On the Stability of Equation-Error Estimates of All-Pole Systems

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***Abstract*— It is well known that the standard equation-error (EE) method for the identification of linear systems yields biased estimates if the data are noise corrupted. Due to this bias, the resulting estimate can be unstable in some cases, depending on the spectral characteristics of the input and the noise, the signal-to-noise ratio (SNR), and the unknown system. In this work, the set of all-pole linear systems whose equation-error estimate is stable for all wide sense stationary inputs and white measurement noise is investigated. Some results concerning the structure of this set are given.**

***Index Terms*—Bias, equation error, identification, stability.**

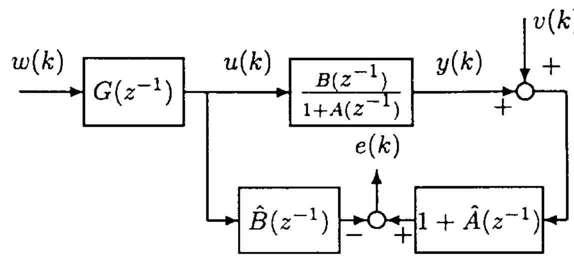

Fig. 1. Equation error configuration.

I. Introduction

It is well known that equation-error (EE) identification of a system

$$H(z^{-1}) = \frac{B(z^{-1})}{1+A(z^{-1})} = \frac{\sum_{i=0}^{m} b_i z^{-i}}{1+\sum_{j=1}^{n} a_j z^{-j}} \qquad (1.1)$$

is biased under measurement noise, and that this could lead to an unstable estimate of $H(z^{-1})$ even if $H(z^{-1})$ is stable [2]. Specifically, consider the setting of Fig. 1 with $w(k), v(k)$ mutually uncorrelated zero mean white processes, with variances 1 and σ^2, respectively; $G(z^{-1})$ is a filter that colors the plant input. The goal is to find $\hat{B}(z^{-1}) = \Sigma_{i=0}^{m} \hat{b}_i z^{-i}$ and $\hat{A}(z^{-1}) = \Sigma_{j=1}^{n} \hat{a}_j z^{-j}$ that minimize $E[e^2(k)]$. It is known that $(1+\hat{A}(z^{-1}))^{-1}$ is guaranteed to be stable under very high or very low SNR's [2], or if $G(z^{-1})$ is all-pole of order no greater than m [3].

This work gives a preliminary characterization of those systems whose EE estimates are stable regardless of $G(z^{-1})$ and σ^2. Attention is restricted to all-pole systems, i.e., $B(z^{-1}) = \hat{B}(z^{-1}) = 1$.

II. Main Results

Call $\boldsymbol{a} = [a_1 \cdots a_n]^t$ stable if $(1+A(z^{-1}))^{-1}$ is stable. Since we are interested in all stable $G(z^{-1})$'s, set $\sigma^2 = 1$ without loss of generality. Then the EE estimate $\hat{\boldsymbol{a}}$ minimizes

$$E[e^2(k)] = (\hat{\boldsymbol{a}}-\boldsymbol{a})^t \boldsymbol{R} (\hat{\boldsymbol{a}}-\boldsymbol{a}) + (1+\hat{\boldsymbol{a}}^t\hat{\boldsymbol{a}}) \qquad (2.2)$$

with $\boldsymbol{R}$ the $n \times n$ autocorrelation matrix of the plant output $y(k) = (G(z^{-1})/1+A(z^{-1}))w(k)$. The set of interest $\mathcal{S}$ obeys: $\boldsymbol{a} \in \mathcal{S}$ iff $\boldsymbol{a}$ is stable and $\hat{\boldsymbol{a}}$ is stable for all $\boldsymbol{R}$.

Lemma 2.1: With $\mathcal{S}$ as above, $\boldsymbol{a} \in \mathcal{S}$ iff for all positive definite symmetric Toeplitz $n \times n$ matrices $\boldsymbol{R}$, the vector $\boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1}\boldsymbol{a}$ is stable.

Proof: This follows from (2.2), noting that $G(z^{-1})$ arbitrary implies that $\boldsymbol{R}$ can be chosen arbitrarily subject to being symmetric, Toeplitz and positive definite. ■

Thus the set $\mathcal{S}$ can be expressed as

$$\mathcal{S} = \{\boldsymbol{a} \in \Re^n | \boldsymbol{T}\boldsymbol{a} \text{ stable } \forall \boldsymbol{T} \in \mathcal{T}\} \qquad (2.3)$$

where

$$\mathcal{T} = \{\boldsymbol{T} = \boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1} | \boldsymbol{R} = \boldsymbol{R}^t > \boldsymbol{0} \text{ |Toeplitz}\}. \qquad (2.4)$$

We now give some properties of $\mathcal{S}$; a complete characterization remains open.

