On the presence of bending stresses in inflated thin-walled structures

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Abstract: It is proposed that, in appropriate circumstances, membrane structures can experience bending moments. On uniformly inflating a thin sheet structure, which has a shape consisting of multiple curvatures, the structure will deform in such a way that the final shape will have a single radius of curvature, assuming that failure does not occur. It is the large change of shape from a multicurvature surface to a single curvature surface that causes bending moments to exist within a membrane. The validity of the hypothesis has been demonstrated using four finite element models, including an elliptical cylinder, an ellipsoid, a 'double' cone and a trileaflet heart valve.

Keywords: bending stresses, inflated thin-walled structures, finite element models

1 INTRODUCTION

In engineering there are many examples of structures that are classified as membranes; these range from pressure vessels and aircraft fuselages to bioprosthetic heart valves. The structures are 'thin walled' in the sense that the thickness of the walls is normally much smaller than the overall size of the structure (**1–4**). Since the walls are thin, when deformation is small the variation of the in-plane stresses through their thickness is also small and is usually assumed to be zero in regions removed from boundary constraints and geometric or loading discontinuities.

Bending arises from the variation of in-plane stress through the thickness of the wall. The magnitude of this variation in a membrane structure will usually be negligible compared with the in-plane or membrane stresses. Poisson and Lagrange, both independently, proposed a definition of a lamina or membrane in the early nineteenth century (**1**). Poisson defined a membrane as comprising two parallel planes which constitute faces, the distance between these faces defining its thickness. Both Poisson and Lagrange paid little attention to the difference between a membrane and a thin plate. In 1961, Goldenveizer (**2**) gave a qualitative description of the limitations of membrane theory. Timoshenko and Woinowsky-Krieger (**3**) showed that if the ratio of the membrane thickness to its characteristic dimension is less than 0.1, the bending stress in the membrane will be small compared to the in-plane stress and the bending moments and shear terms could be neglected in the formulation of membrane theory. In engineering, it is common practice to use the membrane theory in the design of shell-like structures, particularly in the absence of geometric features. The theory may not always be used in its analytical form, but the assumption of constant in-plane stress is frequently used in numerical analysis.

All of the above discussion is based on experience with conventional engineering structures. Although many of these are structurally flexible, they are usually made from materials that can only undergo relatively small strains prior to failure. In other words, the present view of membranes is dominated by those structures that deform through linear strains but may be subjected to geometric non-linearity. However, recent work by the authors and others studying the stress patterns in the leaflets of bioprosthetic heart valves has shown that membrane-like structures can develop bending stresses (**5–9**). The leaflets of a bioprosthetic heart valve experience pressure loading and are both thin and flexible. Intuitively, such a flexible leaflet would be expected to stretch significantly more than it will bend. It is this surmise that led several researchers in the early investigations of stress in bioprosthetic heart valves to use the theory of membranes in their studies of valve leaflets (**10–13**). However, more recent work has shown the presence of bending stresses of the same order of magnitude as the membrane stresses (**5**, **6**). The analysis of the leaflets of bioprosthetic heart valves is different from conventional engineering analyses in that the geometry of the leaflets is complex and the leaflets undergo large shape changes (**9**, **11–15**).

The MS was received on 3 January 1996 and was accepted for publication on 10 February 1997.

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Conventional engineering analysis of a shell-like structure, away from boundary constraints, is dominated by small strain theory, with the addition of some geometric non-linearity in certain circumstances. Recent studies on structures that lie outside these conventional constraints suggest that the results of membrane theory must be interpreted with care when analysing structures that undergo large deformation. Furthermore, membrane-like structures that undergo large deformation may develop bending stresses of similar size to the membrane stresses.

2 HYPOTHESIS ON INFLATED FLEXIBLE STRUCTURES

> A sheet of material of arbitrary initial shape and with its edges fixed from translating which is uniformly pressurized such that it undergoes large deformation will tend to deform towards a shape with a single radius of curvature. A large change in the radius of curvature will cause bending stresses in the membrane-like structure to become significant. A pressurized sheet that has its initial shape in the form of a structure with a single radius of curvature (e.g. a sphere) will deform by stretching. These observations should be true for all materials, including the linearly elastic material studied here and non-linear inelastic materials.

