ be the remaining time before the next connection arrival on route r at time t, and $V_r(i, t)$ $(i = 1, \cdots, N_r(t), r \in R, t \geq 0)$ be the amount of data of ith connection on route

r that has not been transmitted at time t. (One may envision that ongoing connections on a route are lined up in the order of arrival.) Denote $V_r(t) = (V_r(1,t), \cdots, V_r(N_r(t),t), 0, 0, \cdots)$. Then, $(N(t), U(t), V(t)) = \{(N_r(t), U_r(t), V_r(t)) : r \in R\}$ is a *strong* Markov process (in the state space $(\mathcal{Z}_+ \times \mathcal{R}_+ \times \mathcal{R}_+^\infty)^{|R|}$) describing the dynamics of the data network. Readers are referred to Dai (1995) or Davis (1984) for verification. The network model is said to be *stable* if the Markov process $(U(t), V(t))$ is positive Harris recurrent.

A necessary condition for stability is that the normal offered load condition holds, i.e., the average offered load to every link in the network is within the transmission bandwidth capacity of the link,

$$\sum_{r \in R(l)} \rho_r < C_l, \quad \text{for } l \in L. \tag{22}$$

The stationary network model defined here is a processor sharing (PS) system, in which the bandwidth of a link (i.e., the processor) allocated to each route is shared equally by all ongoing connections. As the analysis of a PS system is in general difficult, in stead of further studying the stability of the above stationary network model directly, we consider a stationary network model under the head-of-the-line processor sharing (HOLPS). The HOLPS system is same as the original PS system except that under the HOLPS all the bandwidth allocated to a route goes to the ongoing connection which is established first. Similar to the stationary network model under PS, it suffices to capture the network dynamics under HOLPS by a Markovian state descriptor, also denoted as $(N(t), U(t), V(t))$. We omit its mathematical detail since it is not used explicitly in the rest of the paper. Under the exponential assumption of the connection volume, the ongoing connection process under PS is equal in distribution to the ongoing connection process under HOLPS. This is because, in both systems, the rate at which ongoing connections in a route finish transmissions depends on the bandwidth allocated to the route, which in turn depends only on the number of ongoing connections on each route. Moreover, the other information in the Markov state descriptors (i.e., the residual arrival time $U(t)$ and and the residual connection transmission load $V(t)$) of these two systems are also equivalent in distribution, noting that the distribution of $U(t)$ depends only on the connection arrival, and that the distributions of $V_r(i,t)$ for both systems are the same exponential distributions given the same distribution of the ongoing connection process $N_r(t)$. Then, we can claim that the positive Harris recurrence of the HOLPS system implies the positive Harris recurrence of the PS system since they are the same in distribution. *Therefore, for ease of technical treatment, we assume that the stationary network model is a HOLPS system in the rest of the paper.*

It is useful to introduce more performance measures of the network model (under HOLPS), to gain a better understanding of its dynamics. In particular, we introduce the *transmission load process* $X(t) = \{X_r(t) : r \in R\}$, the *connection arrival process* $E(t) = \{E_r(t) : r \in R\}$, the *transmission load arrival process* $A(t) = \{A_r(t) : r \in R\}$, the *transmission process* $D(t) = \{D_r(t) : r \in R\}$, and the *connection departure process* $S(y) = \{S_r(y) : r \in R\}$. $X_r(t)$ is the immediate remaining transmission load (in terms of the amount of data) embodied in the $N_r(t)$ ongoing connections on route r at time $t \geq 0$, and is given by

$$X_r(t) = \sum_{i=1}^{N_r(t)} V_r(i,t). \tag{23}$$

$E_r(t)$ is the total number of connections that have arrived to route r during the time interval $[0,t]$ for $t \geq 0$, and is given by

$$E_r(t) = \sup\{i : U_r(0) + u_r(1) + \cdots + u_r(i) \leq t\}. \tag{24}$$

$A_r(t)$ is the total amount of transmission load embodied in all the connections that have been established at route r during time interval $[0, t]$ for $t \geq 0$, and is given by

$$A_r(t) = \sum_{i=1}^{E_r(t)} v_r(i). \tag{25}$$

$D_r(t)$ is the total amount of data that has been transmitted via route r during the time interval $[0, t]$ for $t \geq 0$, and is determined by the *bandwidth allocation process/policy* Λ as

$$D_r(t) = \int_0^t \Lambda_r(N(s))ds. \tag{26}$$

$S_r(y)$ is the number of route r connections that have completed transmission if the amount of data that has been transmitted via route r is equal to y, and *under HOLPS*, is given by

$$S_r(y) = \max\{i : v_r(1) + \cdots + v_r(i) \leq y\}. \tag{27}$$

Thus, $S_r(D_r(t))$ is equal to the number of route r connections that have completed transmission up to time t. Processes X, N, A, E, D and S are related by the following data flow balance equations

$$X_r(t) = X_r(0) + A_r(t) - D_r(t), \quad \text{and} \tag{28}$$

$$N_r(t) = N_r(0) + E_r(t) - S_r(D_r(t)), \tag{29}$$

for $t \geq 0$ and $r \in R$.

Bonald and Massoulie (2001) showed with a counterexample that, even under the normal offered load condition, the maximum throughput and some priority based allocation policies may lead to unbounded accumulation of data waiting in the network to be transmitted. Their example has a linear network infrastructure, and the connection arrivals are Poisson with each one carrying i.i.d. exponential amount of data. The maximum throughput allocation policy *unintelligently* prioritizes the connections in allocating the bandwidth on some links, and allocates insufficient bandwidth to some connections of lower priorities so that data volume accumulate unboundedly on these lower priority connection routes. As this example strongly stimulated our work, for emphasis we outline a simplified version as follows.

Example 1. (*Instability of maximum throughput and priority based allocation policies*) Consider a network with two links $L = \{l_1, l_2\}$ with $C_{l_1} < C_{l_2}$, and two routes $R = \{r_1, r_2\}$, where route r_1 traverses both links and route r_2 traverses link l_2 only; see Figure 1. Suppose that the connections on route r_2 are given higher priority than those on route r_1 in the bandwidth allocation at the shared link l_2. Such a policy ensures pathwise maximization of the throughput. Assuming the connection arrival processes are Poisson, then the dynamics of this network is captured by the ongoing connection process $N(t)$, which is a continuous time Markov chain with transition rates

$$q(n, n') = \begin{cases} \lambda_r & \text{for } n' = n + e_r, \\ \nu_{r_2}^{-1} C_{l_2} & \text{for } n' = n - e_{r_2}, n_{r_2} \geq 1, \\ \nu_{r_1}^{-1} C_{l_1} & \text{for } n' = n - e_{r_1}, n_{r_2} = 0, n_{r_1} \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It can be seen that the necessary and sufficient condition for the positive recurrence of the Markov chain $N(t)$ is

$$\rho_{r_1} < (1 - \frac{\rho_{r_2}}{C_{l_2}})C_{l_1} \quad \text{and} \quad \rho_{r_1} + \rho_{r_2} < C_{l_2},$$

which is strictly stronger than the normal offered load condition for this network, which is

$$\rho_{r_1} < C_{l_1} \quad \text{and} \quad \rho_{r_1} + \rho_{r_2} < C_{l_2}.$$

. □

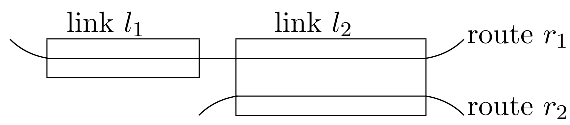

Figure 1: A network with two links and two routes

On the other hand, de Veciana, *et al.* (2001), Fayolle, *et al.* (2001), Bonald and Massoulie (2001), and Ye (2003) proved, under the normal offered condition (22), stability of the stationary stochastic network model in which the connection arrival process $E(t)$ is Poisson and $v_r(n)$ (the amount of data to be transmitted) are i.i.d. exponentials for the bandwidth allocation policies Λ^{arctan}, Λ^{pp}, Λ^{pd}, Λ^{mm}, Λ^{α}, and Λ^{U}. In fact, Ye (2003) does not require the assumption (18) on partial radial homogeneity to establish the stability result for the U-utility maximizing policy.

The theorem below extends these stability results to our more general stationary stochastic network model. One remark we wish to emphasize is that the theorem does not include the *arctan*-utility maximizing policy.

Theorem 3.1 *Suppose the normal offered load condition (22) is satisfied for the stationary stochastic network model* $(L, C, R, M, \lambda, \nu)$*. Then, the bandwidth allocations* Λ^{pp}*,* Λ^{pd}*,* Λ^{mm}*,* Λ^{α} *and* Λ^{U} *ensure the stability of the model.*

For completeness and convenience of comparison, we state below the stability results in Ye (2003) on the *arctan*-utility and U-utility maximizing allocations.

Theorem 3.2 *Suppose the normal offered load condition (22) is satisfied for the stationary stochastic network model* $(L, C, R, M, \lambda, \nu)$ *with the connection arrival process* $A(t)$ *being Poisson. Then, the bandwidth allocations* Λ^{arctan} *and* Λ^{U} *(with the assumption (18) ignored) ensure the stability of the model.*

3.2 Bursty Network Model

In the bursty model, the transmission loads are injected by "an adversary" (a vivid term introduced by Borodin, *et al.* 2001) to the network with no stationarity concerns, and their arrivals to the different routes may be correlated. In this case, we cannot hope for a stationary stochastic model to characterize the network dynamics. Instead, we consider a specific path realization of the network, and capture the dynamics information using three *deterministic* processes, namely, the *transmission load process* $X(t) = \{X_r(t) : r \in R\}$, the *transmission load arrival process* $A(t) = \{A_r(t) : r \in R\}$, and the *transmission process* $D(t) = \{D_r(t) : r \in R\}$. The meanings of these processes are essentially the same as those for the stationary network model but with slightly technical differences, and they are also related by the following flow balance equation:

$$X_r(t) = X_r(0) + A_r(t) - D_r(t), \quad \text{for } t \geq 0, r \in R, \tag{30}$$

which is the same as the balance equation (28) for the stationary model.

