BOUNDS FOR THE NUMBER OF DC OPERATING POINTS OF TRANSISTOR NETWORKS

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Abstract

A transistor network consisting of linear positive resistors, q exponential diodes, and p Ebers-Moll model bipolar transistors has at most $(d+1)^d\, 2^{d(d-1)/2}$ dc operating points, where $d = q+2p$. Bounds are also obtained for the number of operating points in circuits using other models for bipolar transistors.

1. Introduction

Circuits with nonlinear elements may have multiple discrete dc operating points. In contrast, circuits consisting of positive linear resistors possess either one dc operating point, or, in special cases, a continuous family of dc operating points. We consider the problem of estimating upper bounds for the number of operating points of circuits consisting of linear positive resistors, exponential diodes, and Ebers-Moll modeled [2] bipolar transistors. (Inductors and capacitors do not play a role in establishing a circuit's dc operating point and can be removed from the circuits by being short-circuited and open-circuited, respectively.) Lee and Willson [8] showed that a circuit containing two transistors possesses at most three dc operating points. There exist networks with p transistors that have $2^p - 1$ dc operating points [17]. However, it appears that no upper bounds of any kind are known for general circuits having $p > 2$ transistors.

A circuit's operating points are solutions of a system of nonlinear equations

$$\mathcal{F}(\mathbf{x}) = \mathbf{0}. \tag{1}$$

We present explicit upper bounds for the number of isolated zeros of such systems when circuit equations are of Sandberg-Willson form [12], under suitable assumptions on $v - i$ response functions of nonlinear diodes. By applying Theorem 1 given below, we obtain the upper bound $(d+1)^d\; 2^{d(d-1)/2}$ for a system of network equations with q exponential diodes and p Ebers-Moll model transistors, where $d = q+2p$. These bounds for the number of dc operating points of transistor circuits are a direct application of a result of Khovanskii [6] in real algebraic geometry. We also obtain upper bounds for circuits with transistors modeled by piecewise-linear diodes and by piecewise-exponential diodes.

We note that to obtain finite upper bounds on dc operating points, it is necessary to make assumptions on the $v-i$ response curves of nonlinear diodes used in the circuits, because there are examples of circuits having infinitely many operating points, due to Nishi [10]. The bounds we do obtain are, however, very likely far from best possible for the circuits to which they apply. For networks containing p Ebers-Moll model transistors and an arbitrary number of linear positive resistors and independent current and voltage sources, it seems reasonable to conjecture an upper bound of $2^p - 1$ dc operating points.

2. Circuit Equations

We study network equations for a network with q exponential diodes and p bipolar transistors given in the general Sandberg-Willson form [12]–[15]. This is a system of $d = q + 2p$ equations

$$\mathbf{QT}\mathcal{F}(\mathbf{x}) + \mathbf{Px} + \mathbf{c} = \mathbf{0}\ , \tag{2}$$

where

$$\mathcal{F}(\mathbf{x}) := \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_d(x_d) \end{bmatrix} \tag{3}$$

and the $f_i(x_i)$ are monotone increasing functions with $f_i(0) = 0$, characterizing exponential diodes. Here exponential diodes have response curves

$$f_i(x_i) = m_i(e^{n_i x_i} - 1)\ . \tag{4}$$

The $d \times d$ constant matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{T}$ possess the following properties:

(i) The $d \times d$ matrices $(\mathbf{Q}, \mathbf{P})$ form a *passive pair*. That is,

$$\text{for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \quad \text{if } \mathbf{Qx} = \mathbf{Py}, \quad \text{then } \langle \mathbf{x}, \mathbf{y} \rangle \geq 0\ , \tag{5}$$

where $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{d} x_i y_i$.

(ii) $\mathbf{T}$ is a block-diagonal matrix whose first p blocks are 2×2 block matrices of the form

$$\begin{bmatrix} 1 & -\alpha_{i+1} \\ -\alpha_i & 1 \end{bmatrix} , \quad 1 \le i \le p , \quad i \text{ odd} , \qquad (6)$$

followed by q 1×1 blocks, each equal to 1. The controlled-source current-gains satisfy $0 \le \alpha_i, \alpha_{i+1} < 1$.

