MMSE Analysis of Certain Large Isometric Random Precoded Systems.

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Abstract — **Linear Precoding consists in multiplying by a $N \times K$ matrix a K-dimensional vector obtained by serial to parallel conversion of a symbol sequence to be transmitted. In this paper, we analyse the performance of MMSE receivers for certain large random isometric precoded systems on fading channels. Using new tools, borrowed from the so-called *Free Probability Theory*, it can be shown that the Signal to Interference plus Noise Ratio at the equalizer output converges almost surely to a deterministic value depending on the probability distribution of the channel coefficients when $N \to +\infty$ and $K/N \to \alpha \leq 1$. These asymptotic results are used to optimally balance the redundancy introduced between Linear Precoding and classical Convolutional Coding, while preserving a simple MMSE equalization scheme at the receiver.**

I. INTRODUCTION

Transmitting data at a high rate over a microwave channel is possible when some kind of diversity is exploited. To alleviate the effect of channel attenuations, the principle is to transmit various replicas of the information signal which will be appropriately combined by the receiver. Recently, Giraud and Belfiore [1, 2], and then Boutros and Viterbo [3] introduced an attractive new diversity scheme called *signal space diversity*. Contribution [3] focuses on a particular modulation scheme depicted in figure 2, in which the input symbol stream is serial to parallel converted, then the resulting K-dimensional symbol vector $\mathbf{s}(n)$ (a white vector process with $E\left(\mathbf{s}(n)\mathbf{s}^H(n)\right) = \mathbf{I}_K$) is multiplied by an isometric $N \times K$ matrix $\mathbf{W}_N$ (i.e. $\mathbf{W}_N^H\mathbf{W}_N = \mathbf{I}_K$) where $N \geq K$. This N-dimensional vector $\mathbf{W}_N\mathbf{s}(n)$ is parallel to serial converted, and the corresponding generated data stream is sent across a frequency non selective Rayleigh fading channel. After serial to parallel conversion, the N–dimensional received vector $\mathbf{y}(n)$ can be written as:

$$\mathbf{y}(n) \;=\; \mathbf{H}_N(n)\,\mathbf{W}_N\,\mathbf{s}(n) \;+\; \mathbf{n}(n) \qquad (1)$$

where $\mathbf{n}(n)$ is a white additive Gaussian noise such that $E\left(\mathbf{n}(n)\mathbf{n}(n)^H\right) = \lambda\mathbf{I}_N$, and where $\mathbf{H}_N(n) = \mathrm{diag}([h_1(n),\ldots,h_N(n)])$ is the $N \times N$ diagonal complex matrix bearing on its diagonal the channel gains. Thanks to Linear Precoding matrix $\mathbf{W}_N$, each component of $\mathbf{s}(n)$ is sent over a duration N times longer than if $\mathbf{W}_N$ were reduced to $\mathbf{I}_N$.

An important problem lies in the choice of the amount of redundancy introduced by Linear Precoding, i.e. the ratio K/N, and also in the choice of matrix $\mathbf{W}_N$. [1] and [3] considered the case where $K/N = 1$, i.e. $\mathbf{W}_N$ is unitary. They assumed the entries of $\mathbf{H}_N(n)$ independent and identically distributed, and proposed to derive an upper bound of the error probability for the Maximum Likelihood (ML) detector of $\mathbf{s}(n)$. The high computational cost of the ML detector prevents its use in practical contexts. Actually, due to its lower complexity, MMSE detection is often preferred. Therefore, in this paper, we study the impact of the choice of $\mathbf{W}_N$ and of parameter K/N on the asymptotic performance of the MMSE receiver when N and K converge toward $+\infty$ in such a way that $K/N \to \alpha \leq 1$. Several papers [4][5][6] have recently analyzed the behaviour of the SINR at the output of the MMSE detector when the entries of $\mathbf{W}_N$ are independent and identically distributed random variables (to be referred to in the sequel as the i.i.d. case) in the case where $\mathbf{H}_N(n)$ is reduced to $\mathbf{I}_N$ and the various users can have different powers. The originality of our contribution lies in the fact that the linear random precoder $\mathbf{W}_N$ is isometric instead of being i.i.d. Such a choice is justified by the fact that, when synchronization is ensured, isometric precoders provide much better results than i.i.d. ones as will be seen below. Furthermore, our model applies to several other situations: for instance, precoded OFDM and downlink MC-CDMA schemes.