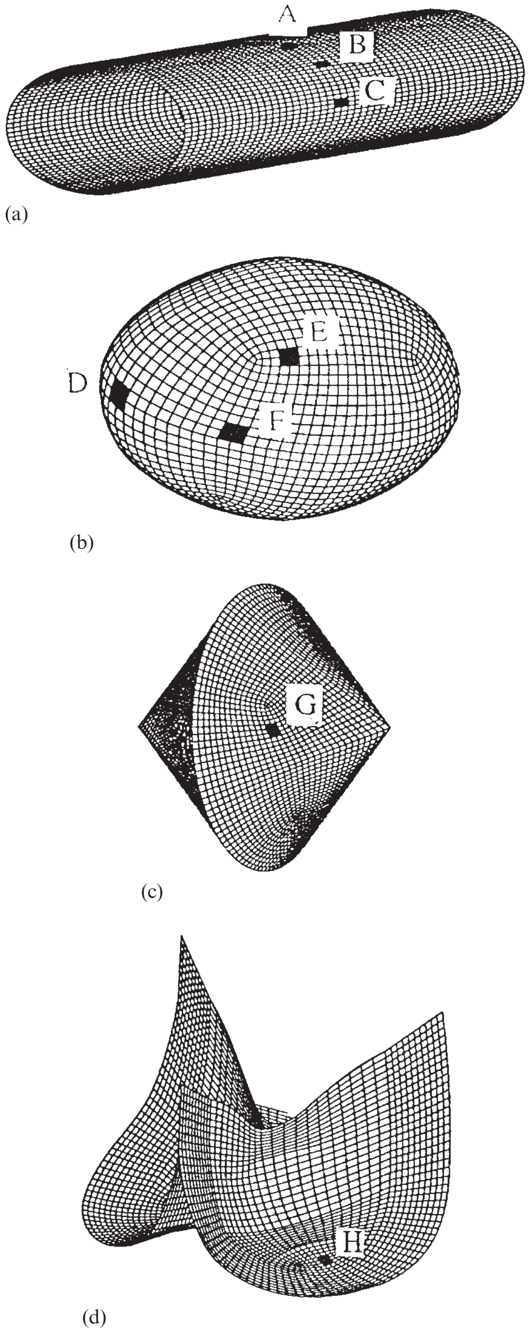
Several examples were used to demonstrate the reasonableness of this hypothesis. However, a formal proof is beyond the present scope of the work. Four membrane-like structures were chosen as exemplars. They were an elliptical cylinder, an ellipsoid, a ‘double’ cone and a trileaflet heart valve. The elliptical cylinder is effectively a two-dimensional model with its radius of curvature varying only in one plane, while the ellipsoid is a three-dimensional model with its radius of curvature varying in all directions. The ‘double’ cone was used to demonstrate that a peculiar shape will adhere to this hypothesis. The bioprosthetic heart valve model is an example of a three-dimensional structure with a complex non-uniform geometry. A dimensional analysis was carried out to determine the relationship between the bending stresses, the stiffness and the radii of curvature of the membrane structures.

3 THE MODELS

All of the geometric modelling was carried out with the commercial stress analysis suite, ANSYS (Version 5.0a). This modeller can be used to generate complex geometry with relative ease. The same suite was used to generate a mesh of elements for subsequent analysis by the finite element method. In simple terms, this method discretizes a continuous system into suitable small elements, to which the physical laws of matter can be applied. A commercial finite element package, OASYS LS-DYNA3D, was used to analyse each of the four models. This package is an explicit, time-stepping, finite element code capable of handling geometric non-linearity and which has a robust set of contact algorithms (**16**). The code was chosen because all of the four models were expected to undergo large shape changes, thereby requiring the strain–displacement (geometric) relationship to be non-linear. Also, effective contact algorithms would be essential because the leaflets of the heart valve come into contact with each other when the valve is closed. Four-noded Belytchko–Tsay quadrilateral shell elements (**16**) were used for all of the models. There are three integration points through the thickness of each shell element, thereby allowing the stress distribution through the thickness to be studied. Each integration point represents a layer in the shell element, i.e. top, middle and bottom layers. A membrane structure will have negligible variation of stress through its thickness. In this study, the maximum principal stresses of the top, middle and bottom layers of the shells were compared and the ratio of bending stress to membrane stress was calculated.