The Ebers-Moll model [2] for a bipolar junction transistor consists of two nonlinear exponential diodes described by (4) in which $m_i n_i > 0$ and $m_i > 0$ for a *pnp* transistor, and $m_i < 0$ for an *npn* transistor. The 2×2 blocks in $\mathbf{T}\mathcal{F}(\mathbf{x})$ in (2) model a bipolar junction transistor as a pair of linearly coupled exponential diodes.

The model parameters m_i, n_i for $1 \le i \le 2p$ are also required to satisfy the passivity, no-gain, and reciprocity conditions given below.

An Ebers-Moll model bipolar transistor is *passive* [4] if and only if $f_i(x_i)$ and $f_{i+1}(x_{i+1})$ and transistor current-gains α_i and α_{i+1} satisfy

$$\begin{aligned} \alpha_{i+1} &\le \frac{m_i}{m_{i+1}} \le 1/\alpha_i \\ \alpha_{i+1} &\le \frac{n_i}{n_{i+1}} \le 1/\alpha_i . \end{aligned} \qquad (7)$$

The assumption of the no-gain property, i.e., no temperature difference between the two transistor pn junctions, implies a common functional form (4) for $f_i(x_i)$ and $f_{i+1}(x_{i+1})$. The necessary and sufficient conditions for an Ebers-Moll model bipolar transistor to possess the *no-gain* property [16] are:

$$\begin{aligned} \alpha_{i+1} \le & \frac{m_i}{m_{i+1}} \le 1/\alpha_i \\ n_i &= n_{i+1} . \end{aligned} \qquad (8)$$

The Ebers-Moll model parameters also satisfy the *reciprocity* condition [2]

$$m_i \alpha_i = m_{i+1} \alpha_{i+1} . \qquad (9)$$

3. Upper Bounds

We consider systems of nonlinear equations of the form

$$\mathbf{A}\mathcal{F}(\mathbf{x}) + \mathbf{B}\mathbf{x} + \mathbf{c} = \mathbf{0} , \qquad (10)$$

where $\mathbf{x} = (x_1, \ x_2, \ \ldots, \ x_d)$ and

$$\mathcal{F}(\mathbf{x}) = \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_d(x_d) \end{bmatrix} . \qquad (11)$$

This equation is equivalent to (2) with constant matrices $\mathbf{A} = \mathbf{QT}$ and $\mathbf{B} = \mathbf{P}$.

We discuss three different types of functional forms for the nonlinear diodes $f_i(x_i)$: exponential functions, piecewise-linear, and piecewise-exponential functions. The simplest model for a bipolar transistor is the Ebers-Moll model, which uses the exponential form. Piecewise-linear functions have often been used to approximate nonlinear resistors, see [1, p. 76].

For exponential functions we obtain an upper bound using the following result of Khovanskii [6], [8, p. 12].

Theorem 1. (Khovanskii) *Consider a system of n polynomial equations*

$$P_i(x_1, \ldots, x_n, y_1, \ldots, y_d) = 0 , \quad 1 \le i \le n , \ (12)$$

in $n + d$ variables $x_1, \ldots, x_n, y_1, \ldots, y_d$, in which each P_i is of total degree k_i. Suppose in addition that

$$y_i = \exp\left(\sum_{j=1}^{n} n_{ij} x_j \right) , \quad 1 \le i \le d , \qquad (13)$$

in which all coefficients n_{ij} are real. The number of isolated real zeros in $\mathbb{R}^{n+d}$ of this system is at most

$$k_1 k_2 \cdots k_n (\sum_{i=1}^{n} k_i + 1)^d \ 2^{d(d-1)/2} . \qquad (14)$$

Proof. See [7], p. 12 and Chapters 2 and 3. □

The proof method of Khovanskii has inefficiencies which suggest that the bound (14) may not be the right order of magnitude in its dependence on d. Perhaps the right order of magnitude would replace the term $2^{d(d-1)/2}$ with c^d for some positive constant c.