The set $\mathcal{T}$ has some nice structural properties. Any matrix $\boldsymbol{T} \in \mathcal{T}$ satisfies:

1) it is symmetric about both diagonals: $\boldsymbol{T} = \boldsymbol{T}^t, \boldsymbol{TJ} = \boldsymbol{JT}$, where $\boldsymbol{J}$ is the $n \times n$ matrix with ones in the antidiagonal and zeros elsewhere;
2) its eigenvalues lie in the open interval (0,1);
3) there exists a basis of eigenvectors of $\boldsymbol{T}$ whose elements are either symmetric (i.e., $\boldsymbol{v} = \boldsymbol{Jv}$) or antisymmetric (i.e., $\boldsymbol{v} = -\boldsymbol{Jv}$);
4) $\boldsymbol{v}$ symmetric (resp. antisymmetric) $\Rightarrow$ $\boldsymbol{Tv}$ symmetric (resp. antisymmetric).

Property 1 follows from (2.4) and that $\boldsymbol{RJ} = \boldsymbol{JR}$; Property 2, from $\boldsymbol{R} > \boldsymbol{0}$; Property 3, from the fact that the eigenvectors of $\boldsymbol{T}$ and $\boldsymbol{R}$ are the same, and [4]; Property 4 follows from Property 1.

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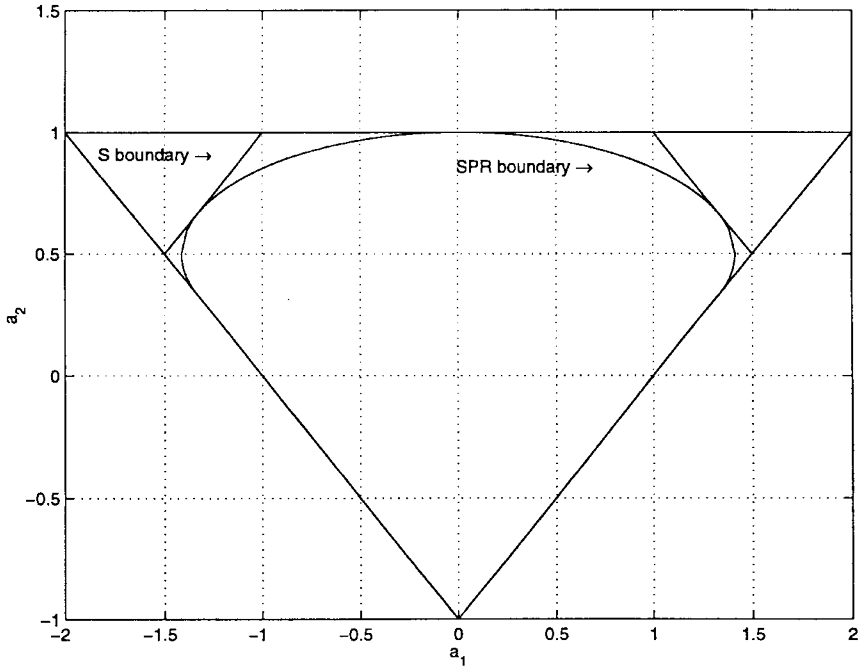

Fig. 2. The set $\mathcal{S}$ for $n = 2$. Also displayed are the stability triangle and the SPR region.

Using these facts, the set $\mathcal{S}$ for $n = 2$ can be found to be as in Fig. 2, which also depicts the stability triangle and the region of strict positive real (SPR) polynomials (i.e., the stable polynomials $1 + A(z^{-1})$ that satisfy $\Re[1 + A(e^{-j\omega})] > 0 \forall \omega$). This SPR condition arises when studying the convergence of output error algorithms [1].

Note that for $n = 2$ $\boldsymbol{a} \in \mathcal{S}$ *if* $1 + A(z^{-1})$ *is SPR. However, this is not true in general: one can find SPR polynomials of order three which are not in* $\mathcal{S}$. *For example,* $1 + A(z^{-1}) = 1 + .05z^{-1} - .9z^{-2} - .1z^{-3}$ *is SPR. Yet, with* $\boldsymbol{R}$ the matrix

$$\begin{bmatrix} 6.7780 & 2.7217 & -4.4444 \\ 2.7217 & 6.7780 & 2.7217 \\ -4.4444 & 2.7217 & 6.7780 \end{bmatrix},$$

which is positive definite Toeplitz, it turns out that $\boldsymbol{R}(\boldsymbol{I}+\boldsymbol{R})^{-1}\boldsymbol{a}$ *is unstable*.