3.1 The elliptical cylinder

The cross-section of this cylinder had a major axis of 15.0 mm and a minor axis of 10.0 mm. The ellipse was ‘extruded’ over a length of 1000.0 mm to form an elliptical cylinder. The cylinder did not have endcaps, so the initial geometry was essentially two-dimensional, with an initial wall thickness of 0.3 mm, although the model and analysis were three dimensional. The cylinder was assumed to be made from a purely elastic material with a Young’s modulus of 2.8 MPa and a Poisson’s ratio of 0.3. The choice of these particular values for the material properties was arbitrary. However, a low modulus of elasticity increases the speed at which the time-stepping code analyses a model. The elliptical cylinder, ellipsoid, ‘double’ cone and trileaflet valve models were assigned the same material properties. A uniform pressure was applied to inflate the cylinder and increased linearly to produce a change of shape. Figure 1 shows the finite element mesh that was used. It consisted of 6000 quadrilateral shell elements. Three regions on the cylinder were chosen at which the ratios of bending stress to membrane stress were monitored. Initially, region C had the smallest radius of curvature and region A had the largest, while region B had an intermediate value. The three regions were selected to be away from the free edges of the cylinder in order to avoid any end effects. Subsequently, a parametric study was conducted in which the Young’s modulus and thickness were varied, as described later.



(a)

(b)

(c)

(d)

Fig. 1 Finite element meshes of the initial shape of (a) the elliptical cylinder, (b) the ellipsoid, (c) the 'double' cone and (d) the trileaflet valve. The regions where the ratios of bending stresses and the membrane stresses were monitored are marked as regions A to H

3.2 The ellipsoid

This model was generated in very much the same way as the elliptical cylinder. It had a cross-section with a major axis of 15.0 mm and a minor axis of 10.0 mm. This profile was rotated about the major axis to generate the model of the ellipsoid. Figure 1b shows the finite element mesh that was used and also the regions where the bending stress–membrane stress ratios were monitored. This mesh was made of 2400 four-noded quadrilateral shell elements. Again, three regions were chosen, namely those with the smallest, the largest and an intermediate radius of curvature (regions D, E and F respectively in Fig. 1b). The ellipsoid was pressurized in a similar manner to the inflation of a rugby ball, using a ramped pressure–time load curve. The dimensions of the initial cross-section and the material properties were the same as those used for the cylinder.

3.3 The 'double' cone

This model was created in a similar manner to the elliptical cylinder and the ellipsoid. It was generated by placing two cones base-to-base. Each cone had a height of 15.0 mm and a base diameter of 30.0 mm. The thickness of this model was also 0.3 mm. The initial shape of the 'double' cone is shown in Fig. 1c. It has 5400 four-noded quadrilateral shell elements. This model has two points of singularity, i.e. at the tips of the cones. When looking at the stress patterns for this model, the stresses in the regions of singularity were ignored. The element chosen for monitoring of the bending stress is marked as region G in Fig. 1c.

3.4 The trileaflet valve

The fourth model, the leaflets of a trileaflet heart valve, had the most complicated geometry. It was generated using dimensions measured from a Wessex porcine valve (**17**). Only the leaflets of the valve were modelled. The frame on to which the leaflets were sewn and which itself is sewn into the patient by the surgeon was assumed to be rigid. This frame is known as the stent. The nodes of the shell elements at the leaflet-stent boundaries were fixed from translating and rotating. The mesh is shown in Fig. 1d and consisted of 4650 four-noded quadrilateral shell elements. Only a single region, marked as region H in Fig. 1d, was monitored for bending. This region corresponds to the belly of the leaflet. It was in this region that all the normal conditions for the applicability of the theory of membranes are satisfied (**1**, **2**), yet in previous analyses (**6**, **17**) it showed significant bending stresses. A uniform pressure was applied to the leaflets to simulate the end of the closing phase of the cardiac cycle when the leaflets coapt to prevent the reversal of blood flow. The leaflets experience the highest pressure in this phase of the cardiac cycle (**6**, **17**). The complete cardiac cycle was not simulated because, in the opening and early closing phases, the leaflets undergo a reversal of curvature (**9**). A ramped pressure–time load curve similar to the *in vivo* physiological pressure difference across the aortic valve at the end of systole was used. The pressure difference started at zero pressure and was ramped up to 12.8 kPa in 8 ms. The leaflets of the valve have material properties that are non-linear and elastic. The stress–strain curve is J-shaped with low elasticity at small strains and increasing stiffness as the strains become larger (**9**). Typically, the low end of the stress–strain curve has a stiffness of about 3.0 MPa and the high end of the stress–strain curve has a stiffness of approximately 30.0 MPa. In this study, the leaflets were assumed to have linear elastic properties with a Young's modulus of 2.8 MPa, Poisson's ratio of 0.3 and a thickness of 0.3 mm.

