Partial k-trees with Maximum Chromatic Number

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Abstract

The paper is concerned with partial k-trees whose chromatic number is maximal, i.e. equal to $(k+1)$. We have proved that any such graph contains a triangle (if $k \geq 3$), but need not contain a clique on $\lfloor \frac{k+5}{2} \rfloor$ vertices as a subgraph.

Keywords: partial k-tree, treewidth, chromatic number, clique number

All graphs in this paper are nonempty, finite and without loops or multiple edges. $G = (V(G), E(G))$ denotes a graph G with the set of vertices $V(G)$ and the set of edges $E(G)$.

For the graph G let $\omega(G)$ denote the cardinality of the maximal clique, $\chi(G)$ the chromatic number, $\alpha(G)$ the maximal size of an independent set of vertices, $d_G(v)$ the degree of the vertex v in G, and $\delta(G)$ the minimal degree among all vertices of G. For any subset W of $V(G)$ we denote by $G \backslash W$ the graph obtained from G deleting the vertices of W (and all the edges adjacent with them). K_k stands for the complete graph on k-vertices and $K_{k_1,\ldots,k_r}$ for the complete r-partite graph ($r \geq 2$) defined in obvious way. $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In this paper the difference between isomorphism and equality of graphs is ignored.

The following definition of partial k-trees is taken from [3]:

Definition 1 *k-trees are defined recursively as follows: a clique with $(k+1)$ vertices is a k-tree; given a k-tree G with n vertices, a k-tree with $(n+1)$ vertices is constructed by taking G and creating a new vertex v which is made adjacent to a k-clique of G and nonadjacent to the $(n-k)$ other vertices of G.*

A **partial k-tree** *is any subgraph of a k-tree with the same vertex set as a k-tree.*

Another approach reflecting the possibility to construct a given graph as a tree-like graph from graphs of specified size uses the notion of the treewidth of a graph.

The following definition of the treewidth was introduced by Robertson and Seymour in [5].

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Definition 2 *A* **tree-decomposition** *of a graph G is a pair $(T, \mathcal{X})$, where T is a tree and $\mathcal{X} = (X_t,\ t \in V(T))$ is a family of subsets of $V(G)$ with the following properties:*

(i) $\cup_{t \in V(T)} X_t = V(G)$;

(ii) for every edge $\{u, v\} \in E(G)$ there exists a node $t \in V(T)$ such that both u and v are contained in X_t;

(iii) for t, t', $t'' \in V(T)$, if t' is on the path of T between t and t'' then

$$X_t \cap X_{t''} \subseteq X_{t'}.$$

The width of the tree-decomposition $(T, \mathcal{X})$ is

$$\max_{t \in V(T)} (|X_t| - 1).$$

The **treewidth** *of the graph G, $TW(G)$, is the minimal nonnegative integer k such that G has a tree-decomposition of the width k.*

The tree-decomposition $(T, \mathcal{X})$ of the graph G is said to be **fundamental** *if $(T, \mathcal{X})$ has the width $TW(G)$ and for any tree-decomposition $(T', \mathcal{X}')$ of G of the same width the inequality $|V(T')| \geq |V(T)|$ holds.*

Lemma 3 *Let G be a graph. Then $TW(G)$ is the smallest nonnegative integer k for which G is a partial k-tree.*

Proof. See Section 2.2 in [3]. □

The second possible definiton of $TW(G)$ is based on the notion of chordal graph. A graph H is said to be chordal if each cycle of length at least 4 has a chord in H.

Lemma 4 *Let G be a graph. Then $TW(G)$ is the smallest nonnegative integer k for which there exists a chordal supergraph H of G that does not contain K_{k+2} as a subgraph. (Clearly, we can confine ourselves to supergraphs H of G with $V(H) = V(G)$.)*

Proof. See [1]. □

The simple result of the following lemma is very often used in our proof methods.

Lemma 5 *Let $(T, \mathcal{X})$ be a fundamental tree-decomposition of a graph G. If vertices $t_1, t_2 \in V(T)$ are adjacent in T, then $X_{t_1} \setminus X_{t_2} \neq \emptyset$.*

Proof. See [1]. □

Theorem 6 *Let G be a graph. Then the following statements hold:*

(i) *If $|V(G)| \geq 2$, then there are (at least) two vertices v in G such that $d_G(v) \leq TW(G)$.*

(ii) $\chi(G) \leq TW(G) + 1$

(iii) $\alpha(G) + TW(G) \leq |V(G)|$.

Proof. Parts (i) and (ii) are easy consequences of known results on chordal graphs (see [2, Chapter 4]). For completeness we provide simple independent proofs.

