DICTATORIAL DOMAINS IN PREFERENCE AGGREGATION

by

Ugur Ozdemir*

and

M.Remzi Sanver*

ABSTRACT[1]

We call a domain of preference orderings "dictatorial" if there exists no Arrovian (Pareto optimal, IIA and non-dictatorial) social welfare function defined over that domain. In a finite world of alternatives where indifferences are ruled out, we identify a condition which implies the dictatoriality of a domain. This condition –to which we refer as "being essentially saturated"- is fairly weak. In fact, independent of the number of alternatives, there exists an essentially saturated (hence dictatorial) domain which consists of precisely six orderings. Moreover, this domain exhibits the superdictatoriality property, i.e., every superdomain of it is also dictatorial. Thus, given m alternatives, the ratio of the size of a superdictatorial domain to the size of the full domain may be as small as 6 / m! – converging to zero as m increases.

* **Department of Economics, Istanbul Bilgi University; uozdemir@bilgi.edu.tr and sanver@bilgi.edu.tr**

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1. INTRODUCTION

Given a society confronting a set of alternatives, a social welfare function (SWF) is a mapping which assigns to every preference profile a rational ordering of alternatives. We know thanks to Arrow (1951) that when there are at least three alternatives, the only way to ensure the Pareto optimality of a SWF satisfying a pairwise independence condition (known as independence of irrelevance alternatives or simply IIA) is dictatoriality. In other words, there exists no Arrovian (Pareto optimal, IIA and non-dictatorial) SWF.

This impossibility result rests on a universal domain assumption, which requires from the SWF to be defined over all preference profile profiles consisting of rational individual orderings. However, there are restrictions of the universal domain which allow escaping from aggregation impossibilities. For example, the *single-peakedness* condition of Black (1948), the *single-cavedness* condition of Inada (1964) and the more general *value-restriction* condition identified by Sen (1966) lead to restricted domains over which the majority relation[2] is transitive - hence an Arrovian SWF exists.

Nevertheless, there are many other restricted domains over which Arrow's impossibility theorem would continue to hold. This leads us to define a dictatoriality concept over domains in a very natural way. We call a domain dictatorial whenever it exhibits no Arrovian SWF. For example the universal domain is dictatorial while single-peaked domains are not.

There is a vast literature exploring the properties of domains which render them dictatorial. What is called the *free triple condition* is at the center of this analysis. A triple[3] of alternatives is said to be free in a given domain if and only if this domain admits all possible orderings of these three alternatives. In fact, one of the earlier results of our direction of research has been established by Arrow (1963) himself: A domain where every triple is free has to be dictatorial. We refer to this as the standard result of the literature.

There are two ways to strengthen the standard result. One is by weakening the definition of a free triple. Kelly (1994a) shows that one does not need all 13 weak orderings of a triple in order to establish an Arrovian impossibility. The result relies on the existence of indifferences

[2] as is defined in May (1952) and further explored by Asan and Sanver (2002) and Woeginger (2003).

while we admit only strict orderings. In fact, in this world where indifferences are ruled out, a triple is free if and only if the domain admits all six orderings of the three alternatives. In other words, with three alternatives only, a domain is dictatorial if and only if it is the full domain. Thus, it is not possible to weaken the definition of a free triple when agents have strict orderings. So the only possibility for strengthening the standard result is to weaken the qualification "every".

The major result in this regards is by Kalai and Muller (1977) who show that a domain is not dictatorial if and only if it satisfies a condition they call nondictatorial decomposability. Thus, the negation of nondictatorial decomposability fully characterizes dictatorial domains. However, this negation leads to a condition which is not very easy to work with. In fact, it is Kalai, Muller and Satterthwaite (1979) who introduce a (stronger) saturation condition which is sufficient (but not necessary) for the dictatoriality of a domain. They show that every saturated domain is dictatorial – a result which they use to establish Arrovian impossibilities in economic environments. The idea behind their result is as follows: Although not all triples have to be free in a saturated domain, every free triple admits a local dictator. The saturatedness of the domain "connects" the free triples in such a way that they all exhibit the same dictator which turns out to be the global dictator of the social choice problem.