To better understand the bursty network model and relate it to the stationary network model, we also define an *ongoing connection process* $N(t) = \{N_r(t) : r \in R\}$ with $N_r(t)$ given by

$$N_r(t) = \frac{X_r(t)}{\nu_r}, \quad \text{for } t \geq 0, r \in R, \tag{31}$$

which approximately represents the number of connections on route r. Such an approximation would be justified for backbone networks with large numbers of ongoing connections, and the average transmission load of connections on the route $r \in R$ is ν_r. We use $x = \{x_r : r \in R\}$ and $n = \{n_r = x_r/\nu_r : r \in R\}$ interchangeably to represent the generic state of the bursty network model.

Similar to those in Cruz (1991a, b) and Borodin, *et al.* (2001), we assume that the arrival process $A(t)$ need not have regularity properties like continuity or differentiability, but just satisfies the following *bursty constraint*:

$$0 \leq A_r(t) - A_r(s) \leq \rho_r(t-s) + w, \quad \text{for } t \geq s \geq 0, r \in R, \tag{32}$$

where ρ_r is a constant in units of data amount per unit time, and w is a constant in units of data amount. ρ_r can be viewed as the average offered transmission load to route r. We define $\lambda_r = \rho_r/\nu_r$ and call it the average arrival rate of connections to route $r \in R$. Denote $\rho = \{\rho_r : r \in R\}$, $\nu = \{\nu_r : r \in R\}$ and $\lambda = \{\lambda_r : r \in R\}$.

The transmission process $D(t)$ is determined by the chosen state dependent bandwidth allocation policy $\Lambda(\cdot)$. In fact, we have

$$D_r(t) = \int_0^t \Lambda_r(N(s))ds \quad \text{for } t \geq 0, r \in R, \tag{33}$$

where $\Lambda(N(t)) = \{\Lambda_r(N(t)) : r \in R\}$ with $\Lambda_r(N(t))$ being the bandwidth or transmission rate allocated to route r at time t when the network state is $n = N(t)$.

In summary, the bursty model is represented by the octuple $(L, C, R, M, \lambda, \nu, \rho, w)$. After the arrival process $A(t)$ and the bandwidth allocation $\Lambda(\cdot)$ are specified, its dynamics is characterized by relations (30)-(33). The busty network model is said to be *stable* if for any $X(0)$ there exists a constant $M_{X(0)}$ such that

$$\sup_{t \geq 0} ||X(t)|| \leq M_{X(0)}.$$

In other words, it is stable if the transmission load process $X(t)$ (or equivalently the ongoing connection process $N(t)$) is bounded for any given initial state $X(0)$ (or $N(0)$). We wish to caution that, since $N(t)$ is only an approximation to the actual number of ongoing connections and since the bandwidth allocation in this model in fact depends on the workload $X(t)$, the examples or the stability result we present below *cannot* be taken to explicitly imply the infiniteness or finiteness of the actual number of ongoing connections in the network. Thus, readers should keep in mind that our bursty network model should be regarded as the *workload approximation* of a connection level model.

It is a necessary condition for the stability of the bursty network model that the normal offered load condition (22) holds, the same as that for the stochastic network model. Again we will show by counterexamples that the normal offered load condition is not sufficient for a bursty network model to be stable if the bandwidth allocation policy is not chosen intelligently. In particular, the examples will show that some priority based, the maximum throughput and the arctan-utility maximizing allocation policies may not ensure stability of the bursty network model even under the normal offered condition. These counterexamples provide strong incentives to study the stability problem of the bursty model of data networks. The positive results for the bursty network model with other bandwidth allocation policies are then given in Theorem 3.3 at the end of this subsection.

The first example is a modification of the stochastic version of the counterexample in Bonald and Massoulie (2001). It shows that the same instability occurs for the bursty network model with a priority based allocation or a maximum throughput allocation, even when the normal offered load condition (22) holds.

Example 2. (*Instability of maximum throughput and priority based allocation policies*) The infrastructure of the network is the same as that of the network in Example 1 illustrated by Figure 1. The arrival processes of the two routes are as follows

$$\begin{aligned} &A_{r_1}(t) = \rho_{r_1} t, \quad \text{and} \\ &A_{r_2}(t) = k \cdot w \quad \text{for } t \in \left[(k-1)\frac{w}{\rho_{r_2}}, k\frac{w}{\rho_{r_2}}\right), \ k = 1, 2, \dots , \end{aligned} \tag{34}$$

where ρ_{r_1}, ρ_{r_2}, and w are constant. It can be checked that both processes satisfy the bursty constraint (32). Suppose that the capacities of the two links are chosen so that

$$\begin{aligned} &\rho_{r_1} < C_{l_1}, \\ &\rho_{r_1} + \rho_{r_2} < C_{l_2}, \ \text{but} \\ &\rho_{r_1} > \left(1 - \frac{\rho_{r_2}}{C_{l_2}}\right) C_{l_1}. \end{aligned}$$

If we want to maximize the throughput of the network, a choice of bandwidth allocation may give priority to data traffic along route r_2 over that along route r_1 on link l_2. Specifically, the bandwidth allocation process $\Lambda(N(t))$ is given by

$$\Lambda_{r_1}(N(t)) = \begin{cases} C_{l_1} & \text{for } t \in [(k-1)\frac{w}{\rho_{r_2}} + \frac{w}{C_{l_2}}, k\frac{w}{\rho_{r_2}}), \ k = 1, 2, \dots \\ 0 & \text{otherwise,} \end{cases}$$

$$\Lambda_{r_2}(N(t)) = \begin{cases} C_{l_2} & \text{for } t \in [(k-1)\frac{w}{\rho_{r_2}}, (k-1)\frac{w}{\rho_{r_2}} + \frac{w}{C_{l_2}}), \ k = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$D_{r_1}(t) = \int_0^t \Lambda_{r_1}(N(s))ds \le \left(1 - \frac{\rho_{r_2}}{C_{l_2}}\right) C_{l_1} t \quad \text{for all } t \ge 0,$$

and thus

$$\begin{aligned} X_{r_1}(t) &= X_{r_1}(0) + A_{r_1}(t) - D_{r_1}(t) \\ &\ge X_{r_1}(0) + \rho_{r_1} t - (1 - \frac{\rho_{r_2}}{C_{l_2}}) C_{l_1} t \to \infty \quad \text{as } t \to \infty. \end{aligned}$$

□

The second example shows that for the bursty network model the arctan-utility maximizing bandwidth allocation policy can cause network instability even when the normal offered load condition (22) holds.

Example 3. (*Instability of arctan-utility maximizing allocation policy*) The infrastructure of the network is illustrated in Figure 2. There are $2m$ links, one long route and $2m$ short routes in the network. The long route traverses all the $2m$ links while each short route passes through one link. We also assume that all the links have bandwidth capacity 1 (in terms of the amount of data per unit time), that all the short routes have offered transmission load of $\rho_s \in (\frac{1}{m}, 1)$, and that the long route has offered transmission load $\rho_0 \in (0, 1-\rho_s)$. Then the normal offered load condition (22) is satisfied for all the $2m$ links in this network. Suppose the

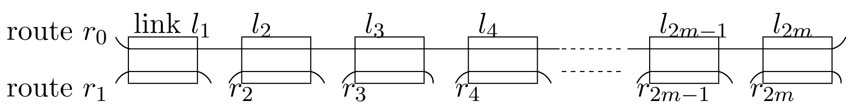

Figure 2: A network with $2m$ links, one long route and $2m$ short routes

"adversary" specifies the arrival processes as

$$A_{r_0}(t) := \rho_0 t,$$
$$A_{r_1}(t) = A_{r_2}(t) := k\rho_s T \quad \text{for } t \in [(k-1)T, kT), \quad \text{for } k = 1,\ 2, \cdots,$$
$$A_{r_{2i-1}}(t) = A_{r_{2i}}(t) := \begin{cases} 0 & \text{for } 0 \le t < \frac{(i-1)T}{m} \\ A_{r_{2(i-1)}}(t - \frac{T}{m}) & \text{for all } t \ge \frac{(i-1)T}{m}, \end{cases} \quad \text{for } i = 2, 3, \cdots,\ m,$$

where $T > 0$ is a constant satisfying:

$$1 + \frac{1}{(\frac{w_r}{\nu_r})^2(\rho_s - 1/m)^2 T^2} \le 2, \quad \text{for } r = 1, \cdots, 2m. \tag{35}$$

Now we can see how the network evolves in time intervals $[kT, (k+1)T)$ for $k = 0,\ 1,\ 2, \cdots$. First, on the sub-interval $[kT, kT + \frac{T}{m})$, we look at links l_1 and l_2. At $t = kT$, the amount of data to be transmitted along the two short routes r_1 and r_2 satisfies

$$X_r(kT) \ge A_r(kT) - A_r(kT-) = \rho_s T, \quad r = r_1,\ r_2.$$

Consequently, for all $t \in [kT, kT + \frac{T}{m})$,

$$X_r(t) \ge X_r(kT) - \frac{T}{m} \ge (\rho_s - \frac{1}{m})T, \quad r = r_1,\ r_2,$$

or

$$N_r(t) \ge \frac{1}{\nu_r}(\rho_s - \frac{1}{m})T, \quad r = r_1,\ r_2.$$

According to (6), the arctan-utility maximizing bandwidth allocation intends to maximize the function

$$\sum_r U_r(N_r(t), \Lambda_r) = \sum_r w_r N_r(t) \arctan(\frac{\Lambda_r}{w_r N_r(t)}).$$