From a technical stand point, the i.i.d. case study of [6] is based on mathematical results that concern the "limiting distribution of eigenvalues" of some large random matrices with independent and identically distributed entries (see e.g. [7]). Analysis of the isometric case requires advanced mathematical concepts. The results given here rely on the so-called *Free Probability Theory* initially developed by D. Voiculescu. Due to the lack of space, the corresponding tools and derivations are not introduced here. The interested reader may consult [8] (this paper can be downloaded from the address http://www-syscom.univ-mlv.fr/∼loubaton/index.html). Note that Evans and Tse already introduced free probability theory in [5], but for solving quite different problems.

II. ASYMPTOTICAL RESULTS

In the following, since the time index is not relevant, we simply omit it. We assume that $\mathbf{H}_N = \mathrm{diag}([h_1,\ldots,h_N])$ has identically distributed centered random diagonal entries. $|h_i|^2$ is supposed to have a probability density $p(t)$ with finite moments of all orders. We set $E(|h_i|^2) = 1$, so that λ represents the inverse of the SNR at the receiver input. Notice that random variables $\{h_i\}_{i=1,N}$ are not assumed to be independent. However, we assume that for each $l \geq 1$,

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{k=1}^{N} |h_k|^{2l} = E(|h_i|^{2l}) \text{ almost surely.} \qquad (2)$$

which implies some kind of asymptotic independence between the random variables h_i and h_j if $|i-j| \to \infty$. This hypothesis is quite realistic in the context of the signal space diversity schemes of [3] if large interleavers/deinterleavers are inserted in the scheme represented in figure 2.

We now explain how the random matrix $\mathbf{W}_N$ is generated. A random unitary matrix Θ is said *Haar distributed* if its probability distribution is invariant by left multiplication by constant unitary matrices (this invariance condition specifies the distribution). Such a matrix can be generated the following way: let $\mathbf{X} = [x_{i,j}]_{1\leq i,j\leq N}$ be a $N \times N$ random matrix with independent complex Gaussian centered unit variance entries, then the unitary matrix $\mathbf{X}(\mathbf{X}^H\mathbf{X})^{-1/2}$ is Haar distributed (see [8]). In the sequel, it is assumed that $\mathbf{W}_N$ is generated by extracting any K columns from a $N \times N$ Haar distributed unitary matrix Θ_N independent of $\mathbf{H}_N$.

Before going further, let us recall the expression of the SINR at one of the K outputs of the MMSE detector. Denote by $\mathbf{w}_N$ be the column of $\mathbf{W}_N$ associated to some element of $\mathbf{s}$, and by $\mathbf{U}_N$ the $N \times (K-1)$ isometric matrix which remains after extracting $\mathbf{w}_N$ from $\mathbf{W}_N$. The SINR $\beta_{\mathbf{w}_N}$ at the output of the MMSE detector is given by (see e.g. [6]):

$$\beta_{\mathbf{w}_N} = \mathbf{w}_N^H \mathbf{H}_N^H \left(\mathbf{H}_N \mathbf{U}_N \mathbf{U}_N^H \mathbf{H}_N^H + \lambda \mathbf{I}_N\right)^{-1} \mathbf{H}_N \mathbf{w}_N \ . \qquad (3)$$

We are now in position to state the main results of this contribution:

Theorem 1 (Isometric Case) *Assume that matrices $\mathbf{W}_N$ and $\mathbf{H}_N$ are chosen as above and moreover, that the probability density $p(t)$ of the random variables $(|h_i|^2)_{i\in\mathbb{N}}$ has a compact support included in the interval $[0,c]$ (which implies that $\sup_{i\in\mathbb{N}} |h_i|^2 \leq c < +\infty$ almost surely).*
When N grows towards infinity and $K/N \to \alpha \leq 1$, the SINR $\beta_{\mathbf{w}_N}$ at the output of a MMSE equalizer converges almost surely to a value $\overline{\beta}$ that is the unique solution of the equation

$$\int_0^\infty \frac{t}{\alpha t + \lambda(1-\alpha)\overline{\beta} + \lambda}\, p(t)\, dt = \frac{\overline{\beta}}{\overline{\beta}+1} \ . \qquad (4)$$