4 DIMENSIONAL ANALYSIS

There are several parameters that affect the level of bending in membrane structures. A dimensional analysis was conducted in order to observe the effects on the bending stress of the radius of curvature, the thickness of the structure, the applied pressure and the Young's modulus of the material. The Buckingham pi method (**18**) was used to produce three dimensionless parameters from the six variables—bending stress, membrane stress, Young's modulus (E), radius of curvature (r), thickness (t) of the membrane and the pressure (P) at which the maximum ratio of bending to membrane stress occurs. The three dimensionless parameters were the stress ratio (ratio of the bending to membrane stress), the stiffness ratio (ratio of Young's modulus to the applied pressure) and the curvature ratio (ratio of the initial radius of curvature to the thickness). These parameters were used to generate a set of curves by varying the Young's modulus of the elliptical cylinder between 2.0 and 50.0 MPa, in increments of 2.0 MPa from 2.0 to 10.0 MPa and increments of 5.0 MPa from 10.0 to 50.0 MPa, and by varying the thickness of the cylinder in the following steps: 0.05, 0.10, 0.20 and 0.30 mm.

5 RESULTS AND DISCUSSION

The ratio of the bending stress to the membrane stress in each specified region was monitored and the results are shown in Figs 2 to 4. It is clear from the results that each of the four models showed signs of significant bending at some stage in the analysis. All four models were dominated by membrane responses when the applied pressure and the deformation were small. The shape changes at this stage were insignificant, with the current shape differing little from the initial shape. However, all of the models started to deform when the pressure was increased further, as predicted by the hypothesis. The elliptical cylinder deformed into an approximately circular cylinder with only one radius of curvature in its cross-section, as shown in Fig. 5a. At high pressure, the ellipsoid deformed into a sphere, as shown also in Fig. 5b. The 'double' cone was almost spherical in its fully inflated state, with the exception of the regions at the tips of the cones. However, as the mesh of this 'double' cone was refined, the tips of the cones became less pronounced. In the case of the more complex trileaflet valve, the belly of the leaflet formed a segment of a spherical bowl (Fig. 5d), which is also a shape with only a single radius of curvature. None of the models underwent further shape change once an approximately constant radius of curvature had been achieved.

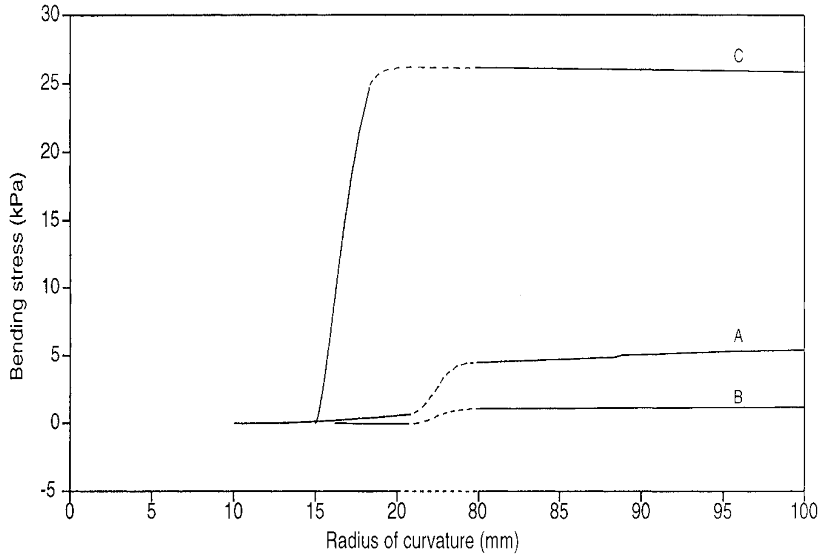


Fig. 2 The development of bending stress with increasing radius of curvature at regions A, B and C in the elliptical cylinder shown in Fig. 1