The assumption (35) yields

$$\frac{\partial U_r(N_r(t), \Lambda_r)}{\partial \Lambda_r} \ge \frac{\partial U_r(\frac{1}{\nu_r}(\rho_s - \frac{1}{m})T, \Lambda_r)}{\partial \Lambda_r} \ge \frac{1}{2}, \quad \text{for } t \in [kT, kT + \frac{T}{m}), \quad r = r_1,\ r_2.$$

However,

$$\frac{\partial U_{r_0}(N_{r_0}(t),\Lambda_{r_0})}{\partial \Lambda_{r_0}} = \frac{1}{1+\frac{\Lambda_{r_0}^2}{w_{r_0}^2 N_{r_0}(t)^2}} < 1, \quad \text{for all } t \geq 0.$$

It is concluded that

$$\frac{\partial U_{r_0}(N_{r_0}(t),\Lambda_{r_0})}{\partial \Lambda_{r_0}} < \frac{\partial U_{r_1}(N_{r_1}(t),\Lambda_{r_1})}{\partial \Lambda_{r_1}} + \frac{\partial U_{r_2}(N_{r_2}(t),\Lambda_{r_2})}{\partial \Lambda_{r_2}}, \quad \text{for } t \in [kT, kT+\frac{T}{m}).$$

Thus, to maximize $\sum_r U_r(N_r(t),\Lambda_r)$, one should decrease Λ_{r_0} as much as possible. Such a decrease brings an increase to Λ_{r_1} and Λ_{r_2}, respectively. Therefore, the optimal solution to the optimization problem (6) requires

$$\Lambda_{r_1}(N(t)) = \Lambda_{r_2}(N(t)) = 1 \text{ and } \Lambda_{r_0}(N(t)) = 0, \quad \text{for } t \in [kT, kT+\frac{T}{m}).$$

The same argument will show that, on sub-interval $[kT+(i-1)\frac{T}{m}, kT+i\frac{T}{m}], i = 2,\cdots,m$, we have

$$\Lambda_{r_{2i-1}}(N(t)) = \Lambda_{r_{2i}}(N(t)) = 1 \text{ and } \Lambda_{r_0}(N(t)) = 0.$$

In summary, we have $\Lambda_{r_0}(N(t)) = 0$ for all $t \geq 0$. Hence,

$$X_{r_0}(t) = X_{r_0}(0) + A_{r_0}(t) = X_{r_0}(0) + \rho_0 t \to +\infty,$$

which implies the instability of the network model. □

It is interesting to note that, in the above counterexample, the arctan bandwidth allocation is in effect equivalent to a priority allocation in which the long route r_0 is always given the lowest priority. Such a priority effect leads to ever increasing transmission load on the route r_0 which is never allocated to any bandwidth. The same priority effect would not be possible for the proportionally fair, the minimal potential delay, the (p,α)-proportionally fair, the max-min fair or the U-utility maximizing allocation (with partial radial homogeneity property). Under these allocation policies, we can see that the bandwidth allocated to each route depends on the *relative* transmission loads on all routes rather than their absolute values, noting the partial radial homogeneity property of these allocation policies. *Intuitively*, when the transmission load on the route r_0 increases to a level relatively much higher than the transmission loads on other routes, the route r_0 will grab the bandwidth to drive down its transmission load eventually. Furthermore, such intuition is reinforced by the following stability results for the bursty network model, which is parallel to Theorem 3.1 for the stationary network model.

Theorem 3.3 *Suppose the normal offered load condition (22) is satisfied for the bursty network model $(L, C, R, M, \lambda, \nu, \rho, w)$. Then, the bandwidth allocations Λ^{pp}, Λ^{pd}, Λ^{mm}, Λ^{α} and Λ^{U} ensure the stability of the model.*

4 Proofs of The Stability Results

In this section, we prove Theorems 3.1 and 3.3 by fluid model approach. By this approach, we first introduce a fluid network model for each given bandwidth allocation policy and prove its stability (the definition of which will be precisely given below) in Subsection 4.1. Then, we show that, if properly scaled, the stationary and bursty network models will converge to the limits that satisfy the fluid network model, and that the stability of the fluid model implies the stability of the stationary and the bursty models in Subsections 4.2 and 4.3, respectively.

According to Remark 2 in Section 2, it suffices to prove Theorems 3.1 and 3.3 for bandwidth allocations Λ^{mm} and Λ^{U}, where the latter includes the others. We remind readers that the same notation such as the processes X, N, A, D and Λ are used for both the stationary and the bursty network models in Subsections 3.1 and 3.2 above, and also in Subsections 4.2 and 4.3 below. Their meanings can be distinguished from the context.

4.1 Fluid Network Model and Its Stability

We describe a fluid network model corresponding to the stationary and the bursty data network models with bandwidth allocations Λ^{mm} and Λ^{U}. One obtains the former by replacing the connections and data transmission load in the latter with continuous fluids. The fluid network model has the same infrastructure as the stationary or the bursty network model. That is, the infrastructure of the fluid network model is also characterized by the quadruple (L, C, R, M) where L is the set of links, C the transmission bandwidth capacity, R the set of routes, and M the 0-1 incidence matrix. But in the fluid network the routes carry continuous fluid flows. Specifically, on route $r(\in R)$ the fluid flows exogenously into the network at a rate less than or equal to ρ_r, and is transmitted through route r at a rate subject to the given bandwidth allocation policy.

To describe the dynamics of the fluid network model, we introduce the $|R|$-dimensional *fluid level process* $\bar{X}(t) = \{\bar{X}_r(t) : r \in R\}$, the *connection level process* $\bar{N}(t) = \{\bar{N}_r(t) : r \in R\}$, the *fluid arrival process* $\bar{A}(t) = \{\bar{A}_r(t) : r \in R\}$, the *transmission process* $\bar{D}(t) = \{\bar{D}_r(t) : r \in R\}$, and the *transmission bandwidth allocation* $\bar{\Lambda}(n, q) = \{\bar{\Lambda}_r(n, q) : r \in R\}$. For any $r \in R$, $\bar{X}_r(t)$ is the amount of fluid waiting to be transmitted along route r at time t; $A_r(t)$ is the cumulative amount of fluid that has arrived to route r during the time interval $[0, t]$; $D_r(t)$ is the total amount of fluid that has been transmitted via route r during the time interval $[0, t]$; and $\bar{\Lambda}_r(n, q)$ is the transmission rate allocated to route r when the connection level state is $\bar{N}(t) = n, n \in \mathcal{R}_+^{|R|}$ and the fluid inflow rate is $\dot{\bar{A}}(t) = q, q \in \mathcal{R}_+^{|R|}$. In particular, for the fluid network model, we define max-min fair bandwidth allocation $\bar{\Lambda}^{mm}(\cdot, \cdot)$ and U-utility maximizing bandwidth allocation $\bar{\Lambda}^{U}(\cdot, \cdot)$ as follows:

$$\bar{\Lambda}_r^{mm}(n, q) := \begin{cases} \Lambda_r^{mm}(n) & \text{if } n_r > 0; \\ q & \text{if } n_r = 0, \end{cases} \tag{36}$$

$$\bar{\Lambda}_r^{U}(n, q) := \begin{cases} \Lambda_r^{U}(n) & \text{if } n_r > 0; \\ q & \text{if } n_r = 0, \end{cases} \tag{37}$$

where $\Lambda^{mm}(\cdot)$ and $\Lambda^{U}(\cdot)$ are defined in Subsections 2.5 and 2.7, respectively.

Given a bandwidth allocation $\bar{\Lambda}(\cdot, \cdot)$, either $\bar{\Lambda}^{mm}(\cdot, \cdot)$ or $\bar{\Lambda}^{U}(\cdot, \cdot)$, the dynamics of the fluid network model is characterized by the following system of equations: for any $r \in R$ and all $t \geq 0$,

$$\bar{X}_r(t) = \bar{X}_r(0) + \bar{A}_r(t) - \bar{D}_r(t), \tag{38}$$

$$\bar{N}_r(t) = \bar{X}_r(t)/\nu_r, \tag{39}$$

$$\bar{A}_r(t) \text{ is Lipschitz continuous and } 0 \leq \dot{\bar{A}}_r(t) \leq \rho_r \quad \text{a.s.}, \tag{40}$$

$$\bar{D}_r(t) = \int_0^t \bar{\Lambda}_r(\bar{N}(s), \dot{\bar{A}}_r(s))ds, \tag{41}$$

where the relation (38) is the flow balance equation. We introduce the connection level process $\bar{N}(t)$ here so as to maintain the similarity of fluid network models to the stationary and bursty network models. The condition (40) is a regularity property of fluid arrival processes, and the equation (41) is self-explanatory.

Remark: 1. Noting that the bandwidth allocation $\bar{\Lambda}(\cdot,\cdot)$ is bounded by the link capacity, the departure process $\bar{D}(t)$ is also Lipschitz continuous. The Lipschitz continuity of processes $\bar{X}(t)$ and $\bar{N}(t)$ then follows from the Lipschitz continuity of processes $\bar{A}(t)$ and $\bar{D}(t)$. Consequently, the processes $\bar{X}(t)$, $\bar{N}(t)$, $\bar{A}(t)$ and $\bar{D}(t)$ defined above are differentiable *a.s.* for $t \geq 0$. We call the time t a *regular* point if all the processes involved are differentiable at t.