(i) Let $(T, \mathscr{X})$ be a fundamental tree-decomposition of the graph G. If $|V(T)| = 1$, the assertion is obvious. If $|V(T)| \geq 2$ then due to Lemma 5 in any end-vertex t of T exists a vertex v of G such that $v \in X_t \setminus X_{t'}$ for any $t' \in V(T) \setminus \{t\}$. It implies $d_G(v) \leq TW(G)$.

(ii) We show the inequality by induction on $|V(G)|$, $|V(G)| = n$. Fix $n \geq 2$ and assume that (ii) holds true for any graph with at most n vertices. Let G be a graph with $|V(G)| = n + 1$. By (i), there exists a vertex v of G such that $d_G(v) \leq TW(G)$. By induction, $\chi(G\setminus\{v\}) \leq TW(G\setminus\{v\}) + 1$ that is clearly at most $TW(G) + 1$. Due to $d_G(v) \leq TW(G)$ we have $\chi(G) \leq TW(G) + 1$.

(iii) If $\alpha(G) = |V(G)|$ then G is a discrete graph and (iii) obviously holds. Otherwise we define a partition of the set $V(G)$ into subsets $V_1, \ldots, V_r$ $(r \geq 2)$ such that V_1 is an independent set of vertices of G of the size $\alpha(G)$ and $|V_i| = 1$ for $i = 2, \ldots, r$. In such a way we understand the graph G is a subgraph of the complete $(|V(G)| - \alpha(G) + 1)$-partite graph $K_{\underbrace{\alpha(G), 1, \ldots, 1}_{r}}$. It implies $TW(G) \leq TW(K_{\alpha(G),1,\ldots,1}) = |V(G)| - \alpha(G)$, cf. [1], which completes the proof. □

Remark 7 If G is a non-discrete partial k-tree ($k \in \mathbb{N}$) then $2 \leq \chi(G) \leq k + 1$ due to Theorem 6. For each positive integer k these bounds on $\chi(G)$ are strict.

As examples of 2-colorable graphs of treewidth k we can take complete bipartite graphs $K_{k,n}$, and planar $k \times n$ grids ($n \geq k$ in both cases). Any k-tree is an example of a graph with chromatic number $(k + 1)$.

We will show that graphs with chromatic number $(k + 1)$ and treewidth at most k need not be so dense as k-trees but, on the other hand, any such graph has to contain a triangle if $k \geq 3$.

Definition 8 *Let k, n be nonnegative integers such that $1 \leq n \leq k + 1$. By $g(k, n)$ we denote the smallest positive integer m with the following property: there exists a partial k-tree G with $\chi(G) \geq n$ and $\omega(G) = m$.*

Remark 9 Obviously, partial 0-trees are only discrete graphs and partial 1-trees are exactly forests, hence $g(k, 1) = 1$ for any $k \in \mathbb{N}_0$ and $g(k, 2) = 2$ for any $k \in \mathbb{N}$. It can be easily seen that any cycle C is a partial 2-tree and if C is of odd length it fulfils $\chi(C) = 3$. It implies that $g(k, 3) = 2$ whenever $k \geq 2$.

To obtain estimates of $g(k,n)$ we will use the following construction for graphs (see Figure 1).

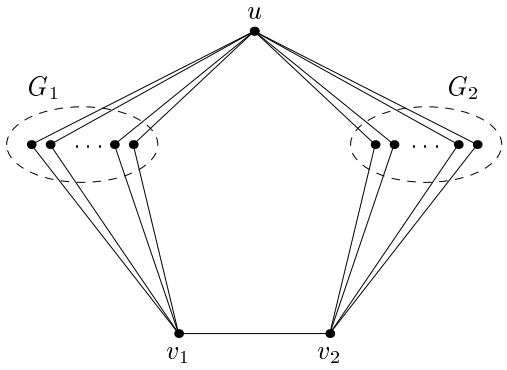

Figure 1:

Definition 10 *Let $G_1 = (V(G_1), E(G_1))$, $G_2 = (V(G_2), E(G_2))$ be two disjoint graphs. Let us denote by $\Phi(G_1, G_2)$ the graph $G = (V(G), E(G))$ defined in the following way:*

$$\begin{aligned} V(G) &= V(G_1) \cup V(G_2) \cup \{v_1, v_2, u\}, \\ &\quad \text{where } u, v_1, v_2 \notin V(G_1) \cup V(G_2) \text{ are distinct;} \\ E(G) &= E(G_1) \cup E(G_2) \cup \{\{u, v\} : v \in V(G_1) \cup V(G_2)\} \cup \{\{v_1, v_2\}\} \\ &\quad \cup_{i=1}^{2} \{\{w, v_i\} : w \in V(G_i)\} \end{aligned}$$

Lemma 11 *Let G_1 and G_2 be disjoint graphs and the graph $G = \Phi(G_1, G_2)$ be defined as above. Then the following implications hold:*

(i) If G_1 and G_2 are partial k-trees, then G is a partial $(k+2)$-tree.