We identify an essential saturatedness condition which exploits the same idea while it is substantially weaker than being saturated. To give a rough idea about the relation between the strength of the conditions, in a world of $m \geq 3$ alternatives, a saturated domain must contain at least (m; 2)/3 free triples[4] while (m-2) free triples may suffice to induce an essentially saturated domain.[5] This enables us to show the existence of an essentially saturated (hence dictatorial) domain consisting of precisely six orderings – a result which is independent of the number of alternatives m. Thus, the ratio of the size of a dictatorial domain to the size of the full domain may be as small as 6 / m! – converging to zero as m increases.

We also explore the *superdictatoriality* property. A dictatorial domain D is superdictatorial whenever every superdomain of D is also dictatorial.[6] We identify a sufficient condition for a

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[3] By a triple, we mean three distinct alternatives.
[4] We write (m; k) for "m choose k" which equals m! (m-k)! / k!
[5] It may also worth noting that the standard result requires (m; 3) free triples.
[6] We know thanks to Bordes and Le Breton (1990) as well as Kelly (1994b) that not every dictatorial domain has to be superdictatorial.

dictatorial domain to be superdictatorial. In fact, our essentially saturated (hence dictatorial) domain consisting of precisely six orderings is also superdictatorial – hence showing that 6 / m! is also the ratio of the size of a superdictatorial domain to the size of the full domain. This result substantially strengthens Fishburn and Kelly (1997) who explore the cardinality of the minimal domain of superdictatorial preference profiles as a function of the number of agents and alternatives.

We present the basic notions in Section 2. Dictatorial domains are analyzed in Section 3. We first show that in the case of three alternatives, dictatorial domains coincide with those admitting all six orderings - hence the impossibility of relaxing the free triple assumption. We then prove that essentially saturated domains are dictatorial. By making use of this result, we are able to establish that independent of the number of alternatives, there exists a dictatorial domain which consists of precisely six orderings. Section 4 explores the superdictatoriality property. Section 5 makes some closing remarks.

2. BASIC NOTIONS

Taking any integers $n \geq 2$, we fix a society $\mathbf{N} = \{1,.., n\}$ confronting a non-empty finite set of candidates $\mathbf{A}$ with $\#\mathbf{A} = m \geq 3$. We write Π for the set of complete, transitive and antisymmetric binary relations over $\mathbf{A}$. Every $R \in \Pi$ stands for some individual preference over $\mathbf{A}$, which we write as R_i when it belongs to a particular individual $i \in \mathbf{N}$.[7] A preference profile is an n-tuple $\underline{R} \in \Pi^{\mathbf{N}}$ of individual preferences.

We let $D \subseteq \Pi$ stand for an arbitrary non-empty subdomain of Π and $\Re$ be the set of complete and transitive binary relations over $\mathbf{A}$. A SWF defined over D is a mapping $\alpha: D^{\mathbf{N}} \rightarrow \Re$. So given any $\underline{R} \in D^{\mathbf{N}}$, $\alpha(\underline{R})$ is a complete and transitive ordering over $\mathbf{A}$. We write $\alpha*(\underline{R})$ for the strict counterpart of $\alpha(\underline{R})$. We qualify a SWF α as *Arrovian* if and only if α is Pareto optimal

[7] For any x, y $\in$ **A**, x R y means " x at least as good as y". As R is antisymmetric, unless x and y coincide, we have x R y $\Rightarrow$ not y R x. For distinct alternatives, we use the strict counterpart P of R where x P y is interpreted as x being preferred to y.

(PO), independent of irrelevant alternatives (IIA) and non-dictatorial (ND).[8] We call a domain $D \subseteq \Pi$ *dictatorial* if it admits no Arrovian SWF (abbreviated as ASWF from now on).

A pair of alternatives[9] {x, y} is said to be non-trivial in a domain $D \subseteq \Pi$ if and only if there exists R, R' $\in$ D such that x R y and y R' x. So we say that {x, y} is trivial in D if and only if we have x R y $\Leftrightarrow$ x R' y for all R, R' $\in$ D. We consider domains which contain at least one non-trivial pair – hence at least two orderings.