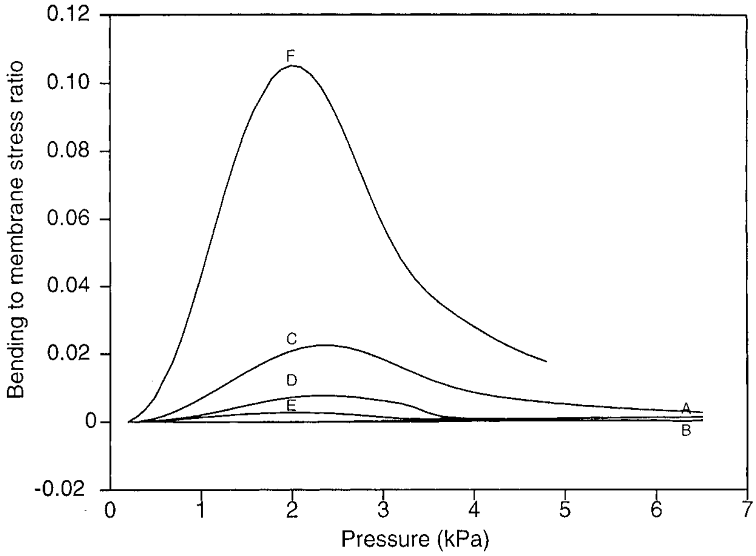


Fig. 3 A plot showing the variation of bending stress–membrane stress ratio with pressure in regions A to F for the cylinder and ellipsoid in Fig. 1

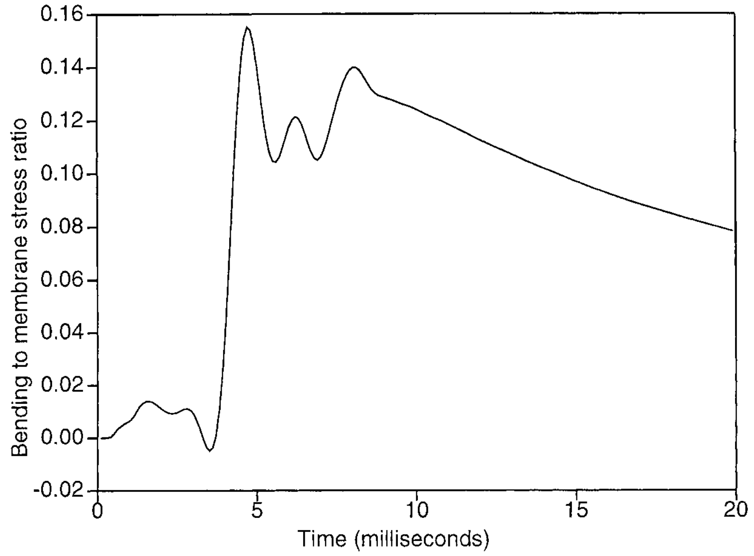


Fig. 4 A graph of the ratio of the bending stress to membrane stress against time for region H of the trileaflet valve

The emergence of bending stresses in these examples is strongly associated with large changes in the radius of curvature. The change of the bending stress with radius of curvature for the elliptical cylinder (Fig. 2) shows that bending stress increases with increasing radius of curvature. The bending stress remained almost constant once the cross-sectional shape with a single curvature was achieved.