The limit SINR neither depends on the particular realization of $\mathbf{W}_N$ if $N \to +\infty$ and $K/N \to \alpha$, nor on the particular entry of $\hat{\mathbf{s}}$. In some sense, the precoded system equipped with a MMSE receiver allows to transform a flat fading Rayleigh channel into a Gaussian channel with signal to noise ratio $\overline{\beta}$. For large blocks, it is also irrelevant to optimize the performance of a MMSE receiver with respect to $\mathbf{W}_N$.
In the statement of theorem 1, $p(t)$ is assumed to be compactly supported. This technical hypothesis is needed because the most powerful results of free probability theory applied to random matrices require compactly supported measures. Although the usual channel probability distributions like the Rayleigh or the Rice distributions are not compactly supported, in practice, formula (4) predicts quite well the performance of our precoded modulation scheme using MMSE detection.

For comparison sake, it is useful to give the expression of the asymptotic SINR when $\mathbf{W}_N$ is i.i.d. :

Theorem 2 (i.i.d. Case) *Assume that the entries of $\mathbf{W}_N$ are centered i.i.d. random variables with variance $1/N$, that the elements $\{h_i\}$ are identically distributed random variables such that $|h_i|^2$ has a probability density $p(t)$ with compact support, and that for each bounded continuous function φ,*

$$\lim_{N\to+\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi\left(\frac{1}{|h_i|^2}\right) = E\left(\varphi\left(\frac{1}{|h|^2}\right)\right) = \int \varphi\left(\frac{1}{t}\right) p(t)\, dt$$

almost surely.
When N grows towards infinity and $K/N \to \alpha \leq 1$, the SINR $\beta_{\mathbf{w}_N}$ at the output of a MMSE equalizer converges almost surely to a value $\overline{\beta}_1$ that is the unique solution of the equation

$$\int_0^\infty \frac{t}{\alpha t + \lambda \overline{\beta_1} + \lambda}\, p(t)\, dt = \frac{\overline{\beta}_1}{\overline{\beta}_1 + 1} \ . \qquad (5)$$

It is clear that for each $\beta > 0$,

$$\int_0^\infty \frac{t}{\alpha t + \lambda\beta + \lambda}\, p(t)\, dt \ \leq \ \int_0^\infty \frac{t}{\alpha t + \lambda(1-\alpha)\beta + \lambda}\, p(t)\, dt \ .$$

This implies that for a fixed value of α, $\overline{\beta}_1 \leq \overline{\beta}$. Moreover, the asymptotic performance of the MMSE receiver in the isometric case is all the more better with respect to the i.i.d. case that α is close to 1. Conversely, $\overline{\beta}_1 \simeq \overline{\beta}$ if α is close to 0.

III. Numerical Illustration

Simulations have been performed assuming a QPSK constellation, independent Rayleigh channel attenuations and perfect channel knowledge at the receiver. Figure 1 shows the BER in the isometric case for various spectral efficiencies $\alpha = 1, \frac{1}{2}$, and $\frac{1}{4}$. The curves closely match the simulation results using a realistic number of subchannels ($N = 256$). The "Gaussian Channel" curve is provided as a reference and corresponds to $\mathbf{H}_N = \mathbf{I}_N$. In this situation, the receiver output SINR is easily shown to be $1/\lambda$.
In order to determine the optimal amount of redundancy that should be spent on Linear Precoding, the throughput of an end-to-end system equipped with a MMSE receiver is analyzed. The throughput $\gamma(\alpha,\lambda)$ is the total number of bit/s/Hz that can be reliably transmitted with this system. It is defined by $\gamma(\alpha,\lambda) = \alpha C(\alpha,\lambda)$ where the capacity $C(\alpha,\lambda)$ is given by $C(\alpha,\lambda) = \log_2(1 + \mathrm{SINR}(\alpha,\lambda))$ and $\mathrm{SINR}(\alpha,\lambda)$ is $\overline{\beta}$ or $\overline{\beta}_1$. Note that $E_b/N_0 = (C\lambda)^{-1}$ (see [9] for more details). Figure 3 shows the behaviour of the optimum value of α (i.e. for which the throughput is maximum) with respect to E_b/N_0 for both isometric and i.i.d. cases. For maximizing the throughput, nearly no redundancy should be spent on the Linear Precoder in the isometric case. In contrast, in the i.i.d case, a significant amount of redundancy is required when $\frac{E_b}{N_0} > 4$dB. Figure 4 shows the maximum throughput vs E_b/N_0 for isometric and i.i.d. linear precoders. The throughput for a Gaussian channel is also provided. Isometric precoding increases the throughput with respect to i.i.d. precoding.
This throughput analysis is now used to study the performance of a system where Linear Precoding of rate α is combined with a classical Convolutional Coding of rate R. Assuming an overall coding rate αR of $1/2$, the purpose is to determine the optimum balance between α and R. Figure 3 suggests that when isometric precoding is used, $\alpha_{opt} \approx 1$ and with i.i.d. precoding,