2. All the processes defined for the fluid network model here are parallel to those for stationary and bursty network models. Following the convention in the literature on fluid network models in queueing theory, we append a bar to the fluid processes for ease of comparison. □

The system of equations (38) -(41) is called a fluid network model in short, and any solution $(\bar{X}, \bar{N}, \bar{A}, \bar{D})$ to the fluid model is called a *fluid solution.* The process $\bar{X}$ (or $\bar{N}$) alone is also called a fluid solution if there exist processes $\bar{N}$ (or $\bar{X}$), $\bar{A}$ and $\bar{D}$ such that $(\bar{X}, \bar{N}, \bar{A}, \bar{D})$ is a fluid solution. The fluid model (38)-(41) is said to be *stable* if there exists a time $\tau \geq 0$ such that $\bar{X}(\tau + \cdot) \equiv 0$ (or equivalently $\bar{N}(\tau + \cdot) \equiv 0$) for any fluid solution $\bar{X}(t)$ with $||\bar{X}(0)|| = 1$. As for the stationary and the bursty network models, the normal offered load condition is necessary for the fluid network model to be stable under any bandwidth allocation. The following proposition proves stability of the fluid network model with either the max-min fair allocation $\bar{\Lambda}^{mm}(\cdot,\cdot)$ or the U-utility maximizing allocation $\bar{\Lambda}^{U}(\cdot,\cdot)$ under the normal offered load condition (22). The proof can be found in the appendix.

Proposition 4.1 *Suppose the normal offered load condition (22) is satisfied for the fluid network model (38)-(41). Then, both the max-min fair allocation $\bar{\Lambda}^{mm}(\cdot,\cdot)$ and the U-utility maximizing allocation $\bar{\Lambda}^{U}(\cdot,\cdot)$ ensure the stability of the fluid network model.*

4.2 Proof of Theorem 3.1

We now go to the second step for the stationary network model. Consider a sequence of such stationary models, indexed by $k = 1, 2, \cdots$. We append superscript k to all the processes associated with the k-th model. Specifically, for the k-th model, we have $X^{(k)}(t)$, $N^{(k)}(t)$, $E^{(k)}(t)$, $A^{(k)}(t)$, $D^{(k)}(t)$ and $S^{(k)}(y)$. The bandwidth $\Lambda(N(t))$ can be either $\Lambda^{mm}(N(t))$ or $\Lambda^{U}(N(t))$. The fluid model approach to proving stability of the stationary network model makes use of the limits of these processes with the scale defined below. Let $\{z_k\}$ be an increasing sequence of positive numbers with $z_k \to \infty$ and let

$$\begin{aligned}
\bar{N}^{(k)}(t) &= \frac{1}{z_k} N^{(k)}(z_k t), \\
\bar{X}^{(k)}(t) &= \frac{1}{z_k} X^{(k)}(z_k t), \\
\bar{A}^{(k)}(t) &= \frac{1}{z_k} A^{(k)}(z_k t), \\
\bar{D}^{(k)}(t) &= \frac{1}{z_k} D^{(k)}(z_k t).
\end{aligned}$$

We have the following proposition on the limits of these scaled processes.

Proposition 4.2 *Given the bandwidth allocation $\Lambda(\cdot)$ (either $\Lambda^{mm}(\cdot)$ or $\Lambda^{U}(\cdot)$), and suppose $\bar{X}^{(k)}(0)$ converges as $k \to \infty$. Then for almost all sample paths and any subsequence of $\{k\}$, there exists a further subsequence, also denoted by $\{k\}$, such that*

$$(\bar{N}^{(k)}(t), \bar{X}^{(k)}(t), \bar{A}^{(k)}(t), \bar{D}^{(k)}(t)) \to (\bar{N}(t), \bar{X}(t), \bar{A}(t), \bar{D}(t)) \qquad u.o.c.,$$

and $(\bar{N}(t), \bar{X}(t), \bar{A}(t), \bar{D}(t))$ is a fluid solution to the fluid network model (38)-(41).

Proof: The u.o.c. convergence of the scaled processes $\bar{A}^{(k)}(t)$ (along a subsequence of k) follows from the functional strong law of large numbers (FSLLN); see for example Theorem 5.10 in Chen and Yao (2001). The u.o.c. convergence of the scaled processes $\bar{D}^{(k)}(t)$ is simply due to the fact that the processes are pointwise bounded ($0 \leq \bar{D}_r^{(k)}(t) \leq \min_{l \in r} C_l t$) and are equicontinuous ($|\bar{D}_r^{(k)}(t) - \bar{D}_r^{(k)}(s)| \leq \min_{l \in r} C_l |t-s|$); see Rudin (1987), page 245. The u.o.c. convergence of $\bar{X}^{(k)}(t)$ to $\bar{X}(t)$ and the relation (38) follow from the relation (28) and the u.o.c. convergence of $\bar{A}^{(k)}(t)$ and $\bar{D}^{(k)}(t)$. To argue for the u.o.c. convergence of $\bar{N}^{(k)}(t)$ to $\bar{N}(t)$ and the relation (39), we express each $\bar{N}_r^{(k)}(t), r \in R$, as

$$\bar{N}_r^{(k)}(t) = \bar{N}_r^{(k)}(0) + \bar{E}_r^{(k)}(t) - \bar{S}_r^{(k)}(\bar{D}_r^{(k)}(t)) \tag{42}$$

where

$$\bar{E}^{(k)}(t) = \frac{1}{z_k} E^{(k)}(z_k t) \quad \text{and} \quad \bar{S}^{(k)}(y) = \frac{1}{z_k} S^{(k)}(z_k y).$$

The equation (42) is similar to equation (6.3) in Chen and Yao (2001), while the expression for connection departure process $S(\cdot)$ under FIFO in a route (i.e., the equation (27)) to the equation (6.2) and the equation (38) to the equation (6.5) ibid. Then, the same proof as used in Theorem 6.5 of Chen and Yao (2001) (which holds for FIFO queues) yields the u.o.c. convergence of $\bar{N}_r^{(k)}(t)$ to $\bar{N}_r(t)$. The relation (39) can also be proven by the similar argument to the corresponding part in Theorem 6.5 of Chen and Yao (2001).

To verify the relation (41), it suffices to show

$$\dot{\bar{D}}_r(t) = \dot{\bar{A}}_r(t) (= \rho_r), \quad \text{if } \bar{N}_r(t) = 0, \tag{43}$$

and

$$\dot{\bar{D}}_r(t) = \bar{\Lambda}_r(\bar{N}(t), \dot{\bar{A}}_r(t)) = \Lambda_r(\bar{N}(t)), \quad \text{if } \bar{N}_r(t) > 0, \tag{44}$$

for any $t \geq 0$ such that all the related processes are differentiable, according to the definitions of $\bar{\Lambda}^{mm}(\cdot,\cdot)$ and $\bar{\Lambda}^{U}(\cdot,\cdot)$ in (36) and (37). (Note that processes $\bar{X}_r(t)$, $\bar{N}_r(t)$, $\bar{A}_r(t)$ and $\bar{D}_r(t)$ are differentiable for almost all $t \geq 0$ since they are Lipschitz continuous.) When $\bar{N}_r(t) = 0$ or $\bar{X}_r(t) = 0$, we have $\dot{\bar{X}}_r(t) = 0$ (if $\bar{X}_r(t)$ is differentiable at t) since t is a local minimum of the function $X_r(\cdot)$. Thus, the equality (43) follows. When $\bar{N}_r(t) > 0$, let us evaluate

$$\left| \frac{1}{h} (\bar{D}_r^{(k)}(t+h) - \bar{D}_r^{(k)}(t)) - \Lambda_r(\bar{N}(t)) \right|$$

for sufficiently small positive number h and sufficiently large index number k along a convergent subsequence. By the identity (33), it is equal to

$$\left| \frac{1}{h} (\bar{D}_r^{(k)}(t+h) - \bar{D}_r^{(k)}(t)) - \Lambda_r(\bar{N}(t)) \right|$$

$$
\begin{aligned}
&= \left|\frac{1}{h}\int_0^h \Lambda_r(N(z_k(t+s)))ds - \Lambda_r(\bar{N}(t))\right| \\
&\leq \frac{1}{h}\int_0^h |\Lambda_r(N(z_k(t+s))) - \Lambda_r(\bar{N}(t+s))|ds \\
&\quad + \frac{1}{h}\int_0^h |\Lambda_r(\bar{N}(t+s)) - \Lambda_r(\bar{N}(t))|ds. \qquad (45)
\end{aligned}
$$

Choose h sufficiently small that $\bar{N}_r(t+s) > 0$ for $s \in (0, h)$, then, the partial radial homogeneity property of $\Lambda^{mm}(\cdot)$ or $\Lambda^U(\cdot)$ (see Lemma 6.2) and the convergence of process $\bar{N}^{(k)}(\cdot)$ imply

$$
\begin{aligned}
& \Lambda_r(N(z_k(t+s))) - \Lambda_r(\bar{N}(t+s)) \\
= \;& \Lambda_r(\bar{N}^{(k)}(t+s)) - \Lambda_r(\bar{N}(t+s)) \to 0,
\end{aligned}
$$

as $k \to \infty$, for all $s \in (0, h)$. Therefore, by Lebesgue dominated convergence theorem, we take $k \to \infty$ in (45) to obtain

$$
\left|\frac{1}{h}(\bar{D}_r(t+h) - \bar{D}_r(t)) - \Lambda_r(\bar{N}(t))\right| \leq \frac{1}{h}\int_0^h |\Lambda_r(\bar{N}(t+s)) - \Lambda_r(\bar{N}(t))|ds.
$$