Theorem 1 immediately yields the following upper bound:

Theorem 2. *Consider a system of nonlinear equations*

$$\mathbf{A}\mathcal{F}(\mathbf{x}) + \mathbf{B}\mathbf{x} + \mathbf{c} = \mathbf{0} \qquad (15)$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and

$$\mathcal{F}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_d(\mathbf{x}) \end{bmatrix} \qquad (16)$$

has entries $f_i(\mathbf{x}) = \exp(\sum_{j=1}^{d} n_{ij} x_j)$ for $1 \le i \le d$. This system has at most

$$(d+1)^d \ 2^{d(d-1)/2} \qquad (17)$$

isolated zeros.

Proof. This follows by applying Theorem 1 with $n = d$ and $k_i = 1$, for $1 \le i \le d$. □

Theorem 2 applies to circuit equations with Ebers-Moll modeled bipolar transistors and exponential diodes. This is achieved by absorbing the constant terms in the nonlinear diodes (4) in the Ebers-Moll model into the constant term $\mathbf{c}$ in (15).

Theorem 2 also applies to circuit equations with Gummel-Poon transistor models [3], [5]. In the Gummel-Poon model additional functional dependence of the emitter and collector currents on the "base charge" is introduced. This dependence manifests itself in the transistor currents now being sums of exponential functions with three distinct exponents.

It may well be that a stronger upper bound than (17) is valid for system (15) with

$$f_i(\mathbf{x}) = \exp\left(\sum_{j=1}^{d} n_{ij} x_j\right) \ , \quad 1 \le i \le d \ . \qquad (18)$$

For $d = 2$, the upper bound given by Theorem 2 is 18 real solutions. However, it can be shown that for $d = 2$ there can be at most 6 real solutions. (The argument, due to Poonen [11], is complicated and we omit it.) There are simple examples of this type having 4 real zeros. For example, the "decoupled" pair of equations

$$\begin{aligned} e^{x_1} - 2x_1 - 1 &= 0 \\ e^{x_2} - 3x_2 - 1 &= 0 \end{aligned} \qquad (19)$$

has 4 real solutions. We do not know if there exist such systems having 5 or 6 real solutions.

We now consider circuits with piecewise-linear modeled bipolar transistors [1]. In the piecewise-linear case there is very simple upper bound for the number of solutions. The bound does not assume monotonicity of the diode functions $f_i(x_i)$, nor that $f_i(0) = 0$.

Theorem 3. *Consider a system of d nonlinear equations*

$$\mathbf{A}\mathcal{F}(\mathbf{x}) + \mathbf{B}\mathbf{x} + \mathbf{c} = \mathbf{0} \qquad (20)$$

in which $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ *and*

$$\mathcal{F}(\mathbf{x}) = \begin{bmatrix} f_1(x_1) \\ \vdots \\ f_d(x_d) \end{bmatrix} \ , \qquad (21)$$

where each function $f_i(x_i)$ *is piecewise-linear with* k_i *pieces. This system has at most*

$$k_1 k_2 \cdots k_d \qquad (22)$$

isolated zeros.

Proof. A continuous piecewise-linear function with k pieces has the form

$$f(y) = a_\ell y + b_\ell \quad \text{for} \ \ y_\ell < y < y_{\ell+1} \ , \quad 1 \le \ell \le k \ , \qquad (23)$$

where $-\infty = y_1 < y_2 < \ldots < y_{k+1} = +\infty$, together with continuity requirement that

$$f(y_{\ell+1}) = a_\ell y_{\ell+1} + b_\ell = a_{\ell+1} y_{\ell+1} + b_{\ell+1} \ , \ 1 \le \ell \le k \ .$$