3. DICTATORIAL DOMAINS

We start by considering the case where we have three alternatives only. So we let **A** = {a, b, c}. Our first result announces the equivalence between dictatoriality and the full domain:

Theorem 3.1: Let **A** = {a, b, c}. A domain $D \subseteq \Pi$ is dictatorial if and only if $D = \Pi$.

Proof: The "if" part directly follows from Arrow's impossibility theorem. To prove the "only if" part, we first show that no domain D with #D = 5 is dictatorial. Take such a domain where, without loss of generality, the ordering c a b is removed.[10] Consider the SWF α: $D^{\mathbf{N}} \rightarrow \Re$ where given any $\underline{R} \in D^{\mathbf{N}}$ we have

(i) a $\alpha^*(\underline{R})$ b if there exists i $\in$ **N** with a P_i b and b $\alpha^*(\underline{R})$ a otherwise

(ii) b $\alpha^*(\underline{R})$ c if there exists i $\in$ **N** with b P_i c and c $\alpha^*(\underline{R})$ b otherwise

(iii) a $\alpha^*(\underline{R})$ c if there exists i $\in$ **N** with a P_i c and c $\alpha^*(\underline{R})$ a otherwise

We know by Maskin (1995) that α is both well-defined and Arrovian. We now show that no domain D with #D = 4 is dictatorial.[11] Suppose not, i.e., there exists some dictatorial D with

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[8] The concepts are used in their standard definitions. First, α is PO iff given any x, y $\in$ **A** and any $\underline{R} \in D^n$ with x R_i y for all i $\in$ **N**, we have x $\alpha*(\underline{R})$ y. Next, α is IIA iff given any $\underline{R}$, $\underline{R}' \in D^n$ and any x, y $\in$ **A** with x R_i y $\Leftrightarrow$ x R_i' y for all i $\in$ **N**, we have x $\alpha(\underline{R})$ y $\Leftrightarrow$ x $\alpha(\underline{R}')$ y. Finally, an individual d $\in$ **N** is said to be a dictator of α iff given any $\underline{R} \in D^n$ and any distinct x, y $\in$ **A**, we have x R_d y $\Rightarrow$ x $\alpha*(\underline{R})$ y. We say that α is ND if it exhibits no dictator.

[9] By a pair, we mean two distinct alternatives.

[10] We abuse notation by writing c a b instead of c P a P b.

[11] Remark that being a subdomain of a non-dictatorial domain does not ensure non-dictatoriality, a matter which we adress in Section 4.

#D = 4. As every domain which contains five orderings is non-dictatorial, by Corollary 4.1 of our paper, D must contain at least one trivial pair which is not possible. Thus, every D with #D = 4 is non-dictatorial. It is straightforward to check that domains with three orderings either do not have any trivial pair or satisfy the value-restriction condition of Sen (1966). Finally, domains with two orderings are always value-restricted. Thus, we conclude that in a world of three alternatives, the full domain assumption is necessary for Arrow's Impossibility Theorem to hold. Q.E.D.

Theorem 3.1 has an implication for the general case with at least three alternatives: In a world where indifferences are ruled out, it is not possible to weaken the definition of a free triple. In other words, a triple $\{a, b, c\}$ is free in a domain D if and only if given any $x, y, z \in \{a, b, c\}$, there exists $P \in D$ with $x\ P\ y\ P\ z$.

Before stating our main result for the general case, we introduce a few concepts as well as the related notation. We first define what we mean by the *local dictatoriality graph* G(D) of a domain D. The vertices of G(D) are the non-trivial pairs in D. Any two vertices $\{x, y\}$ and $\{z, w\}$are linked if and only if $\{x, y\} \cup \{z, w\}$ forms a free triple in D.[12] Let k stand for the number of connected components G(D) has.[13] So we write $G(D) = \{G_i\}_{i=1}^{k}$ where each G_i is a connected component of G(D). A connected component G_i may consist of a single vertex in which case we qualify it as *simple*. Note that a connected component which is not simple contains at least three vertices.

