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# A NEW BASIS FOR TRADES *

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ABSTRACT

A very fast algorithm to generate a semitriangular basis for trades consisting of minimal trades (sparsest basis) is given. By augmenting the elements of this basis, we construct an infinite family of 2-designs.

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## 1. INTRODUCTION AND NOTATIONS

Let $v, k, t$ and $\lambda$ be positive integers such that $t \leq k \leq v$. Let $v\Sigma i$ denote the set of all $i$-subsets of a $v$-set $V(0 < i < v)$. An element of $v\Sigma i$ consisting of elements $x_1, \ldots, x_i$ is denoted by $x_1 \ldots x_i$. A $t-(v,k,\lambda)$ design is a collection $\mathcal{B}$ of elements of $v\Sigma k$ (called blocks) with the property that every element of $v\Sigma t$ occurs in exactly $\lambda$ blocks of $\mathcal{B}$. Any collection of blocks is usually referred to as an incomplete block design. A $(v,k,t)$ *trade* of *volume* $s$ (or simply a trade) consists of two incomplete block designs $T_1$ and $T_2$ each of $s$ blocks such that for every element of $v\Sigma t$, the number of blocks containing this element is the same as in $T_1$ and $T_2$. A trade is void if $s = 0$. The definitions of $t$-designs and trades allow repeated blocks. Both $T_1$ and $T_2$ must cover the same set of elements of $V$ which is called the *foundation* of the trade.

Hwang [5] has shown that

(1) the minimum foundation size of non-void $(v,k,t)$ trade is $k+t+1$, and

(2) for $v \geq k+t+1$, the minimum volume of a non-void $(v,k,t)$ trade is $2^t$,

(3) the non-void trades with foundation size $k+t+1$ and volume $2^t$, exist and have unique structures. These trades are called *minimal trades.*

Following Graver and Jurkat [3], Graham, Li and Li [2], in a fundamental paper cast these minimal trades in terms of polynomials: Consider the polynomial

$$(*) \quad \phi(x_1, \ldots, x_v) = (x_1 - x_2)(x_3 - x_4)\ldots(x_{2t+1} - x_{2t+2})x_{2t+3}\ldots x_{k+t+1}.$$

If we multiply the factors out and identify each $x_i$ with $i$, then the result is clearly a minimal trade.

Now, let $M$ be the free $\mathbb{Z}$-module generated by all subsets of $V$. In this terminology, an incomplete block design $D$, is $\sum_{B \in v\Sigma k} f_B B$, where $f_B$ denotes the multiplicity of the block $B$ in $D$. Thus an element $\Sigma f_B B$ is a $t$-design if $f_B \geq 0$ and such that for $t$-subsets $X$ of $V$,

$$\sum_{B \supseteq X} f_B = \lambda.$$

A submodule of $M$ which is defined by

$$N_k = \left\{ \sum_{B \in v\Sigma k} t_B B | \Sigma t_B = 0 \quad \forall\, t - subsets\ X \subset V \right\},$$

is precisely the collection of $(v,k,t)$ trades. It has been shown [2] that for $v \geq k+t+1$ and $k \geq t+1$, we have

$$dim N_k = \binom{v}{k} - \binom{v}{t}.$$

Finally we consider the $\binom{v}{t}$ by $\binom{v}{k}$ "$t$-set inclusion" matrix $P_{v,k,t} = (p_{X,Y})$ with $|X| = t, |Y| = k$ and

$$p_{X,Y} = \begin{cases} 1 & \text{if} \quad X \subseteq Y, \\ 0 & \text{otherwise.} \end{cases}$$

$P_{v,k,t}$ is a full rank matrix [1,2]. The mininal trades form a sparsest basis for the $ker P_{v,k,t}$.

In [3] a generating system for a special structure called "$(k,t)$-pod" was given. In [2], linearly dependent generators were ruled out and a basis for trades was obtained. In [4], Hedayat and Hwang presented a computer algorithm for generating such a basis. They generated $v!$ permutations and chose from them only $\binom{v}{k} - \binom{v}{t}$ ones which satisfy the criterion given in [2] and then applied it to ($*$). Clearly, the number of computations performed in this algorithm is of the order of an exponential function and the algorithm suffers from a complexity of computations in time. Later on, Khosrovshahi and Mahmoodian [6] gave another algorithm which utilizes the Gauss elimination procedure to reduce $P_{v,k,t}$ to echelon form. Since these computations are performed in rational arithmetic, it requires a large space of computer memory to store many large matrices. Therefore, this algorithm suffers from a complexity of computations too, not in time, but in space and besides, part of the elements of the basis are not minimal trades.