$\alpha_{opt} \approx 2/3$ for the most common values of E_b/N_0. The optimum values of R are thus close to $1/2$ and to $3/4$ respectively. These claims are sustained by figures 5 and 6.

IV. Conclusion

This contribution extends the pioneering work of [6] devoted to asymptotic performance analysis of DS-CDMA systems employing i.i.d. signatures. The theoretical asymptotic SINR at the output of a MMSE receiver for a system using isometric Linear Precoding (LP) is derived using new mathematical tools, borrowed from the so-called Free Probability theory. It is shown, in particular, that in a system where Linear isometric Precoding is combined with Convolutional Coding, nearly no redundancy should be spent on LP to maximize the throughput. However, for Linear i.i.d Precoders, redundancy is required at the emitter side. Finally, in all the cases, isometric Linear Precoders always outperform i.i.d ones.

References

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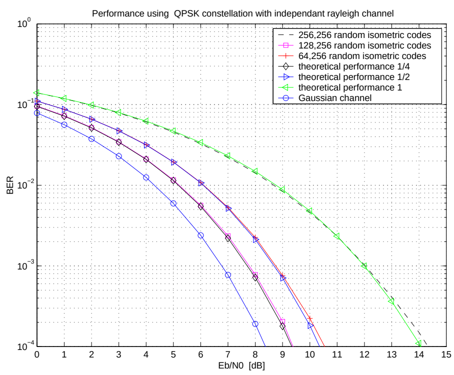

Figure 1: Probability of error

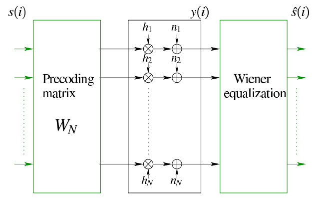

Figure 2: System Model

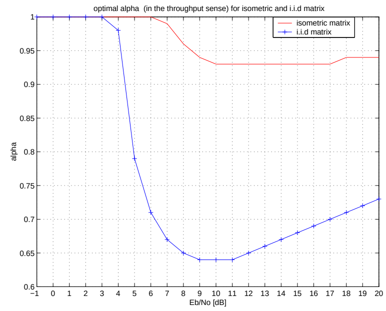

Figure 3: Optimum α

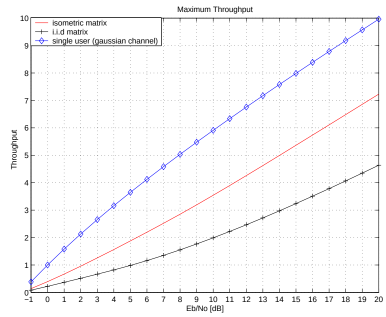

Figure 4: Optimum Throughput

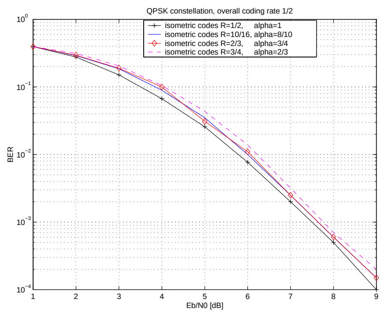

Figure 5: Isometric precoding matrix

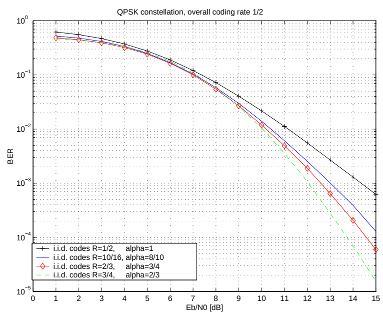

Figure 6: i.i.d precoding matrix