In the 'dotted' region in Fig. 2, the results are sensitive to the rate at which the pressure is increased because the cylinder tends to oscillate on reaching its final shape as a consequence of its low stiffness. The point to note from Fig. 2 is that the bending stress was small when the radius of curvature was small, but increased significantly as the structure deformed to its final shape. The oscillation was also observed in the double-cone model. As expected, this model also showed signs of significant bending. The point of observation (region G in Fig. 1) experienced a bending stress as large as 16 per cent of the membrane stress. The bending stress–membrane stress ratio for this model is shown in Fig. 6. Figure 4 shows a plot of the bending stress in the belly of the trileaflet valve (region H in Fig. 1) with time, since the pressure is directly proportional to time. The peaks and troughs seen in this plot correspond to the fluttering of the leaflet as it was pressurized. This behaviour is similar to that of a soap bubble wobbling as it is inflated. When a sheet of material is forced to change its radius of curvature by a large amount, the layer of the material furthest away from the neutral plane of the sheet will stretch (or compress) significantly. This stretching (or compressing) of the layers away from the neutral axis will increase (or decrease) the in-plane stress of that layer, thus producing a variation in the stress through the thickness of the sheet. Figure 3 shows the ratio of bending stress to membrane stress for the elliptical cylinder and the ellipsoid. With increasing pressure, the ratio increases to a maximum and then decreases to some small value. The maximum corresponds to the transition between the multicurvature shape and the shape with a single radius of curvature. Once the latter shape was achieved, all the models remained in this shape but continued to expand with increasing pressure. The expansion of the single-curvature shapes increases the membrane stress, thus reducing the ratio of bending stress to membrane stress.

The relative importance of material stiffness has been studied in a series of numerical simulations of the elliptical cylinder. The results are expressed in Fig. 7 in dimensionless form. They show that bending exists in membrane structures which deform in the manner stated by the hypothesis. The amount of bending decreases with increasing curvature ratio. However, for a fixed curvature ratio, the effect of bending increases with the stiffness of the membrane. The data of Fig. 7 can be used as a guide to estimate the amount of bending that might exist in membrane structures where bending stresses may be important. The results from the analysis of the elliptical cylinder in the previous section correspond to the lower end of curve ‘50’ in Fig. 7.

The presence of bending stresses in membrane structures is a phenomena that is independent of the material properties. Bending is generated as a result of the large shape changes and is not dependent on the material properties. Hence, a membrane constructed from nonlinear material or inelastic material that undergoes large shape changes will generate bending stresses. However, the magnitude of the bending stress is dependent on the material properties.

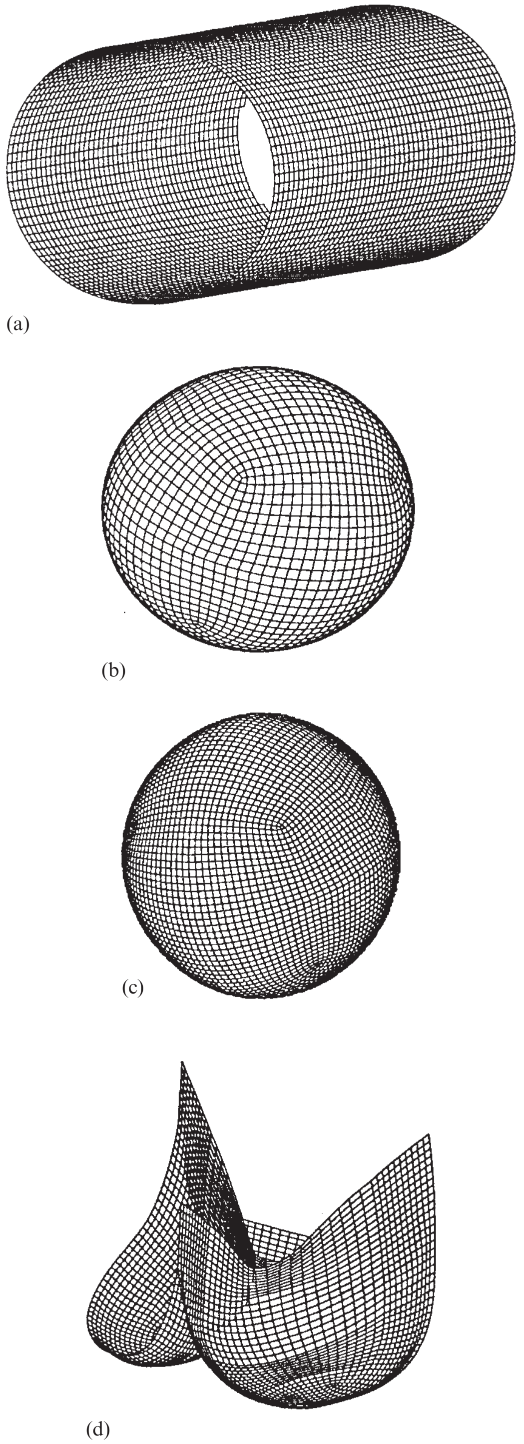


Fig. 5 Finite element meshes corresponding to the final shapes of (a) the elliptical cylinder, (b) the ellipsoid, (c) the ‘double’ cone and (d) the trileaflet valve. The elliptical cylinder has become a circular cylinder and the ellipsoid and the cone have become spherical. The belly of the trileaflet valve has formed a bowl, which also has a single radius of curvature