In this paper, we take a fresh look at the elements of the basis of trades and we give an algorithm which (i) does not involve any kind of complexity, (ii) can be easily carried out manually, (iii) the elements of the basis are minimal trades, the whole basis is in *semitriangular* form, and the elements of the basis have some sort of canonical property, (iv) and finally by augmenting the elements of the basis it is possible to construct $t$-designs. We give an example.

Finally, we would like to note that trades are very important and complicated combinatorial objects. In fact, they can be viewed as building blocks of $t$-designs. If all the trades of all possible volumes could be catalogued, then the problem of existence of $t$-designs could become much more tractable. Recently, Ray-Chaudhuri and Singhi [7] utilized simple trades (called by them elementary signed $t$-designs) to prove a very important existence theorem on $t$-designs.

## 2. A BASIS FOR TRADES

In this section, we will estbalish the existence of a semitriangular basis for trades.

**Definition 1** Let $w = (w^1, w^2, \ldots, w^n)$ be a nonzero element of $\mathbb{Z}^n$ and suppose $j, 1 \leq j \leq n$, is the smallest natural number for which $w^j \neq 0$. Then we call $j$ the *starting index* of $w$.



**Definition 2** Let $B = \{w_1, \ldots, w_k\}$ be an ordered set of $k$ nonzero vectors in $\mathbb{Z}^n$ and suppose $\lambda_i$ is the starting index of $w_i$. Then we call $B$ *semitriangular* if

$$1 \leq \lambda_1 < \ldots < \lambda_k \leq n.$$

**Lemma 1** *Every semitriangular set of vectors is linearly independent.*

We are looking for a semitriangular basis, consisting of minimal trades, for trades. Every minimal trade $T$ is associated with the following type of polynomial [2]:

$$(^{**}) \quad P(x_1, \ldots, x_0) = (x_{b_1} - x_{c_1}) \ldots (x_{b_{t+1}} - x_{c_{t+1}}) x_{b_{t+2}} \, x_{b_{t+3}} \ldots x_{b_k},$$

in which all the $b_i'$s and $c_i'$s are distinct. First we look for the starting index of $T$. Since the starting indices of $T$ and $-T$ are equal, therefore, we can assume that

$$\begin{cases} b_i < c_i, & 1 \leq i \leq t+1, \\ b_1 < b_2 < \ldots < b_{t+1}, & \\ b_{t+2} < \ldots < b_k. & \end{cases} \tag{1}$$

Now, the "*smallest block*" (i.e. in lexicographical ordering) of $T$ is $B_s = b_1 \ldots b_k$. Therefore, the starting index of $T$ is the number of $B_s$ in the lexicographic order.

**Lemma 2** *The block $B = j_1 \ldots j_k$ such that $j_1 < j_2 < \ldots < j_k$ is the "smallest block" of a minimal trade $T$ if and only if*

$$\begin{cases} j_\ell \leq v - k - t + 2\ell - 2, & 1 \leq \ell \leq t+1, \\ j_\ell \leq v - k + \ell, & t+2 \leq \ell \leq k. \end{cases} \tag{2}$$

**Proof** (Necessity) Let $T$ be a minimal trade with "smallest block" $B_s$, then the distinct elements $c_1, \ldots, c_{t+1}; b_1, \ldots, b_k$ exist which satisfy (1) and $B_s = j_1 \ldots j_k = b_1 \ldots b_k$.

Let $1 \leq m \leq t+1$. From (1) and the definition of $B_s$, it follows that

$$j_m \leq b_n < c_n, \qquad n = m, m+1, \ldots, t+1.$$

Therefore, there are at least $t - m + 2, c_i'$s which are greater than $j_m$. Moreover, there are exactly $k - m$, $b_i'$s greater than $j_m$. Hence, at least $k - 2m + 2 + t$ postive integers $s$ exist such that $j_m < s \leq v$. Hence