Due to the Lipschitz continuity of $\bar{N}(t)$, the partial continuity property of $\Lambda^{mm}(\cdot)$ and $\Lambda^U(\cdot)$ (see Lemma 6.2 again) implies that the right-hand side of the above inequality goes to zero as $h \to 0+$. That is,

$$
\dot{\bar{D}}_r(t+) = \bar{\Lambda}_r(t),
$$

and hence

$$
\dot{\bar{D}}_r(t) = \bar{\Lambda}_r(t) \qquad a.e.
$$

□

Proof of Theorem 3.1: The theorem now follows from Proposition 4.1, Proposition 4.2, and Theorem 3 of Bramson (1998). □

Remark: 1. We have omitted a minor detail on the choice of the sequence of scaling parameter $\{z_k\}$. In Dai (1995), it is chosen as the norm of the initial state of the Markov process (N, U, V) and the resulting fluid network model is a delayed model where the fluid arrival process is in the form of

$$
\bar{A}_r(t) = \rho_r(t - \bar{U}_{0,r})
$$

where $\bar{U}_{0,r}$ is a positive number obtained as a limit of scaled initial remaining arrival time $U_r(0)/z_k$. For the delayed fluid network model, an alternative is to apply the refinement in Chen(1995), where it is shown that the stability of the delayed fluid network model follows from that of the corresponding non-delayed fluid network model. One way to avoid the delayed model is presented in Bramson (1998). In this method, the scaling parameters are chosen so that the sequence $\{U_r(0)/z_k\}$ converges to 0 and thus a non-delayed fluid network model is obtained. We use this method in this section implicitly.

2. In a contemporary independent work, Kelly and Williams (2003) also derive and study the fluid network for the data network model with Poisson arrivals, exponential connection volumes and the (p, α)-proportionally fair bandwidth allocation. □

4.3 Proof of Theorem 3.3

The second step for the bursty model is parallel to that for the stationary model. First we scale all the processes in the bursty network model and take proper limits. Let $\{t_k\}$ and $\{z_k\}$ be an increasing sequence of times with $t_k \to \infty$ and an increasing sequence of positive numbers with $z_k \to \infty$. Define

$$\begin{aligned}
\bar{X}^{(k)}(t) &= \frac{1}{z_k} X(t_k + z_k t), \\
\bar{N}^{(k)}(t) &= \frac{1}{z_k} N(t_k + z_k t), \\
\bar{A}^{(k)}(t) &= \frac{1}{z_k} \left(A(t_k + z_k t) - A(t_k) \right), \\
\bar{D}^{(k)}(t) &= \frac{1}{z_k} \left(D(t_k + z_k t) - D(t_k) \right).
\end{aligned}$$

We have the following proposition on the limits of these scaled processes.

Proposition 4.3 *Given the bandwidth allocation $\Lambda(\cdot)$ (either $\Lambda^{mm}(\cdot)$ or $\Lambda^U(\cdot)$), and suppose that $\{\bar{X}^{(k)}(0)\}$ (or $\{\frac{1}{z_k} X(t_k)\}$) has convergent subsequences. Then, there exists a subsequence of $\{k\}$, also denoted by $\{k\}$, such that*

$$(\bar{N}^{(k)}(t), \bar{X}^{(k)}(t), \bar{A}^{(k)}(t), \bar{D}^{(k)}(t)) \to (\bar{N}(t), \bar{X}(t), \bar{A}(t), \bar{D}(t)) \qquad u.o.c.,$$

where $(\bar{N}(t), \bar{X}(t), \bar{A}(t), \bar{D}(t))$ is a fluid solution to the fluid network model (38)-(41).

Proof: The u.o.c. convergence of scaled processes $\bar{A}^{(k)}(t)$ and $\bar{D}^{(k)}(t)$ along a subsequence to $\bar{A}(t)$ and $\bar{D}(t)$ is a direct result from Lemma 6.3, which is a variation or generalization of Arzela-Ascoli Theorem. (Also note that it is sufficient to show the u.o.c. convergence of the scaled departure process $\bar{D}^{(k)}(t)$ by using Arzela-Ascoli Theorem, which is the Lemma 6.3 with σ_n being set to zero.) Then the u.o.c. convergence of scaled processes $\bar{X}^{(k)}(t)$ and $\bar{N}^{(k)}(t)$, relations (38)-(40) follow consequently. The verification of the relation (41) is almost a word for word repetition of the proof of Theorem 3.1 on the same relation except for a slight difference in the scaling and shifting of the time space. That is, the time space is scaled and shifted as $t_k + z_k t$ here rather than as $z_k t$ in the proof of Theorem 3.1. We omit the details. □

Finally, the above proposition combined with the stability of fluid model is used to prove Theorem 3.3.

Proof of Theorem 3.3: We prove the theorem by contradiction. Suppose $X(\cdot)$ is not bounded. Let τ be the time specified in Proposition 4.1. We first show by induction that there exist two increasing sequences $\{t_k, k \geq 1\}$ and $\{z_k, k \geq 1\}$, with $t_k \to \infty$ and $z_k \to \infty$, satisfying

$$||X(t_k)|| \leq z_k \leq ||X(t_k + z_k \tau)||. \tag{46}$$

For a fixed $k \geq 1$, suppose t_i and z_i, $1 \leq i \leq k-1$, are found. For convenience, define $t_0 = 0$ and $z_0 = 0$ when $k = 1$. Then, z_k can be chosen so that

$$z_k \geq \sup\{||X(t)|| : 0 \leq t \leq t_{k-1} + z_{k-1}\tau + 1\} + 1.$$

For such z_k, we have

$$z_k \geq ||X(t_{k-1} + z_{k-1}\tau)|| + 1 \geq z_{k-1} + 1.$$

Let

$$s_0 = t_{k-1} + z_{k-1}\tau + 1 \qquad \text{and} \qquad s_i = s_{i-1} + z_k\tau \ \text{ for } i = 1, 2, \dots .$$

Then, there exists an $j \geq 1$ such that $||X(s_{j-1})|| \leq z_k \leq ||X(s_j)||$. Otherwise, we have $||X(s_i)|| \leq z_k$ for all $i \geq 0$ since $||X(s_0)|| \leq z_k$, and thus $X(\cdot)$ is bounded. Now, t_k can be chosen as s_{j-1}.

Next, according to Proposition 4.3, we know that there exists a subsequence of $\{k\}$, denoted by $\{k\}$, such that $\frac{1}{z_k}X(t_k + z_k t)$ converges ($u.o.c.$) to a fluid solution $\bar{X}(t)$ of the fluid network model (38)-(41). Due to the inequality (46), we have $||\bar{X}(0)|| \leq 1$ and $||\bar{X}(\tau)|| \geq 1$. However, according to Proposition 4.1, $||\bar{X}(\tau)|| = 0$. This contradiction proves Theorem 3.3. □

Remark: Since the arctan-utility maximizing bandwidth allocation policy is stable for the stationary network model with Poisson arrival of connections (Theorem 3.2) under the normal offered load condition, one might expect that the bursty model with arctan-utility maximizing allocation were also stable and a fluid model could be used to prove its stability. However, this is not the case (see Example 3). After careful examination, the most plausible corresponding fluid model would be a fluid network model with max-throughput bandwidth allocation. This is because that the utility $w_r n_r \arctan(\Lambda_r / w_r n_r)$ is approximately Λ_r as $n_r \to \infty$, and that the bandwidth allocation for the corresponding fluid network model is determined mainly by increasing the ongoing connection number n_r to ∞ in the original network model with proper scaling in time and space. It is direct to see that a fluid network model with max-throughput bandwidth allocation may not be stable under the normal offered load condition. From this perspective, it would not be surprising that the bursty network model with the arctan-utility maximizing allocation may not be stable under the normal offered load condition. We note that we are not able to exactly identify the fluid network model corresponding to either the stationary or bursty data network model with arctan-utility maximizing allocation. An interesting technical problem for further study would be how to identify such fluid network model. □

5 Concluding Remarks

In this paper we have provided an overview on the stability of a wide collection of bandwidth allocation policies with respect to two different network models (see Table 5 for a summary of the existing stability results). The stationary data network model is a connection level model while the bursty model can be regarded as a workload approximation of the connection level model. The stationary network model with the assumption of general connection arrival processes is a natural extension of the Markovian models previously studied by de Veciana, *et al.* (2001), Bonald and Massoulie (2001), and Ye (2003) amongst others. However, we have to retain the assumption that connection volumes are i.i.d. exponential. Relaxing this assumption would lead to tremendous technical difficulties in proving convergence of the fluid limit of the connection level process due to processor-sharing nature of the link bandwidth. For possible approaches to surmount the difficulties, readers are referred to Chen, *et al.* (1997) and Gromoll, *et al.* (2001) and the literature there on recent developments in the research on the fluid model of processor-sharing queues.