6 CONCLUSION

The work described has shown that structures which are usually classed as membranes can develop bending stresses if the structure is pressurized such that it deforms severely. Any structure when sufficiently inflated will deform into a shape with a single radius of curvature. The bending stresses are generated due to the large change in the radius of curvature from the initial to the final shape. The four models analysed using the finite

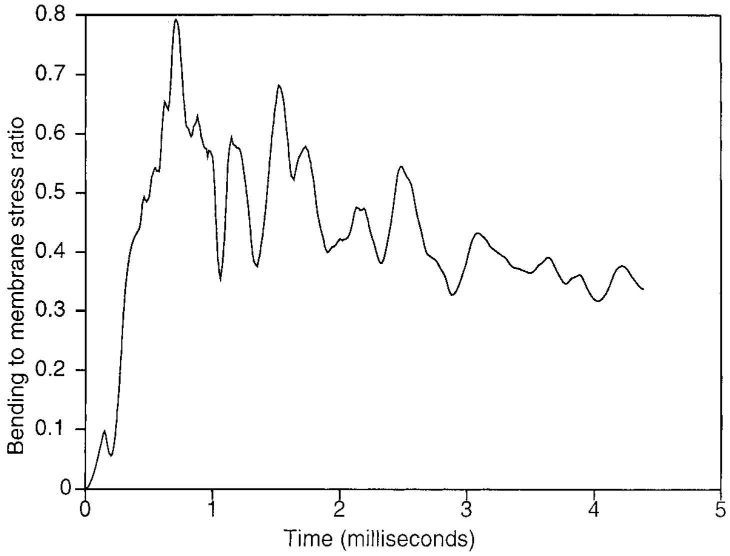


Fig. 6 A graph of the ratio of the bending stress to membrane stress against time for region G of the 'double' cone model

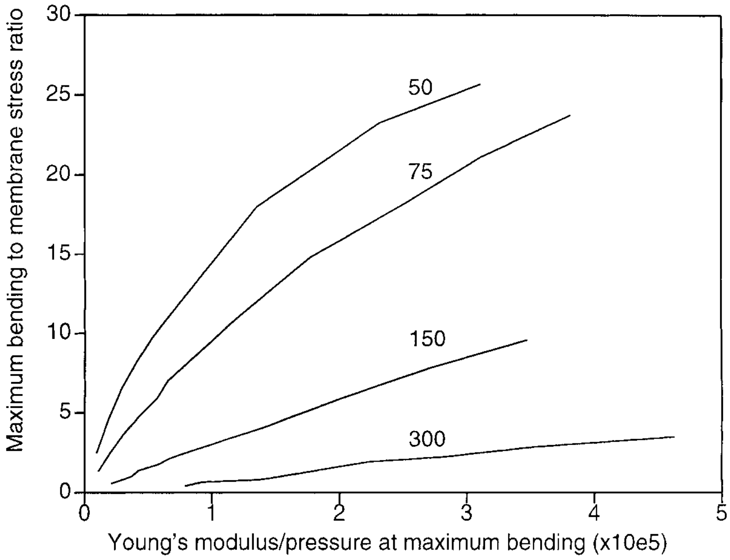


Fig. 7 A plot, for various ratios of radius to thickness of the cylinder, of the ratio of the bending stress to membrane stress at region C against the ratio of Young's modulus to the pressure corresponding to maximum bending

element method showed that the hypothesis on the large deformation of membrane structures was sound.

ACKNOWLEDGEMENTS

The authors acknowledge with thanks the support of Dr J. Miles and Ove Arup and Partners in providing access to software and for technical assistance, and the CVCP and the University of Sheffield in funding a studentship for G. Chew.

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