To conclude we present a general result that is valid for any order n. Since $\boldsymbol{a}$ is unknown one cannot *a priori* determine if $\boldsymbol{a} \in \mathcal{S}$. Now suppose it is known that $\boldsymbol{a} \in \mathcal{A}$ where

$$\mathcal{A} = \left\{ \boldsymbol{a} = \boldsymbol{a}_0 + \sum_{i=1}^{p} \lambda_i \boldsymbol{a}_i | \lambda_i \in [\lambda_i^-, \lambda_i^+] \right\} \tag{2.5}$$

with $\boldsymbol{a}_0$ arbitrary and for each $1 \leq i \leq p, \boldsymbol{a}_i$ is a symmetric or antisymmetric vector; $\boldsymbol{a}_0, \boldsymbol{a}_i, \lambda_i^-, \lambda_i^+$ are known. In effect this constitutes partial knowledge of $H(z^{-1})$, and $\mathcal{A}$ can be viewed as an uncertainty set. This is a fairly natural characterization as in speech processing one models $\boldsymbol{a}$ in terms of $\boldsymbol{R}$, and the eigenvectors of $\boldsymbol{R}$ are symmetric or antisymmetric.

Our result states that in order to check if $\mathcal{A} \subset \mathcal{S}$ it suffices to check if the set

$$\mathcal{A}^* = \left\{ \boldsymbol{a} = \boldsymbol{a}_0 + \sum_{i=1}^{p} \lambda_i \boldsymbol{a}_i | \lambda_i \in \{\lambda_i^-, \lambda_i^+\} \right\} \tag{2.6}$$

(the set of corners of $\mathcal{A}$), is contained in $\mathcal{S}$. Observe, $\mathcal{A}^*$ has a finite number of members, though $\mathcal{A}$ does not. First we give the following lemma.

Lemma 2.2: Symmetric and antisymmetric vectors constitute convex directions for stability, i.e., if $\boldsymbol{v} \in \Re^n$ is symmetric or antisymmetric, $\boldsymbol{a} \in \Re^n$ is stable and $\boldsymbol{a} + \boldsymbol{v}$ is stable, then $\boldsymbol{a} + \lambda \boldsymbol{v}$ is stable for all $0 \leq \lambda \leq 1$.

Proof: Let $1 + \tilde{A}(\lambda, z^{-1})$ be the polynomial corresponding to $\tilde{\boldsymbol{a}} = \boldsymbol{a} + \lambda \boldsymbol{v}$. Consider the bilinear transformation $z = (s + 1/s - 1)$. Then $\boldsymbol{a} + \lambda \boldsymbol{v}$ is stable iff the roots of

$$C(\lambda, s) = (s+1)^n \left[1 + \tilde{A}\left(\frac{s-1}{s+1} \right) \right] \tag{2.7}$$

have negative real parts. Write this as $C(\lambda, s) = C(0, s) + \lambda(s-1)P(s)$ with $P(s) = \Sigma_{i=1}^{n} v_i (s-1)^{i-1}(s+1)^{n-i}$. For $\boldsymbol{v}$ symmetric or antisymmetric, $P(s)$ contains either only odd, or only even, powers of s. Then $(s-1)P(s)$ is a convex direction for stability with respect to the imaginary axis [5] and therefore the result follows. ■

Theorem 2.1: With $\mathcal{A}, \mathcal{A}^*$, and $\mathcal{S}$ as above, $\mathcal{A} \subset \mathcal{S} \Leftrightarrow \mathcal{A}^* \subset \mathcal{S}$.

Proof: Suppose $\mathcal{A}^* \subset \mathcal{S}$. Then for any $\boldsymbol{a} \in \mathcal{A}$ and any $\boldsymbol{T} \in \mathcal{T}$

$$\boldsymbol{T}\boldsymbol{a} = \boldsymbol{T}\boldsymbol{a}_0 + \sum_{i=1}^{p} \lambda_i (\boldsymbol{T}\boldsymbol{a}_i). \tag{2.8}$$

Letting each λ_i vary in $[\lambda_i^-, \lambda_i^+]$, (2.8) represents a family of polynomials whose coefficient vectors lie in a polytope. Its corners are stable (because $\mathcal{A}^* \subset \mathcal{S}$), and the edges are defined by the vectors $\boldsymbol{T}\boldsymbol{a}_i$, which are either symmetric or antisymmetric and therefore convex directions. Thus the edges are stable. By the so-called edge theorem [5], this implies stability of the whole family (2.8). Therefore $\mathcal{A} \subset \mathcal{S}$. ■

III. Conclusion

We have presented a new operator condition that ensures that equation error identification of a given all-pole system yields a stable estimate for all WSS inputs and white WSS measurement noise. Preliminary results characterizing the set of operators that obey this condition have been given. In low-order cases, this set is convex. We conjecture that convexity is true in higher dimensions as well, just as the set of operators satisfying the SPR condition, featuring in the analysis of output error algorithms, is also convex.

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