$$j_m \leq v - k - t - 2 + 2m.$$

Moreover, we have $j_{t+2} < j_{t+3} < \ldots < j_k \leq v$. Therefore,

$$j_m \leq v - k + m, \quad t+2 \leq m \leq k.$$

(Sufficiency) Suppose $j_1, \ldots, j_k (j_1 < j_2 < \ldots < j_k)$ satisfy (2). We let $b_m = j_m$, for $m = 1, \ldots, k$. Now, it suffices to choose distinct $c_1, \ldots, c_{t+1}$ such that

$$b_m < c_m \leq v, \quad \text{for} \quad 1 \leq m \leq t+1.$$

First, we observe that

$$b_{t+1} = j_{t+1} \leq v - k - t - 2 + 2t = v - k + t.$$

Hence, there are at least $k - t$ postive integers $n$ such that $b_{t+1} < n \leq v$. From these integers, we have already chosen $k - t - 1$ numbers to be equal to $b_i'$s. Hence, there is at least one integer distinct from all $b_i'$s and such that $b_{t+1} < n \leq v$. Therefore, we can define $c_{t+1}$ to be one of these integers. Now, suppose that $c_{m+1}, \ldots, c_{t+1}$ are chosen. Since $b_m \leq v - k - t + 2m - 2$, hence there are at least $k + t + 2 - 2m$ positive integers such that $b_i < b_m$, and since for $i < m$, we have $b_i < b_m$ and so far we have chosen $t + 1 - m$ of $c_i'$s and exactly $k - m$ of these $b_i'$s are greater than $b_m$. So there are at least $k + t + 2 - 2m - (t + 1 - m) - (k - m) = 1$ positive integers, $n$, distinct from $b_i'$s and $c_i'$s which satisfies $b_m < n \leq v$. Thus we can choose $C_m$. The end of the proof.

**Theorem** *There exists a semitriangular basis for trades.*

**Proof** It is sufficient to prove that there are exactly $\binom{v}{k} - \binom{v}{t}$ blocks which satisfy (2). To accomplish this, we let $A(v,k,t) = \{j_1 \ldots j_k | j's \text{ satisfy}(2)\}$. In addition we require that $k \geq t+1$, and $v \geq k+t+1$, and also we suppose that $t \geq 0$. Although for $t = 0$ trades are not defined but the set $A(v,k,t)$, for $t = 0$ makes sense.

The proof follows by induction on $t$.

For $t = 0$, (2) becomes

$$\begin{cases} j_i < v - k, \\ j_\ell \leq v - k + \ell, \quad 1 < \ell \leq k, \end{cases}$$

and the only block which does not satisfy the above conditions is $(v-k)(v-k+1)\ldots v$. Therefore, $|A(v,k,0)| = \binom{v}{k} - 1 = \binom{v}{k} - \binom{v}{0}$.

Now assume that for $t < n$, we have $|A(v,k,t)| = \binom{v}{k} - \binom{v}{t}$. For $t = n$, we define the following sets:

$$H_i = \{j_1 \ldots j_k | j_1 \ldots j_k \in A(v,k,t),\ j_1 = i\},$$
$$K_i = \{i_1 \ldots i_{k-1} | i(i_1 + i)(i_2 + i) \ldots (i_{k-1} + i \in H_i\},$$



for $1 \leq i \leq v - k - n$. Clearly, $H_i'$s are disjoint, $|H_i| = |K_i|$,

$$A(v,k,n) = \bigcup_{i=1}^{v-k-n} H_i,$$

and

$$|A(v,k,t)| = \sum_{i=1}^{v-k-n} |H_i|.$$

**Proposition** $K_i = A(v-i, k-1, n-1)$.

**Proof**

$$i_1 \ldots {}_{k-1} \in K_i \Longleftrightarrow i(i+i_1)(i+i_2)\ldots(i+i_{k-1} \in A(v,k,t)$$

$$\overset{by(2)}{\Longleftrightarrow} \begin{cases} i \leq v - k - n, \\ i + i_m \leq v - k - n - 2 + 2(m+1), \quad 1 \leq m \leq n, \\ i + i_m \leq v - k + (m+1), \quad n+1 \leq m \leq k-1. \end{cases}$$

$$\Longleftrightarrow \begin{cases} i_m \leq (v-i) - (k-1) - (n-1) - 2 + 2m, \\ \qquad 1 \leq m < n = (n-1) + 1, \\ i_m \leq (v-i) - (k-1) + m, \\ \qquad (n-1) + 2 = n + 1 \leq m < k - 1. \end{cases}$$

$$\Longleftrightarrow i_1 i_2 \ldots i_{k-1} \in A(v-i, k-1, n-1).$$

From induction hypothesis, we have

$$|A(v-i, k-1, n-1)| = \binom{v-i}{k-1} - \binom{v-i}{n-1}.$$

From the Proposition it follows that

$$|H_i| = |K_i| = \binom{v-i}{k-1} - \binom{v-i}{n-1}.$$

Now, we finish up the proof of the theorem.