The bursty model for data networks does not require the time-invariant probability assumption and the other regularity assumptions on the arrival processes of the data transmission load to the networks. Thus, the bursty model can be a more realistic representation of data networks in some aspects. Our work reveals that some bandwidth allocation policies that ensure stability

Bandwidth Allocation	Stationary Model (Poisson Arrivals)	Stationary Model (General Arrivals)	Bursty Model
Priority	$\times$	$\times$	$\times$
Maximum Throughput	$\times$	$\times$	$\times$
Arctan	$\surd$	?	$\times$
Proportional Fair	$\surd$	$\surd$	$\surd$
Min Potential Delay	$\surd$	$\surd$	$\surd$
Maxmin	$\surd$	$\surd$	$\surd$
(p, α)-Proportional Fair	$\surd$	$\surd$	$\surd$
U-Utility	$\surd$	$\surd$	$\surd$

Note: "$\surd$" and "$\times$" mean "stable" and "instable", respectively. The U-utility maximizing allocation in the stationary model (Poisson arrival) does not require the partial radial homogeneity property.

Table 1: Summary of Existing Results

of the stationary network model may cause instability to the bursty model. One of such policies is the arctan-utility maximizing allocation policy, which is of particular interest because it is thought to be a good approximation of a type of the TCP rate control adopted by many applications in the Internet. We now know that this allocation is stabilizing for the stochastic model with Poisson arrivals of connections but not for the bursty model. In particular, we believe that the instability example (Example 3) is at least theoretically important, though the arrival process is not representative of the real situation. It is yet to gain more perspective about the instability result and to show the impact of such an instability result on real networks. As for the general stationary network model under the arctan-utility maximizing allocation, if to use the fluid network model approach to investigate its stability, the most plausible fluid network model would be the fluid model with a maximum throughput bandwidth allocation (cf. Remark in Section 4.3). However, we have not been able to identify the fluid model exactly.

We have essentially investigated stability of all the bandwidth allocations that are related to the class of (p, α)-proportionally fair policies. In particular, we have found that at $\alpha = 0$ the policy of maximum throughput allocation leads both the stationary network model and the bursty network model to become instable. On the other hand, all policies with $\alpha > 0$ ensure the stability of both models. The $\arctan(\cdot)$ utility maximizing policy may be viewed as corresponding to $\alpha = 0+$ in light of our conjecture on the form of its corresponding fluid model. There are also other bandwidth allocations that cannot be approximated by the (p, α)-proportionally fair policies. A prominent example is the policy that would give rise to the TCP throughput function of the loss probability as derived by Padhye, *et al.* (2000), which is a refinement of the well known formula of Mathis, *et al.* (1997). (As exposed in Kelly (2001) and Low (2002), the formula in Mathis, *et al.* (1997) relates to the $\arctan(\cdot)$ utility maximizing policy.) The stability of these allocations for a network model poses challenging issues for future research.

The network models we introduce in the paper view the network at the higher level of connections, and focus on the dynamic nature of data traffic in the network, but ignore some details on how the connections are established and stabilized at the data packet level. Idealized assumptions have to be made on the behavior of the network at the actual packet transmission level. For example, we implicitly assume that all the bandwidth allocations can respond instantaneously to evolving traffic patterns in the network and the route for a connection staying unchanged during the transmission session. Attempts to interrelated properties of the data network at the two levels of modelling are summarized in Kelly (2001) and Low (2002). The results we present in this paper on the connection level stability of the data networks are complement to the analysis of the network at the packet level.

A final and important remark is on the very notion of stability itself. In the operations of real data networks supporting elastic traffic (in particular the Internet), most transmission protocols would reduce the data transmission rate automatically when the network approaches instability due to extremely heavy load. In other words, by design, instability caused network crash would not happen. In this case, we may interpret the stability results of the paper in terms of admissible throughput. We then conclude that the normal offered load condition defines the complete set of admissible throughput for the proportionally fair, the minimum potential delay, the max-min fair and the U-utility maximizing bandwidth allocation policies. The characterization of the sets of admissible throughput for priority based, the maximum throughput, and the arctan-utility maximizing allocation policies is an open question.

6 Appendix

Proof of Proposition 4.1 for allocation $\bar{\Lambda}^{mm}$: Define a candidate Lyapunov function $f(t)$ as

$$f(t) = \max_{r \in R} f_r(t)$$

with

$$f_r(t) = \frac{\bar{N}_r(t)}{\rho_r}.$$

Fix the time $t \geq 0$ and assume t is a regular point from now on. Let $\bar{r} \in R$ be a route that achieves the maximum, i.e.,

$$\frac{\bar{N}_{\bar{r}}(t)}{\rho_{\bar{r}}} = \max_{r \in R} \frac{\bar{N}_r(t)}{\rho_r}. \tag{47}$$

We claim that

$$\dot{f}(t) = \dot{f}_{\bar{r}}(t) \tag{48}$$

whose proof can be found in Dai and Weiss (1996). Then, it is sufficient to show that

$$\dot{f}_{\bar{r}}(t) \leq -\epsilon$$

for some $\epsilon > 0$ whenever $\bar{N}(t) \neq 0$.

Define

$$\epsilon = \min_{r \in R} \min_{l \in r} \left(v_r \sum_{r' \in l} \rho_{r'} \right)^{-1} \left(C_l - \sum_{r' \in l} \rho_{r'} \right).$$

We have

$$\begin{aligned} \dot{f}_{\bar{r}}(t) &= \frac{d}{dt} \left(\frac{\bar{N}_{\bar{r}}(t)}{\rho_{\bar{r}}} \right) \\ &= \frac{1}{\rho_{\bar{r}} v_{\bar{r}}} \left(\dot{\bar{A}}_{\bar{r}}(t) - \bar{\Lambda}_{\bar{r}}(t) \right) \end{aligned}$$

$$
\begin{aligned}
&\leq \quad \frac{1}{\rho_{\bar{r}} v_{\bar{r}}} \left(\rho_{\bar{r}} - C_{\bar{l}} \frac{\bar{N}_{\bar{r}}(t)}{\sum_{r \in R(\bar{l})} \bar{N}_r(t)} \right) &(49)\\
&= \quad \frac{1}{\rho_{\bar{r}} v_{\bar{r}}} \left(\rho_{\bar{r}} - C_{\bar{l}} \frac{\rho_{\bar{r}}}{\sum_{r \in R(\bar{l})} \rho_r} \frac{\bar{N}_{\bar{r}}(t)/\rho_{\bar{r}}}{(\sum_{r \in R(\bar{l})} \bar{N}_r(t))/(\sum_{r \in R(\bar{l})} \rho_r)} \right)\\
&\leq \quad \frac{1}{\rho_{\bar{r}} v_{\bar{r}}} \left(\rho_{\bar{r}} - C_{\bar{l}} \frac{\rho_{\bar{r}}}{\sum_{r \in R(\bar{l})} \rho_r} \right) &(50)\\
&= \quad -\frac{1}{v_{\bar{r}} \sum_{r \in R(\bar{l})} \rho_r} \left(C_{\bar{l}} - \sum_{r \in R(\bar{l})} \rho_r \right)\\
&\leq \quad -\epsilon.
\end{aligned}
$$

The inequality (49) is due to the condition (40) on the fluid arrival process and the inequality (57) with link $\bar{l}$ instead of l in Lemma 6.1. Inequality (50) follows from the fact that

$$\frac{\bar{N}_{\bar{r}}(t)}{\rho_{\bar{r}}} \geq \frac{\sum_{r \in R(\bar{l})} \bar{N}_r(t)}{\sum_{r \in R(\bar{l})} \rho_r},$$

which is simply implied by the definition of $\bar{r}$ in the equation (47).

Proof of Proposition 4.1 for allocation $\bar{\Lambda}^U(\cdot,\cdot)$: The stability of the fluid network model with allocation $\bar{\Lambda}^U(\cdot,\cdot)$ follows from Theorem 2.3 (i) of Ye and Chen (2001) after the following claims (a)-(c) are verified.
(a) (Scale property) For any fluid solution $\bar{N}(\cdot)$, the process $\frac{1}{z}\bar{N}(z\cdot)$ is also a fluid solution for any fixed $z > 0$.
(b) (Shift property) For any fluid solution $\bar{N}(\cdot)$, the process $\bar{N}(s+\cdot)$ is also a fluid solution for any fixed $s \geq 0$.
(c) (Lyapunov condition) For any fluid solution $\bar{N}(\cdot)$, there is an absolutely continuous function $f(t)$ such that for almost all $t \geq 0$,

$$w_1(||\bar{N}(t)||) \leq f(t) \leq w_2(||\bar{N}(t)||), \tag{51}$$
$$\dot{f}(t) \leq -w_3(||\bar{N}(t)||), \tag{52}$$

where $w_i(\cdot), i = 1,2,3$ are three strictly increasing continuous functions with $w_i(0) = 0, i = 1,2,3$.