(ii) If $\chi(G_1) \geq n$ and $\chi(G_2) \geq n$ then $\chi(G) \geq n + 2$.

(iii) If $\omega(G_1) \leq m$ and $\omega(G_2) \leq m$ then $\omega(G) \leq m + 1$.

Proof. (i) By Lemma 4 there exists a chordal supergraph H_i $(i = 1, 2)$ of the graph G_i which does not contain K_{k+2} and such that $V(H_i) = V(G_i)$.

Let define a graph H in the following way:

$$\begin{aligned} V(H) &= V(G), \\ E(H) &= E(G) \cup E(H_1) \cup E(H_2) \cup \{\{v_1, u\}, \{v_2, u\}\}. \end{aligned}$$

The graph H is a supergraph of G which does not contain K_{k+4}. Now we will show that the graph H is chordal.

Let C be a cycle of length at least 4 in the graph H. If both v_1 and v_2 belong to $V(C)$, then necessarily $u \in V(C)$ and one of edges $\{v_1, u\}$, $\{v_2, u\}$ is a chord of the cycle C. If at most one of vertices v_1 and v_2 belongs to $V(C)$ then $V(C) \subseteq V(H_j) \cup \{u\} \cup \{v_j\}$ for some $j \in \{1, 2\}$. Let $Z = V(C) \setminus V(H_j)$. If $Z = \emptyset$, then C is a subgraph of H_j and the cycle C has a chord in the graph H as H_j is chordal. If $Z \neq \emptyset$ and $z \in Z$ (clearly either $z = u$ or $z = v_j$) then z is adjacent in H to any vertex of $V(C) \setminus \{z\}$ and C has a chord. By Lemma 4, $TW(G) \leq k + 2$, and, due to Lemma 3, G is a partial $(k + 2)$-tree.

(ii) Let us suppose that $\chi(G) \leq n + 1$. Then we can choose a partition of $V(G)$ into $(n + 1)$ subsets V_1, V_2, ... , V_{n+1} independent in G. We denote r, s, $t \in \{1, \ldots, n + 1\}$ such that $v_1 \in V_r$, $v_2 \in V_s$, $u \in V_t$. Clearly $V(G_1) \subseteq \cup\{V_i, i \in \{1, \ldots, n + 1\}, i \neq r, i \neq t\}$. The assumption $r \neq t$ would imply $\chi(G_1) \leq n - 1$, so necessarily $r = t$. Similarly, $s = t$ and consequently, $r = s$. But $\{v_1, v_2\} \in E(G)$, which is a contradiction. This proves $\chi(G) \geq n + 2$.

(iii) It easily follows from the definition of the graph G.

□

Lemma 12 *For nonnegative integers k, n such that $1 \leq n \leq k+1$ the following inequalities hold:*

(i) $g(k + 2, n + 2) \leq g(k, n) + 1$,

(ii) If $n \geq 4$, then $g(k, n) \leq \min\{r + g(k - 2r, n - 2r), r \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}\} \leq \lfloor \frac{n}{2} \rfloor + 1$

Proof. (i) Let G_1 and G_2 be two disjoint isomorphic copies of a graph G with $TW(G) \leq k$, $\chi(G) \geq n$ and $\omega(G) = g(k, n)$. We define $\tilde{G} = \Phi(G_1, G_2)$. By Lemma 11 we get $TW(\tilde{G}) \leq k + 2$, $\chi(\tilde{G}) \geq n + 2$ and $\omega(\tilde{G}) \leq g(k, n) + 1$. It follows that $g(k + 2, n + 2) \leq g(k, n) + 1$.