$$
\begin{aligned}
|A(v,k,n)| &= \sum_{i=1}^{v-k-n} |H_i| \\
&= \sum_{i=1}^{v-k-n} \left[ \binom{v-i}{k-1} - \binom{v-i}{n-1} \right] \\
&= \sum_{i=1}^{v-k-n} \binom{v-i}{k-1} - \sum_{i=1}^{v-k-n} \binom{v-i}{n-1} \\
&= \sum_{i=1}^{v-k-n} \left[ \binom{v-1+1}{k} - \binom{v-i}{k} \right] - \sum_{i=1}^{v-k-n} \left[ \binom{v-i+1}{n} - \binom{v-i}{n} \right] \\
&= \binom{v}{k} - \binom{k+n}{k} - \left[ \binom{v}{n} - \binom{k+n}{n} \right] \\
&= \binom{v}{k} - \binom{v}{n} \\
&= \binom{v}{k} - \binom{v}{t}.
\end{aligned}
$$

## 3. AN ALGORITHM

In this section we present an algorithm to produce a semitriangular basis for trades. The algorithm consists of two parts: first it chooses the starting block one at a time, then it constructs the associated trade based on the starting block. Clearly, there is no time and space computational complexity involved in this algorithm.

The following is a description of the algorithm. As before, we make the following notational conventions. We will denote every starting block of a minimal trade of the form (**) by $(x_{b_1} - x_{c_1}) \dots (x_{b_{t+1}} - x_{c_{t+1}}) x_{b_{t+2}} \cdots x_{b_k}$.

Finally, we define

$$
Bound(i) = \begin{cases} v-k-t-2+2i, & i = 1, \dots, t+1, \\ v-k+i, & i = t+2, \dots, k. \end{cases}
$$

**Table 1** *An algorithm to produce a basis for trades.*

1. For $i = 1, \dots, k$, set $b_i = i$.
2. For $i = t+1, \dots, 1$  
   set $A_i = \{b_i + 1, \dots, v\} - \{c_{i+1}, \dots, c_{t+1}, b_{i+1}, \dots, b_{t+1}\}$  
   and $c_i = min A_i$.
3. Construct a trade using (**).



4. Find the largest value of $j$ such that $1 \le j \le k$ and $b_j <$ bound $(j)$.  
   If such a $j$ exists, then set $m = b_j$.  
   For $i = j, \dots, k$  
   $b_j = m + 1 + j - i$  
   $G_0$ to 2.

   Otherwise stop.

**Notes** (1) In step 2, since we have $b_i \le v - k - t - 2 + 2i$, hence

$$
\begin{aligned}
|A_i| &\ge (k + t + 2 - 2i) - (t + 1 - i) \\
&= k + 1 - i \ge 1.
\end{aligned}
$$

Therefore $A_i \ne \phi$.

(2) Since

$$
\text{Bound}(j) < \text{Bound}(j+1) < \dots < \text{Bound}(k),
$$

and if $b_j < \text{Bound}(j)$, then for $i \ge j$,

$$
\begin{aligned}
b_i = b_j + 1 + j - i &< \text{Bound}(j) + 1 + j - i \\
&\le \text{Bound}(i).
\end{aligned}
$$

(3) By choosing $c_i'$s differently, one can construct other kinds of basis too.