Call $S(\ell) = \{y : y_\ell \le y \le y_{\ell+1}\}$ the *ℓ-th segment* of f. For the functions $f_j(x_j)$ we denote their linear segments by $S_j(\ell)$ for $1 \le \ell \le k_j$. Each solution $x = (x_1, x_2, ..., x_d)$ of the system (20) has variable x_j falling in a particular linear segment of $f_j(x_j)$, for $1 \le j \le d$, call it $S_j(\ell_j)$. For each fixed choice of segments, the equation (20) becomes linear, and, hence, has a unique solution. Thus, any two solutions to (20) have different sets of segments $\{S_j(\ell_j) : 1 \le j \le d\}$. There are exactly $k_1 k_2 \cdots k_d$ possible segment sets, so the bound follows. □

The two methods above can be combined and applied to circuits with piecewise-exponential modeled bipolar transistors. Such models arise when approximating exponential functions in the Gummel-Poon model [5] over certain ranges of voltages where some exponential terms become negligible.

4. Concluding Remarks

The upper bounds stated in Section 2 depend on the particular form assumed for the nonlinear functions $f_i(\mathbf{x})$. This is unavoidable. For example, the equation

$$f(x) - 2x - \frac{1}{4} = 0 \ , \qquad (24)$$

with the monotone increasing function $f(x) = 2x + \sin x$, has infinitely many real solutions. Allowing small perturbations around a "nice" function does not eliminate such examples. For example, for any $\epsilon > 0$ the system

$$f(x) - 2x - \epsilon/4 = 0 \ , \qquad (25)$$

with $f(x) = 2x + \epsilon \sin x$, has infinitely many real solutions. Nishi [10] recently reported that transistor circuits may possess infinitely many solutions in some cases where the diode characteristics have positive first and second derivatives, i.e., are increasing convex functions.

It would be desirable to obtain upper bounds for the number of dc operating points that were insensitive to small perturbations in the $v - i$ response curves of elements in the circuit, because physical devices will contain such imperfections. It seems reasonable to allow only perturbations that preserve the sign of the first two derivatives. However, in view of the Nishi example [10], one still must put additional conditions on the matrices $\mathbf{A}, \mathbf{B}$ and vector $\mathbf{c}$ in (10) to rule out infinitely many solutions. Such conditions surely exist, for if $\mathbf{A} = \mathbf{I}$ and $\mathbf{B}$ is a P_0-matrix, then it is well known that the system (10) has at most one solution for each fixed $\mathbf{c}$ whenever the $f_i(x_i)$ are monotone increasing functions, see Willson [[15], Theorem 12]. It may well be that a finite upper bound can be obtained with the conditions $\mathbf{A}, \mathbf{B}$ weakened further, perhaps to the conditions on $\mathbf{QT}$ and $\mathbf{P}$ given in the Sandberg-Willson form (2).

The upper bounds of Section 2 do not make use of all properties that circuit equations possess. In particular, they make no use of properties (i) and (ii) of the Sandberg-Willson form equations (2), nor do they employ the passivity (7), the no-gain (8), and the reciprocity (9) properties of bipolar junction transistors. Improved upper bounds may be possible for such circuit equations with exponential-type diodes, assuming these properties hold. For example, it is known that the circuit equations in Sandberg-Willson form (2) for a circuit having at most two transistors ($d = 4,\ p = 2,\ q = 0$) have at most three isolated real zeros, see Lee and Willson [8]. However, one can find equations of the form of Theorem 1 for $d = 4$ that have 16 isolated real zeros.

Nishi and Kawane [9] present an interesting alternate approach to obtaining upper bounds for the number of solutions of network equations for nonlinear resistive circuit by assuming that the circuit elements satisfy monotone sign conditions on derivatives such as $\frac{df_i(x)}{dx} > 0$, or a convexity condition like $\frac{d^2 f_i(x)}{dx^2} > 0$, together with restrictions on the form of the matrices $\mathbf{QT}$ and $\mathbf{P}$ in (2). They announce an upper bound of 2^d operating points for systems (10) under such extra hypotheses. However, their arguments appear to be incomplete, even for $d = 2$. The example of Nishi [10] shows that the situation is delicate.

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