(ii) The part (i) implies that for any positive integers k, n with $4 \leq n \leq k+1$ and for any $r \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$ the following inequalities hold:

$$g(k, n) - g(k - 2, n - 2) \leq 1, \tag{1}$$

$$g(k - 2, n - 2) - g(k - 4, n - 4) \leq 1, \tag{2}$$

$$\ldots$$

$$g(k - 2r + 2, n - 2r + 2) - g(k - 2r, n - 2r) \leq 1. \tag{r}$$

Summing up (1) to (r) together we obtain

$$g(k, n) \leq g(k - 2r, n - 2r) + r \tag{$*$}$$

that holds for any $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor - 1$. The second inequality in (ii) easily follows from $(*)$ for the case $r = \lfloor \frac{n}{2} \rfloor - 1$. □

Theorem 13 *For any positive integer k there exists a partial k-tree with chromatic number $(k + 1)$ which does not contain clique on $\lfloor \frac{k+5}{2} \rfloor$ vertices as a subgraph.*

Proof. For theorem holds obviously for $k \leq 2$, see e.g. Remark 9. For $k \geq 3$ we have $g(k, k+1) \leq \lfloor \frac{k+1}{2} \rfloor + 1$ by Lemma 12(ii).

Hence we are able to construct a partial k-tree G with chromatic number at least $(k+1)$ (Theorem 6 implies, $\chi(G) = k+1$) and $\omega(G) \leq \lfloor \frac{k+3}{2} \rfloor$. $\square$

Constructions of triangle-free graphs with arbitrarily large chromatic number are well known, see e.g. [4]. The following theorem shows, that such graphs cannot be partial k-trees with chromatic number $k+1$.

Theorem 14 *Let G be a partial k-tree ($k \geq 3$) with chromatic number $(k+1)$. Then G contains a triangle.*

Proof. Suppose for a contradiction that there exists a integer k, $k \geq 3$, and G a partial k-tree with chromatic number $(k+1)$ and $\omega(G) = 2$. Let us keep fixed one such graph G with the minimal number of vertices be fixed.

The following claims are easy consequence of this minimality property.

Claim 1 *$\chi(G \backslash M) \leq k$ whenever $\emptyset \neq M \subseteq V(G)$. Consequently, $\delta(G) \geq k$.*

Let $(T, \mathcal{X})$, where $\mathcal{X} = (X_t,\ t \in V(T))$, be a fundamental tree-decomposition of G. We denote $t_i^1 \in V(T)$, $i \in \{1, \ldots, r_1\}$ the end-vertices of the tree T. Obviously, $r_1 \geq 2$.

Let $i \in \{1, \ldots, r_1\}$. By Lemma 5 we can choose (and keep fixed in what follows) a vertex v_i of G such that $v_i \in X_{t_i^1}$ and $v_i \notin X_t$ for any $t \in V(T) \setminus \{t_i^1\}$. Let us denote $S_i = \{v \in V(G) : \{v, v_i\} \in E(G)\}$.

Claim 2 *For any $i \in \{1, \ldots, r_1\}$ the following holds: $|S_i| = k$, and for any two vertices $u, v \in S_i$ $\{u, v\} \notin E(G)$.*

Proof of Claim 2. The property (ii) from the definition of a tree-decomposition implies $S_i \subseteq X_{t_i^1} \setminus \{v_i\}$ and due to Claim 1 $|S_i| = k$. Necessarily $S_i = X_{t_i^1} \setminus \{v_i\}$. The second part easily follows from the assumption $\omega(G) = 2$. $\square$

By Claim 2 and $\chi(G) = k \geq 3$ we easily get $|V(T)| \geq 3$.

Claim 3 *The sets S_i, $i \in \{1, \ldots, r_1\}$, are pairwise distinct.*

Proof of Claim 3. Let $S_i = S_j$ for some $i, j \in \{1, \ldots, r_1\}$, $i \neq j$. The graph $G \backslash \{v_i\}$ is k-colorable by Claim 1. Then the vertices v_i and v_j can be colored with the color of the vertex v_j; this contradicts with $\chi(G) = k+1$. $\square$

Let us select $T_2 = T \setminus \{\cup t_i^1 : i \in \{1, \ldots, r_1\}\}$. Due to $|V(T)| \geq 3$ the tree T_2 is nonempty. We denote t_i^2, $i \in \{1, \ldots, r_2\}$ the end-vertices of the tree T_2.

We choose some vertex $t_{i_0}^2$, $i_0 \in \{1, \ldots, r_2\}$. Let us define the set I as follows: $I = \{i \in \{1, \ldots, r_1\} : \{t_i^1, t_{i_0}^2\} \in E(T)\}$. Obviously, $I \neq \emptyset$.

Claim 4 *$\cup_{i \in I} S_i \subseteq X_{t_{i_0}^2}$ and $|X_{t_{i_0}^2}| = k+1$.*

Proof of Claim 4. It easily follows from Claim 1 and the definition of the fundamental tree-decomposition. □

Claim 5 *There exists a vertex w such that $w \in X_{t^2_{i_0}}$ and $w \notin X_t$, for $t \in V(T) \setminus \{\{t^1_i : i \in I\} \cup \{t^2_{i_0}\}\}$.*

Proof of Claim 5. If $r_2 = 1$, then $V(T) = \{\{t^1_i : i \in I\} \cup \{t^2_{i_0}\}$. It means that each vertex of $X_{t^2_{i_0}}$ has the property of vertex w.