Claims (a) and (b) are straightforward to verify, and we omit the details. Claim (c) is verified by choosing the function $f(t)$ as

$$f(t) = \sum_{r \in R} \int_0^{\bar{N}_r(t)} \nu_r \partial_2 U_r(y, \rho_r(1+\delta)) dy$$

where δ is sufficiently small so that $\{\rho_r(1+\delta), r \in R\}$ still satisfies the normal offered load condition (22) with ρ_r replaced by $\rho_r(1+\delta)$. Then, $f(t)$ is absolutely continuous because $\bar{N}(t)$ is Lipschitz continuous and the integrands are uniformly bounded on any compact set of y (cf. conditions (14) and (16)). In addition, we define three strictly increasing continuous functions $w_i(\cdot), i = 1,2,3$ as follows

$$
\begin{aligned}
w_1(y) &= \frac{y}{2|R|} \underline{w}(\frac{y}{2|R|}),\\
w_2(y) &= y \bar{w}(y),\\
w_3(y) &= (\min_{r \in R} \lambda_r) \delta \underline{w}(\frac{y}{|R|}),
\end{aligned}
$$

where

$$\underline{w}(y) = \min_{r\in R} \{\nu_r \cdot \partial_2 U_r(y, \rho_r(1+\delta))\},$$
$$\bar{w}(y) = \max_{r\in R} \{\nu_r \cdot \partial_2 U_r(y, \rho_r(1+\delta))\}.$$

Then, $w_1(0) = w_2(0) = w_3(0) = 0$ by the assumption (14). To verify the left hand side of (51), letting $\hat{r} \in R$ be a route such that $\bar{N}_{\hat{r}}(t) = \max\{\bar{N}_r(t) : r \in R\}$, we have

$$\begin{aligned} f(t) &\geq \sum_{r\in R} \int_{\bar{N}_r(t)/2}^{\bar{N}_r(t)} \underline{w}(y)dy \geq \sum_{r\in R} \frac{\bar{N}_r(t)}{2} \underline{w}(\frac{\bar{N}_r(t)}{2}) \\ &\geq \frac{\bar{N}_{\hat{r}}(t)}{2} \underline{w}(\frac{\bar{N}_{\hat{r}}(t)}{2}) \geq \frac{||\bar{N}(t)||}{2|R|} \underline{w}(\frac{||\bar{N}(t)||}{2|R|}) \\ &= w_1(||\bar{N}(t)||). \end{aligned}$$

To verify the right hand side of (51), due to the assumption (16), we have

$$\begin{aligned} f(t) &\leq \sum_{r\in R} \int_0^{\bar{N}_r(t)} \bar{w}(y)dy \leq \sum_{r\in R} \bar{w}(\bar{N}_r(t))\bar{N}_r(t) \\ &\leq \bar{w}(||\bar{N}(t)||)||\bar{N}(t)|| = w_2(||\bar{N}(t)||). \end{aligned}$$

To verify the inequality (52), we note that

$$\begin{aligned} \dot{f}(t) &= \sum_{r\in R} \dot{\bar{N}}_r(t) \cdot \nu_r \cdot \partial_2 U_r(\bar{N}_r(t), \rho_r(1+\delta)) \\ &= \sum_{r\in R} \dot{\bar{X}}_r(t) \cdot \partial_2 U_r(\bar{N}_r(t), \rho_r(1+\delta)) \\ &= \sum_{r\in R} [\dot{\bar{A}}_r(t) - \bar{\Lambda}_r^U(\bar{N}(t), \dot{\bar{A}}(t))]\partial_2 U_r(\bar{N}_r(t), \rho_r(1+\delta)) \\ &= \sum_{r\in S_t} [\dot{\bar{A}}_r(t) - \bar{\Lambda}_r^U(\bar{N}(t), \dot{\bar{A}}(t))]\partial_2 U_r(\bar{N}_r(t), \rho_r(1+\delta)) \\ &\leq \sum_{r\in S_t} [\rho_r - \bar{\Lambda}_r^U(\bar{N}(t), \dot{\bar{A}}(t))]\partial_2 U_r(\bar{N}_r(t), \rho_r(1+\delta)), \end{aligned} \tag{53}$$

where $S_t := \{r \in R : \bar{N}_r(t) > 0\}$. Now, with a careful thought, it is not difficult to check that $\{\rho_r(1+\delta), r \in R\}$ is a feasible solution to the following optimization problem while $\{\bar{\Lambda}_r^U(\bar{N}(t), \dot{\bar{A}}(t)), r \in S_t\} = \{\bar{\Lambda}_r^U(\bar{N}(t)), r \in S_t\}$ is its unique optimal solution:

$$\begin{aligned} \text{maximize} \quad & \sum_{r\in S_t} U_r(\bar{N}_r(t), \Lambda_r), \\ \text{subject to} \quad & \sum_{r\in S_t\cap R(l)} \Lambda_r \leq C_l, \quad \text{for } l \in L \\ & \Lambda_r \geq 0, \quad \text{for } r \in R. \end{aligned}$$

(Note that the summation in the objective is on all the routes in S_t, rather than R, and that the feasibility condition (3) does not appear here.) Then, the optimality of the allocation $\{\bar{\Lambda}_r^U(\bar{N}(t), \dot{\bar{A}}(t)), r \in S_t\}$, together with the concavity assumption (17) on the utility function, yields

$$\sum_{r\in S_t} \left(\rho_r(1+\delta) - \bar{\Lambda}_r^U(\bar{N}(t), \dot{\bar{A}}(t))\right) \cdot \partial_2 U_r(\bar{N}_r(t), \rho_r(1+\delta)) \leq 0,$$

or

$$\sum_{r\in S_t}\left(\rho_r - \bar{\Lambda}^U_r(\bar{N}(t),\dot{\bar{A}}(t))\right)\cdot \partial_2 U_r(\bar{N}_r(t),\rho_r(1+\delta))$$
$$\leq -\sum_{r\in S_t}\rho_r\cdot\delta\cdot\partial_2 U_r(\bar{N}_r(t),\rho_r(1+\delta)). \tag{54}$$

Consequently, from inequalities (53) and (54), we have

$$\begin{aligned}
\dot{f}(t) &\leq -\sum_{r\in S_t}\rho_r\cdot\delta\cdot\partial_2 U_r(\bar{N}_r(t),\rho_r(1+\delta)) \\
&= -\sum_{r\in R}\rho_r\cdot\delta\cdot\partial_2 U_r(\bar{N}_r(t),\rho_r(1+\delta)) \qquad (55)\\
&\leq -\sum_{r\in R}\lambda_r\cdot\delta\cdot\underline{w}(\bar{N}_r(t)) \\
&\leq -(\min_{r\in R}\lambda_r)\delta\sum_{r\in R}\underline{w}(\bar{N}_r(t)) \\
&\leq -(\min_{r\in R}\lambda_r)\delta\underline{w}(\max_{r\in R}\{\bar{N}_r(t)\}) \\
&\leq -(\min_{r\in R}\lambda_r)\delta\underline{w}(\frac{||\bar{N}(t)||}{|R|}) \\
&= -w_3(||\bar{N}(t)||),
\end{aligned}$$

where the equality (55) is due to the assumption (14). □

Lemma 6.1 *Consider the max-min fair bandwidth allocation $\Lambda^{mm}(\cdot)$ and the max-min algorithm given in Section 2.5. Let n be an ongoing connection state in either $\mathcal{Z}_+^{|R|}$ or $\mathcal{R}_+^{|R|}$, and suppose there are $K(\geq 1)$ levels of non-zero bottleneck, or in other words, the max-min algorithm stops at the $(K+1)$-st iteration for the given state n. Then, we have*

(a) the bandwidth allocated to each connection is increasing by the bottleneck level, i.e.,

$$a^{(1)}(n) < a^{(2)}(n) < \cdots < a^{(K)}(n); \tag{56}$$

(b) for each route $r\in R$, the bandwidth allocation $\Lambda^{mm}_r(n)$ has a lower bound

$$\Lambda^{mm}_r(n) \geq C_l\frac{n_r}{\sum_{r'\in R(l)} n_{r'}} \qquad \text{for some } l\in r. \tag{57}$$

Proof: It is direct to check from the definition of the max-min fair policy or the max-min algorithm that the inequality (56) holds.

To see the lower bound (57), suppose route r traverses a k-th level bottleneck link l (thus $r\in R(l)$), and note that $R(l)$ (the set of all routes using link l) can be partitioned into $R(l) = \left(\cup_{i=1}^k R_B^{(i)}(l)\right)\cup\{r'\in R(l): n_{r'}=0\}$ with $R_B^{(i)}(l) = R(l)\cap R_B^{(i)}$. Then, we have

$$\begin{aligned}
C_l &= \sum_{r'\in R(l)}\Lambda^{mm}_{r'}(n) = \sum_{i=1}^k\sum_{r'\in R_B^{(i)}(l)} n_{r'}a^{(i)}(n) \\
&\leq \sum_{i=1}^k\sum_{r'\in R_B^{(i)}(l)} n_{r'}a^{(k)}(n) = a^{(k)}(n)\sum_{r'\in R(l)} n_{r'}, \qquad (58)
\end{aligned}$$

where the first equality holds since link l is a (k-th level) bottleneck; the second equality is due to the definition (10) noting that $R_B^{(i)}(l) \subset R_B^{(i)}$; and the inequality is due to the inequality (56). Finally, (58) and (10) together imply the inequality (57). □

Lemma 6.2 *The max-min fair allocation $\Lambda^{mm}(n)$ and the U-utility maximizing allocation $\Lambda^U(n)$ (defined in Section 2.5 and 2.7 respectively) have the following properties:*

(a) (Partial radial homogeneity) For a fixed state $n \in \mathcal{R}_+^{|R|}$ and any positive constant c, equation

$$\Lambda_r(cn) = \Lambda_r(n) \tag{59}$$

holds for any $r \in R$ such that $n_r > 0$.

(b) (Partial continuity) Suppose a sequence of states $\{n^j, j = 1, 2, \cdots\} \subset \mathcal{R}^{|R|}$ converges to $n \in \mathcal{R}^{|R|}$ as $j \to \infty$. Then,

$$\Lambda_r(n^j) \to \Lambda_r(n), \quad \textit{as } j \to \infty, \tag{60}$$

for any $r \in R$ such that $n_r > 0$.

Proof: The partial radial homogeneity for allocation Λ^{mm} can be seen directly from its definition in Section 2.5, in particular, equations (9) and (10). On the other hand, the partial radial homogeneity for allocation Λ^U is only a repetition of the assumption (18) in the definition of allocation Λ^U in Section 2.5.