## SAMPLE OUTPUT

```
**********************
*                    *
* A BASIS FOR TRADES *
* V=  7 K=  3 T=  2  *
**********************

-------------------------
|012     |+             |
|013     |-+            |
|014     | -+           |
|015     |  -+          |
|016     |   -          |
|023     | -  +         |
|024     |-+- -+        |
|025     |  +- -+       |
|026     |   +  -       |
|034     |+    - +      |
|035     |    -+--+     |
|036     |      + -     |
|045     |    +   -     |
|046     |       -+     |
|056     |       +      |
|123     |    -    +    |
|124     |    +-   -+   |
|125     |-    +-    -+ |
|126     |      +    -  |
|134     |  -  + -  - + |
|135     |+-+-+-++--+--+|
|136     |   +  - +  + -|
|145     | +  -   ++   -|
|146     |       +-   -+|
|156     |       -    + |
|234     |  +         - |
|235     | +-+        +-|
|236     |   -     -   +|
|245     |+-         - +|
|246     |         +-++-|
|256     |          + - |
|345     |-          +  |
|346     |          +-  |
|356     |         +-   |
|456     |         -    |
-------------------------
```

```
**********************
*                    *
* A BASIS FOR TRADES *
* V=  8 K=  4 T=  2  *
**********************

|0123
|0124
|0125
|0126
|0127
|0134
|0135
|0136
|0137
|0145
|0146
|0147
|0156
|0157
|0167
|0234
|0235
|0236
|0237
|0245
|0246
|0247
|0256
|0257
|0267
|0345
|0346
|0347
|0356
|0357
|0367
|0456
|0457
|0467
|0567
|1234
|1235
|1236
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|2457
|2467
|2567
|3456
|3457
|3467
|3567
|4567
```

```
The blanck is to be considered zero
The "+" is to be considered +1
The "-" is to be considered -1
```



## 4. AN APPLICATION

In this section we demonstrate that by suitable augmentation of the elements of the basis of trades produced in Section 3, one can produce the family of simple

$$BIB(4n+2, \frac{n(4n+1)(4n+2)}{3}, n(4n+1), 3, 2n)$$

designs. The existence of these designs was established in [8]. The idea of the algorithm comes from the fact that the existence of this family of simple designs with $b = \frac{1}{2}\binom{v}{k}$ corresponds to the existence of $(4n+2,3,2)$ trades with volume $\frac{1}{2}\binom{v}{k}$. These trades, of course, are linear combinations of the elements of the basis of trades.

In what follows via a simple algorithm we produce the above mentioned trades, and it is assumed that $T$ is a $\binom{v}{k}$ integral vector.

**Table 2** An algorithm to produce $BIB\left(4n+2, \frac{n(4n+1)(4n+2)}{3}, n(4n+1), 3, 2n\right)$ designs.

1. Set $T = 0$.
2. Find the first nonzero component of $T$. This component corresponds to a starting block $B$ of a base trade. If such a block does not exist, stop.
3. Construct the base trade $T_1$ which has $B$ as its starting block.
4. If $T + T_1$ is a simple trade, then set $T = T + T_1$, otherwise set $T = T - T_1$.
5. $G_0$ to 2.

**Proposition** *The above algorithm generates a trade with volume $\frac{1}{2}\binom{v}{k}$ for the case $v = 4m+2, k = 3, m = 1, 2, \ldots$ .*

**Proof** We use induction on $n$. For $n = 1$, the statement can be easily verified.

Now, suppose that the statement is correct for $n = m$, we will verify it for $n = m+1$. To do this, we study the algorithm very carefully and we analyse the resulting trade in every step of the algorithm.

**Notation** In what follows we will denote a minimal trade represented by $(x_1 - x_2) - (x_3 - x_4)(x_5 - x_6)$ by $(1\ 3\ 5)(2\ 4\ 6)$.

1. Clearly the first two trades on the basis are

$$G_1 = (1\ 2\ 3)(6\ 5\ 4),$$
$$G_2 = (1\ 2\ 5)(4\ 3\ 6).$$

At the end of this step we will have

$$T = T + G_1 - G_2.$$

2. Now, the next trades on the agenda to be considered are

$$T_{1i} = (1\ 2\ 2i+1)(4\ 3\ 2i+2), \quad i = 3, \ldots, 2m.$$

These trades are mutually disjoint and we obtain

$$T = T + \sum_{i=3}^{2m} T_{1i}.$$

**Note** At the end of this step all the blocks of the form $12i$ have been included in $T$.

3. Now we consider

$$G_3 = (1\ 3\ 4)(2\ 6\ 5).$$

Notice that so far 134 and 135 have not appeared in $T$ and this trade is also disjoint from $T$. Therefore,

$$T = T + G_3.$$

4. Up to now, out of the blocks of the form $14j$ only 145 and 146 have appeared in $T$. Hence, the next batch of trades to be considered are of the form