If $r_2 \geq 2$ let $\tilde{t} \in T_2$ be a (unique) vertex such that $\{\tilde{t}, t^2_{i_0}\} \in E(T_2)$. By Lemma 5 there is a vertex $w \in X_{t^2_{i_0}} \setminus X_{\tilde{t}}$. As $t^2_{i_0}$ is the end-vertex of the tree T_2, $w \notin X_t$ for $t \in V(T) \setminus \{\{\{t^1_i : i \in I\} \cup \{t^2_{i_0}\}\}$. □

In the following let $w \in X_{t^2_{i_0}}$ be a fixed vertex as in Claim 5.

Claim 6 $w \in \cup_{i \in I} S_i$.

Proof of Claim 6. If $w \in X_{t^2_{i_0}} \setminus \{\cup_{i \in I} S_i\}$ then necessarily $|I| = 1$. Let $I = \{i'\}$. Clearly, the vertices w and $v_{i'}$ have the same set of neighbors in the graph G. Due to Claim 1, the graph $G \setminus \{v_{i'}\}$ is k-colorable and the vertex $v_{i'}$ can have the color of vertex w. This is a contradiction with $\chi(G) = k + 1$, completing the proof of Claim 6. □

Now we will prove that our claims are inconsistent. Due to Claim 6, $w \in S_j$ for some $j \in I$. By Claims 2 and 4 we have $S_j \subseteq X_{t^2_{i_0}}$, $|S_j| = k$ and for each vertex $v \in S_j$ $\{v, w\} \notin E(G)$. But $d_G(w) \geq k \geq 3$, which implies $|I| \geq 3$. Due to Claim 3 it follows $\cup_{i \in I} S_i = X_{t^2_{i_0}}$. Then for any $u, v \in X_{t^2_{i_0}}$ there exists $i \in I$ such that $u, v \in S_i$ which implies $\{u, v\} \notin E(G)$ (Claim 2).

Let us consider a partition $V_1, \ldots, V_k$ of vertices of the graph $G \setminus \{\{v_i, i \in I\} \cup \{w\}\}$ into k independent sets.

(i) Let the partition be such that the vertices from $X_{t^2_{i_0}} \setminus \{w\}$ are in the sets $V_1, \ldots, V_s$, $s < k$. Clearly, there exists $l \in \{1, \ldots, s\}$ such that the set V_l contains at least two vertices from $X_{t^2_{i_0}} \setminus \{w\}$. Then the following is a partition of $V(G)$ into $(s+1)$ subsets independent in G:

$$\begin{aligned} V'_l &= V_l \cup \{w\}, \\ V'_i &= V_i, \text{ if } i \in \{1, \ldots, s\},\ i \neq l \\ V'_k &= V_k \cup \{v_i : i \in I\}. \end{aligned}$$

This is a contradiction with $\chi(G) \geq k + 1$.

(ii) Assume that for each partition $V_1, \ldots, V_k$ of vertices of the graph $G \setminus \{\{v_i : i \in I\} \cup w\}$ into k independent subsets, $V_l \cap (X_{t^2_{i_0}} \setminus \{w\}) \neq \emptyset$ holds true for each $l \in \{1, \ldots, k\}$. Then we define a graph $\tilde{G}$ in the following way:

$$\begin{aligned} V(\tilde{G}) &= V(G) \setminus \{v_i : i \in I\} \\ E(\tilde{G}) &= E(G) \setminus \{\{v, v_i\} : i \in I\}\} \cup \{\{w, u\} : u \in X_{t^2_{i_0}} \setminus \{w\}\}. \end{aligned}$$

It is easy to see that $\tilde{G}$ is a partial k-tree with $\chi(\tilde{G}) = k+1$ and $\omega(\tilde{G}) = 2$, but $|V(\tilde{G})| < |V(G)|$. This is a contradiction to the minimality of the graph G, completing the proof of theorem.

□

Remark 15 In Theorems 13 and 14 we have proved $3 \leq g(k, k+1) \leq \lfloor \frac{k+3}{2} \rfloor$ for $k \geq 3$. It is an interesting problem to obtain better estimates of minimum clique number among partial k-trees with chromatic number $(k+1)$, when k approaches infinity. Another related problem would be to determine the minimum tree width among triangle-free graphs with chromatic number k, for large k.

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