The partial continuity for allocation $\Lambda^{mm}(n)$ is proved by induction as follows. Let $a^{(k)}(n)$, $L^{(k)}$, $R^{(k)}$, $R^{(k)}(l)$, $L_B^{(k)}$, $R_B^{(k)}$, $C_l^{(k)}$, and K be determined for the given state n using the max-min algorithm presented in Section 2.5. Fix number k ($1 \le k \le K$) for this moment. Assume the convergence in (60) is true for all the routes passing through the first $(k-1)$st level links, or

$$\Lambda_r^{mm}(\hat{n}) \to \Lambda_r^{mm}(n), \quad \text{as } \hat{n} \to n, \quad \text{for } r \in R_B^{(i)} \text{ and } 1 \le i \le k-1. \tag{61}$$

Then, it suffices to prove

$$\Lambda_r^{mm}(\hat{n}) \to \Lambda_r^{mm}(n), \quad \text{as } \hat{n} \to n, \quad \text{for } r \in R_B^{(k)}, \tag{62}$$

to complete the induction argument for all the routes $r \in R \setminus R_B^{(K+1)}$, noting that $R \setminus R_B^{(K+1)} = \cup_{i=1}^K R_B^{(i)} = \{r \in R : n_r > 0\}$.

Define

$$\hat{C}_l^{(k)}(\hat{n}) = C_l - \sum_{1 \le i \le k-1} \sum_{r \in R_B^{(i)}(l)} \Lambda_r^{mm}(\hat{n}),$$

where $R_B^{(i)}(l) := R_B^{(i)} \cap R(l)$ is the set of the routes that are the i-th level bottleneck routes and traverse link l. Let

$$\hat{a}^{(k)}(\hat{n}) = \min\left\{ \frac{\hat{C}_l^{(k)}(\hat{n})}{\sum_{r \in R^{(k)}(l)} \hat{n}_r} : l \in \cup_{i=k}^K L_B^{(i)} \right\}.$$

Then, it is direct to check that

$$\hat{a}^{(k)}(\hat{n}) \to a^{(k)}(n) \quad \text{as } \hat{n} \to n.$$

According to the definition of the max-min allocation policy, we have, for all $r \in R_B^{(i)},\ k \leq i \leq K$,

$$\Lambda_r^{mm}(\hat{n}) \geq \hat{n}_r \hat{a}^{(k)}(\hat{n}).$$

Thus,

$$\liminf_{\hat{n} \to n} \Lambda_r^{mm}(\hat{n}) \geq n_r a^{(k)}(n).$$

Particularly, for $r \in R_B^{(k)}$, we have

$$\liminf_{\hat{n} \to n} \Lambda_r^{mm}(\hat{n}) \geq \Lambda_r^{mm}(n) \left(= n_r a^{(k)}(n) \right). \tag{63}$$

On the other hand, we have the feasibility constraint

$$\sum_{r \in R_B^{(k)}(l)} \Lambda_r^{mm}(\hat{n}) \leq \hat{C}_l^{(k)}(\hat{n}), \quad \text{for } l \in L_B^{(k)}.$$

Hence, for any $r \in R_B^{(k)}$, there exists a link $l \in L_B^{(k)}$ such that

$$\Lambda_r^{mm}(\hat{n}) \leq \hat{C}_l^{(k)}(\hat{n}) - \sum_{r' \in R_B^{(k)}(l): r' \neq r} \Lambda_{r'}^{mm}(\hat{n}),$$

and thus

$$\limsup_{\hat{n} \to n} \Lambda_r^{mm}(\hat{n}) \leq C_l^{(k)} - \sum_{r' \in R_B^{(k)}(l): r' \neq r} \liminf_{\hat{n} \to n} \Lambda_{r'}^{mm}(\hat{n}),$$

which, combining the inequality (63), implies

$$\limsup_{\hat{n} \to n} \Lambda_r^{mm}(\hat{n}) \leq C_l^{(k)} - \sum_{r' \in R_B^{(k)}(l): r' \neq r} \Lambda_{r'}^{mm}(n) = \Lambda_r^{mm}(n). \tag{64}$$

Inequalities (63) and (64) together lead to

$$\lim_{\hat{n} \to n} \Lambda_r^{mm}(\hat{n}) = \Lambda_r^{mm}(n) \ \text{ for } r \in R_B^{(k)}.$$

The partial continuity property for allocation $\Lambda^U(n)$ is proved by an argument of contradiction as follows. Suppose the property does not hold, then there exists a sequence of states $\{n^j, j = 1, 2, \cdots\} \subset \mathcal{R}^{|R|}$ converging to state $n \in \mathcal{R}^{|R|}$ as $j \to \infty$, such that, for some $r \in R$ with $n_r > 0$, $\Lambda_r^U(n^j)$ converges to some $\tilde{\Lambda}_r \neq \Lambda_r^U(n)$. Denote $S = \{r \in R : n_r > 0\}$ and $S_j = \{r \in R : n_r^j > 0\}$, and without lost of generality, assume that $S \subset S^j$ for all $j = 1, 2, \cdots$. Let $\bar{\Lambda} = (\bar{\Lambda}_r)$ with $\bar{\Lambda}_r = \tilde{\Lambda}_r$ for $r \in S$ and $\bar{\Lambda}_r = 0$ otherwise. Then, it is direct to check that $\Lambda^U(n)$ and $\bar{\Lambda}$ are the unique optimal solution and a feasible solution, respectively, to the optimization problem (12), and thus

$$\sum_{r \in S} U_r(n_r, \tilde{\Lambda}_r) < \sum_{r \in S} U_r(n_r, \Lambda_r^U(n)). \tag{65}$$

(Note that the uniqueness on the optimal solution is due to the concavity assumption (17) on the objective function.) On the other hand, it is not difficult to verify that $\Lambda^U(n^j)$ and

$\Lambda^U(n)$ are the unique optimal solution and a feasible solution, respectively, to the optimization problem (12) with n replaced by n^j, noting that $S \subset S_j$ for all j. Therefore, we have

$$\sum_{r \in R} U_r(n_r^j, \Lambda_r^U(n^j)) \geq \sum_{r \in R} U_r(n_r^j, \Lambda_r^U(n)), \tag{66}$$

for all j. Letting $j \to \infty$, from the joint continuity of $U_r(\cdot, \cdot), r \in R$, we have

$$\sum_{r \in S} U_r(n_r, \tilde{\Lambda}_r) \geq \sum_{r \in S} U_r(n_r, \Lambda_r^U(n)), \tag{67}$$

since $n_r = 0$ for $r \in R \setminus S$ and $U_r(0, \cdot) \equiv 0$ for $r \in R$. Finally, the contradiction between inequalities (65) and (67) proves the partial continuity property for the allocation $\Lambda^U(n)$. □

Lemma 6.3 *(A variation of Arzela-Ascoli theorem) Suppose that $f_n : \mathcal{R}_+ \to \mathcal{R}$ ($n = 1, 2, ...$) is a sequence of functions with the following properties:*
(a) $\{f_n(0),\ n = 1, 2, ...\}$ is bounded, and
(b) there are a constant $M > 0$ and a sequence of positive numbers $\{\sigma_n,\ n = 1, 2, ...\}$ with $\sigma_n \to 0$ as $n \to \infty$, such that

$$|f_n(t) - f_n(s)| \leq M(t - s) + \sigma_n, \tag{68}$$

Then, it has a subsequence that converges uniformly on compact set (u.o.c.) to Lipschitz continuous function $f : R_+ \to R$ with Lipschitz constant M.

Proof: Let $\mathcal{T}$ be any compact set (or, for simplicity, any close interval) in $\mathcal{R}_+$. Denote the set of rational numbers as F. Using a standard diagonal process argument (for example, see Rudin (1987)), we can find a subsequence of $\{f_n(\cdot)\}$, also denoted by $\{f_n(\cdot)\}$, such that it converges for any $t \in F \cap \mathcal{T}$. Let

$$f(t) = \lim_{n \to \infty} f_n(t) \qquad \text{for any } t \in F \cap \mathcal{T}. \tag{69}$$

Let ϵ be any given positive real number. Pick an integer N_0 such that $\sigma_n < \epsilon$ for all $n \geq N_0$, and a real number δ such that $0 < \delta < \epsilon/M$. Then, by the condition (68), we have

$$|f_n(t') - f_n(t'')| \leq 2\epsilon, \tag{70}$$

for any $n \geq N_0$ and $t', t'' \in \mathcal{T}$ with $|t' - t''| < \delta$. Now, cover the compact set $\mathcal{T}$ with some open interval $I_1, ..., I_M$ of length δ, and pick $t_i \in I_i \cap F$ ($i = 1, ..., M$). By the convergence assumption (69), there exists an integer $N \geq N_0$ such that

$$|f_m(t_i) - f_n(t_i)| \leq \epsilon, \tag{71}$$

for $i = 1, ..., M$ and any $m, n \geq N$.

Pick any $t \in \mathcal{T}$. Then, there exists an $i \in \{1, ..., M\}$ such that $t \in I_i$, and thus $|t - t_i| < \delta$. By the definitions of δ and N, and inequalities (70) and (71), we have

$$\begin{aligned} |f_m(t) - f_n(t)| &\leq |f_m(t) - f_m(t_i)| + |f_m(t_i) - f_n(t_i)| + |f_n(t_i) - f_n(t)|, \\ &\leq 2\epsilon + \epsilon + 2\epsilon = 5\epsilon, \end{aligned}$$

for any $m, n \geq N$. That is, function f can be extended so that

$$f(t) = \lim_{n \to \infty} f_n(t) \qquad \text{for any } t \in \mathcal{T}.$$

The Lipschitz property of $f(\cdot)$ follows directly from the condition (68). $\square$

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