$$T_{2i} = (1\ 4\ 2i+1)(2\ 5\ 2i+2), \quad i = 3, \ldots, 2m.$$

These trades are mutually disjoint, and they are disjoint from $G_1 - G_2 + G_3$ too. If $i \neq j$, then $T_{1i}$ and $T_{2j}$ are disjoint, but $T_{1i}$ and $T_{2i}$ are not. Therefore, we have

$$T = T - \sum_{i=3}^{2m} T_{2i}.$$

5. At this state of the game, if a block $abc \in T$, then $b < 6$. Hence, the next class of trades are

$$S_{ii} = (1\ 2i\ 2i+1)(2\ 2i+3\ 2i+2)$$
$$S_{ij} = (1\ 2i\ 2j+1)(2\ 2i+1\ 2j+2),$$
$$i = 3, \ldots, 2m; \quad j = i+1, \ldots, 2m+2.$$



Clearly $(i,j) \neq (i',j')$, then $S_{ij}$ and $S_{i'j'}$ are disjoint. If $i \neq i'$, then $S_{ii}$ and $S_{i'i'}$ are also disjoint. But $S_{ii}$ and $S_{ii+1}$ are not. At the end of this stage we end up with the following $T$:

$$T = T + \sum_{i=3}^{2m} \left( S_{ii} - S_{ii+1} + \sum_{j=i+2}^{2m+2} S_{ij} \right).$$

6. Now, the only blocks of the form $1ab$ which have not appeared in $T$ so far are those with $4m+2 \leq b$. And they are

$$V_1 = (1\ 4m+2\ 4m+3)(2\ 4m+5\ 4m+4),$$
$$V_2 = (1\ 4m+2\ 4m+5)(2\ 4m+3\ 4m+6),$$
$$V_3 = (1\ 4m+3\ 4m+4)(2\ 4m+6\ 4m+5).$$

Clearly, to have a simple $T$, we have to have

$$T = T + V_1 - V_2 + V_3.$$

With this addition, all the blocks of the form $1ab$ have appeared in $T$.

7. Out of the blocks of the form $2ab$, only blocks of the form $23i$ and $24i$ have not appeared in $T$. Therefore, we have

$$T_{3i} = (2\ 3\ 2i+1)(5\ 4\ 2i+2),$$
$$i = 3, \ldots, 2m+2.$$

These trades are mutually disjoint and clearly among all those trades used on the structure of $T, T_{1i}$ and $T_{2i}$ are not disjoint from $T_{1i} - T_{2i}$. Hence we can have

$$T = T + \sum_{i=3}^{2m+2} T_{3i}.$$

8. Now, every block $abc$ which is present in $T$ at this stage satisfy the following conditions

$$a < 3 \quad \text{or} \quad c < 7, \quad \text{or} \quad a = 3 \quad \text{and} \quad b \leq 5.$$

The trades which now have to be considered are the following:

$$F_{ii} = (3\ 2i\ 2i+1)(4\ 2i+3\ 2i+2)$$
$$F_{ij} = (3\ 2i\ 2j+1)(4\ 2i+1\ 2j+2)$$
$$i = 3, \ldots, 2m, \quad j = 2i+1, \ldots, 2m+2.$$

These trades are disjoint from $T$ and with a similar argument as in 5, we conclude that

$$T = T + \sum_{i=3}^{2m} \left( F_{ii} - F_{ii+1} + \sum_{j=i+2}^{2m+2} F_{ij} \right).$$

9. Finally we consider the following trades:

$$\begin{aligned}
H_1 &= (3\ 4m+2\ 4m+3)(4\ 4m+5\ 4m+4),\\
H_2 &= (3\ 4m+2\ 4m+5)(4\ 4m+3\ 4m+6),\\
H_3 &= (3\ 4m+3\ 4m+4)(4\ 4m+6\ 4m+5).\\
T &= T + H_1 - H_2 + H_3.
\end{aligned}$$

At this stage of the process of completing $T$, any block $abc$ has appeared in $T$ if and only if $a \leq 4$. The remaining blocks are formed from $4m+2$ remaining elements, namely $\{5, 6, \ldots, 4m+6\}$. According to our induction hypothesis, we can find a simple trade of volume $\frac{1}{2}\binom{v}{3}$ based on the $v = 4m+2$ elements. Clearly this trade, $T'$, is disjoint from $T$ and $T + T'$ will be the desired trade.


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