We first show that given an ASWF, every non-simple connected component of G(D) exhibits a "local" dictator. To state this formally, consider an ASWF $\alpha: D^{\mathbf{N}} \to \mathfrak{R}$. For any pair $\{x,y\} \subset \mathbf{A}$, let $d(\{x, y\}) = \{i \in \mathbf{N} : (x\ R_i\ y \Rightarrow x\ \alpha^*(\underline{R})\ y)$ and $(y\ R_i\ x \Rightarrow y\ \alpha^*(\underline{R})\ x)$ for all $\underline{R} \in D^{\mathbf{N}}\}$ be the set of dictators over $\{x, y\}$ under α.[14]

[12] Thus $\{x, y\}$ and $\{z, w\}$are linked only if they have exactly one element in common.

[13] A graph G is said to be connected if and only if given any two vertices v and v' of G, there is a sequence $v = v_1,..., v_s = v'$ of vertices of G such that v_i and v_{i+1} are linked for each i = 1,..., s-1. So by a connected component of G(D), we mean a maximal connected subgraph G of G(D), i.e., G is connected but no supergraph of G is so.

[14] Note that when a pair $\{x, y\}$ is non-trivial, $d(\{x,y\})$ is a singleton while Pareto optimality implies $d(\{x,y\})=\mathbf{N}$ for trivial pairs.

Lemma 3.1: Let #**A** $\geq$ 3. Take any D $\subseteq \Pi$ and any ASWF α: $D^{\mathbf{N}} \rightarrow \Re$. Consider some non-simple connected component G_i of G(D). Given any two vertices {x,y} and {z,w} of G_i we have d({x,y})= d({z,w}).

Proof: Take **A**, D and α as in the statement of the lemma. We first show that given any triple {a, b, c} which is free in D, we have d({a, b}) = d({b, c}) = d({a, c}) = {i*} for some i* $\in$ **N**. To see this, given any R $\in \Pi$ and any triple {a, b, c}, write R_{abc} for the (usual) restriction of R over {a, b, c}.[15] Similarly, let D_{abc} = {R_{abc} : R $\in$ D}stand for the restriction of D over {a, b, c}.[16] We can thus introduce the restriction of α: $D^{\mathbf{N}} \rightarrow \Re$ over {a, b, c} as the mapping α_{abc}: $D_{abc}^{\mathbf{N}} \rightarrow \Re_{abc}$ which is defined for all $\underline{R}_{abc} \in D_{abc}^{\mathbf{N}}$ and for all x, y $\in$ {a, b, c} as x $\alpha_{abc}(\underline{R}_{abc})$ y if and only if x $\alpha(\underline{R})$ y. It is straightforward to check that α_{abc} is PO and IIA. Moreover, as {a, b, c} is a free triple, D_{abc} contains all strict orderings over {a, b, c}. Thus, by Arrow's impossibility theorem, α_{abc} exhibits a dictator i* $\in$ **N**. Moreover, as α satisfies IIA, we have d({a, b}) = d({b, c}) = d({a, c}) = {i*} as well.

To complete the proof, take any two vertices v ={x, y} and v'={z, w} of a connected component G_i of G(D). As G_i is connected, there exist an integer s and a sequence v = v_1,..., v_s = v' of vertices of G_i such that v_i and v_{i+1} are linked, i.e., $v_i \cup v_{i+1}$ forms a free triple for each i = 1,..., s-1. Thus, given any i = 1,..., s-1, there exists a dictator d(i) over the pairs in $v_i \cup v_{i+1}$. Moreover, $v_i \cup v_{i+1}$ and $v_{i+1} \cup v_{i+2}$ have a common pair. So d(i) is the same agent for every i = 1,..., s-1. Q.E.D.

We now introduce a condition which ensures that the local dictators of the connected subgraphs of G(D) coincide- hence leading to a "global" dictator. To do this, we define another graph G*(D) to which we refer as the *global dictatoriality graph* of D. The vertices of G*(D) are the connected components of G(D). Given any two vertices G_i, G_j of G*(D), we say that G_i is linked to G_j if and only if there exists a vertex {x, y}of G_i such that:

(i) there exists R $\in$ D, some integer k > 3 and a sequence of alternatives x = x_1,…,x_k = y with x_i R x_{i+1} for every i=1,….k-1 while each {x_i, x_{i+1}} is a vertex of G_j

and

[15] i.e., for any x, y $\in$ {a, b, c}, we have x R_{abc} y if and only if x R y.

[16] Note that #D_{abc} $\leq$ #D, as there may exists R, R' $\in$ D with R_{abc} = R'_{abc}.

(ii) there exists $R' \in D$, some integer $k' > 3$ and a sequence of alternatives $y = x_1, \ldots, x_{k'} = x$ with $x_i \; R' \; x_{i+1}$ for every $i=1,\ldots k'-1$ while each $\{x_i, x_{i+1}\}$ is a vertex of G_j

We say that two nodes G_i and G_j are linked in $G^*(D)$ iff G_i is linked to G_j or G_j is linked to G_i.

We call the domain D *essentially saturated* if and only if $G^*(D)$ is a connected graph.

Theorem 3.2: Let $\#\mathbf{A} \geq 3$. Any essentially saturated domain $D \subseteq \Pi$ is dictatorial.

Proof: Consider a Pareto optimal and IIA SWF α defined over an essentially saturated domain D. As the essential saturation of D means that $G^*(D)$ is a connected graph, we will prove the dictatoriality of α by showing that any two nodes G_i and G_j of $G^*(D)$ which are linked exhibit the same dictator. So take any two linked nodes G_i and G_j of $G^*(D)$. Note that at least one of these two nodes must be a non-simple connected component of G(D). First, consider the case where both G_i and G_j are non-simple connected components of G(D). Hence, G_i and G_j exhibit local dictators d and d' respectively. Suppose that d and d' are distinct. Let, without loss of generality, G_i be linked to G_j. So there exists a vertex $\{x, y\}$of G_i, some $R^* \in D$, some integer $k > 3$ and a sequence of alternatives $x = x_1, \ldots, x_k = y$ with $x_i \; R^* \; x_{i+1}$ for every $i \in \{1,\ldots, k-1\}$ while each $\{x_i, x_{i+1}\}$ is a vertex of G_j. Now consider a profile $\underline{R} \in D^{\mathbf{N}}$ with $y \; P_d \; x$ and $R_{d'} = R^*$. Since d is the dictator over $\{x, y\}$, we must have $y \; \alpha^*(\underline{R}) \; x$. Since d' is the dictator of each $\{x_i, x_{i+1}\}$, we have $x_i \; \alpha^*(\underline{R}) \; x_{i+1}$ for every $i \in \{1,\ldots, k-1\}$, hence $x \; \alpha^*(\underline{R}) \; y$ by transitivity of $\alpha(\underline{R})$, giving a contradiction. Thus we conclude that the respective dictators d and d' of G_i and G_j must coincide.

Now consider the case where one of G_i and G_j (say G_i without loss of generality) is a simple connected component of G(D). Hence, it is G_i which must be linked to G_j while G_j exhibits a local dictator d. To see that d is also the dictator over the (unique) vertex $\{x, y\}$of G_i, take any profile $\underline{R} \in D^{\mathbf{N}}$. Consider first the case where $x \; P_d \; y$. As G_i is linked to G_j, there exists some $R^* \in D$, some integer $k > 3$ and a sequence of alternatives $x = x_1, \ldots, x_k = y$ with $x_i \; R^* \; x_{i+1}$ for every $i \in \{1,\ldots, k-1\}$ while each $\{x_i, x_{i+1}\}$ is a vertex of G_j. Now consider the profile $\underline{R}' \in D^{\mathbf{N}}$ with $R_k' = R_k$ for all $k \in \mathbf{N} \setminus \{d\}$ while $R_d' = R^*$. Since d is the dictator of each $\{x_i, x_{i+1}\}$, we

have x_i $\alpha^*(\underline{R}')$ x_{i+1} for every $i \in \{1,\ldots, k-1\}$, hence x $\alpha^*(\underline{R}')$ y by transitivity of $\alpha(\underline{R})$. As α satisfies IIA, this implies x $\alpha^*(\underline{R})$ y as well. Now consider the case where y P_d x. As G_i is linked to G_j, there exists some $R^{**} \in D$, some integer $k' > 3$ and a sequence of alternatives $y = x_1,\ldots,x_{k'} = x$ with x_i R^{**} x_{i+1} for every $i \in \{1,\ldots, k'-1\}$ while each $\{x_i, x_{i+1}\}$ is a vertex of G_j. Now consider the profile $\underline{R}'' \in D^{\mathbf{N}}$ with $R_k'' = R_k$ for all $k \in \mathbf{N} \setminus \{d\}$ while $R_d'' = R^{**}$. Since d is the dictator of each $\{x_i, x_{i+1}\}$, we have x_i $\alpha^*(\underline{R}'')$ x_{i+1} for every $i \in \{1,\ldots, k'-1\}$, hence x $\alpha^*(\underline{R}'')$ y by transitivity of $\alpha^*(\underline{R}'')$. As α satisfies IIA, this implies x $\alpha^*(\underline{R})$ y as well. Q.E.D.

Theorem 3.2 is stronger than the result of Kalai et al. (1979) who show that a saturated domain is dictatorial. For, essential saturation is implied by their saturation condition. In fact, one can check that D is saturated if and only if the local dictatoriality graph G(D) is connected. We also wish to note that if every triple is free then G(D) is trivially connected – hence giving the standard result as an immediate corollary to Theorem 3.2. It is worth noting that the standard result requires (m; 3) free triples; a saturated domain must contain at least (m; 2)/3 free triples while (m-2) free triples may suffice to induce an essentially saturated domain.

The following result follows from Theorem 3.2:

Proposition 3.1: Let $\#\mathbf{A} \geq 3$. Let D be a domain where there exists R, R' $\in$ D such that we have x R y $\Leftrightarrow$ y R' x for all x, y $\in$ **A**. Let a_1 R a_2 R …R a_m without loss of generality. If $\{a_i, a_{i+1}, a_{i+2}\}$is a free triple for every $i \in \{1,\ldots,m-2\}$ then D is dictatorial.

Proof: Simply check that the domain D described in the statement of the proposition is essentially saturated. Q.E.D.

Proposition 3.1 paves the way towards a result which announces that dictatorial domains may be extremely narrow.

Theorem 3.3: Let $\#\mathbf{A} \geq 3$. There exists a domain $D \subseteq \Pi$ with #D = 6 such that D is dictatorial.

Proof: We will prove the theorem by using Proposition 3.1, i.e., showing the existence of a domain $D \subseteq \Pi$ with #D = 6 such that there exists R, R' $\in$ D with x R y $\Leftrightarrow$ y R' x for all x, y $\in$ **A** while, letting a_1 R a_2 R …R a_m without loss of generality, $\{a_i, a_{i+1}, a_{i+2}\}$ is a free triple for every i $\in$ {1,…,m-2}.

To see this, start by the ordering R with a_1 P a_2 P …P a_m. Define a permutation function π over **N** with $\pi(i) = i + 1$ when i is odd and $\pi(i) = i - 1$ when i is even. We obtain the second ordering R' as $a_{\pi(i)}$ P' $a_{\pi(i+1)}$ for i $\in$ {1,…, m-1}. Define another permutation function ρ over **N** with $\rho(1) = 1$, $\rho(i) = i + 1$ when i is odd but different than 1 and $\rho(i) = i - 1$ when i is even. This leads to a third ordering R'' with $a_{\rho(i)}$ P'' $a_{\rho(i+1)}$ for i $\in$ {1,…, m-1}. Now, define three orderings R''', R^{iv} and R^{v} as follows: For each x, y $\in$ **A**, we have (x P''' y $\Leftrightarrow$ y P x), (x P^{iv} y $\Leftrightarrow$ y P' x) and (x P^{v} y $\Leftrightarrow$ y P'' x). It is tedious but straightforward to check that in the domain D consisting of the six orderings R, R', R'', R''', R^{iv} and R^{v}, each $\{a_i, a_{i+1}, a_{i+2}\}$ is a free triple for i $\in$ {1,…,m-2}. Q.E.D.[17]

4. SUPERDICTATORIAL DOMAINS

A dictatorial domain $D \subseteq \Pi$ is said to be *superdictatorial* if and only if every superdomain D' $\supseteq$ D is also dictatorial. The definition makes sense, i.e., not every superdomain of a dictatorial domain is dictatorial. To see this, we refer to the following example:

[17] For example, consider the case where **A** contains 11 alternatives. The orderings described in the proof of Theorem 3.3 are as in the following table:

R	R′	R″	R‴	R^{iv}	R^{v}
a_1	a_2	a_1	a_{11}	a_{10}	a_{11}
a_2	a_1	a_3	a_9	a_{11}	a_{10}
a_3	a_4	a_2	a_{10}	a_8	a_9
a_4	a_3	a_5	a_7	a_9	a_8
a_5	a_6	a_4	a_8	a_6	a_7
a_6	a_5	a_7	a_5	a_7	a_6
a_7	a_8	a_6	a_6	a_4	a_5
a_8	a_7	a_9	a_3	a_5	a_4
a_9	a_{10}	a_8	a_4	a_2	a_3
a_{10}	a_9	a_{11}	a_1	a_3	a_2
a_{11}	a_{11}	a_{10}	a_2	a_1	a_1

Note that any additional alternative can be handled through this pattern without need of reference to an additional ordering.

D

R	R'	R''	R'''	R^{iv}	R^{v}
a	a	b	b	c	c
d	d	a	c	a	b
b	c	d	a	d	a
c	b	c	d	b	d

D'

R	R'	R''	R'''	R^{iv}	R^{v}	R^{vi}
a	a	b	b	c	c	*d*
d	d	a	c	a	b	*a*
b	c	d	a	d	a	*b*
c	b	c	d	b	d	*c*

In domain D which contains six admissible orderings, triples {a, b, c} and {b, c, d} are free. Thus all pairs, but {a, d} have a common dictator. But since {a, d} is a trivial pair, every agent is a dictator over it. Hence D is a dictatorial domain. On the other hand, although {a, b, c} and {b, c, d} remain to be free in D' $\supseteq$ D, the pair {a, d} is no more trivial, ending up in D' being non-dictatorial.[18] Observe that the failure of the dictatorial D to be superdictatorial relies on the addition of a non-trivial pair. The next theorem generalizes this observation.

Theorem 4.1: If a domain D $\subseteq \Pi$ is dictatorial while some D' $\supseteq$ D is not, then there exists a pair {x, y} $\subseteq$ **A** which is trivial in D but non-trivial in D'.

Proof: Take any domain D $\subseteq \Pi$ which is dictatorial. Consider some D' $\supseteq$ D which is not dictatorial. So there exists some ASWF α defined over D'. On the other hand, when α is defined over D, it exhibits a dictator d $\in$ **N**. As α is non-dictatorial over D', there exist a non-trivial pair {x, y} $\subseteq$ **A** and some $\underline{R} \in [D']^{\mathbf{N}}$ such that x $\alpha(\underline{R})$ y while y P_d x . We claim that {x, y} must be trivial in D. Suppose not. Then there exists a profile $\underline{R}' \in [D]^{\mathbf{N}}$ with x R'_i y $\Leftrightarrow$ x R_i y for every i $\in$ **N**. As α satisfies IIA, this implies x $\alpha(\underline{R}')$ y, contradicting that d is the dictator of α defined over D. Q.E.D.

Corollary 4.1: Let D be a domain where every pair {x, y} is non-trivial. If D is dictatorial then D is superdictatorial as well.

[18] For example the SWF which assigns individual i as the dictator of every pair but {a,d} and individual j as the dictator of the pair {a,d} is Arrovian.

Note that the proof of Theorem 3.3 uses two orderings which are the inverse of each other. Thus the domain D with #D = 6 has no trivial pairs. This observation, combined with Corollary 4.1, leads to the following strenghtening of Theorem 3.3:

Theorem 4.2: Let #**A** ≥ 3. There exists a domain D ⊆ Π with #D = 6 such that D is superdictatorial.

5. CLOSING REMARKS

Our results contribute to the literature on the remarkable robustness of Arrow's Impossibility Theorem against domain restrictions. We also think that our proof technique based on graphs introduces a tool which may be useful for further explorations of the subject. In particular, a characterization of dictatorial domains in terms of graphs may lead to a better understanding of the Kalai and Muller (1977) result and the literature which follows. Moreover, such a characterization can allow to define a measure of dictatoriality over domains as a function of the fragmentariness of their dictatoriality graphs. All this shows the existence of further room for research even in this well-crafted area of social choice theory.[19]

[19] See Gaertner (2001) as well as le Breton and Weymark (2003) for two excellent surveys of the literature.

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