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Finite Element Numerical Simulation of Two-dimensional Ground Water Solute Migration Question

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Abstract. The finite element numerical model of 2-D solute transport in ground water with variable C was established under the condition of variety of the velocity of the movement of water is not too fast,and was carried on by the finite element numerical simulation method.the numerical simulation shows that the calculated result is in good agreement with the experiment result.

In some natural aquifer, groundwater are often handled as a constant speed because thevelocity of the movement of water is slow and chang little.At this time, the pollutants(or tracers) in the aquifer have a certain hydrodynamics dispersion in the direction of perpendicular to streamlines in addition to have transport along the direction of streamlines(convection-diffusion).This problem can be adjusted to a two-dimensional convection-diffusion question by the action of one- dimensional consant velocity field. In this paper, two-dimensional solute transport problems was carried on by the finite element numerical simulation method**.**

1. Mathematical Model

1.1 The equation of two-dimensional groundwater solute transport

Two-D groundwater solute transport problem with variable C can be described as eq.1 [1-4]

$$\frac{\partial C}{\partial t}=\frac{\partial}{\partial x}\left(\alpha v\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(D_y\frac{\partial C}{\partial y}\right)-v\frac{\partial C}{\partial x}\qquad (x,y\in\Omega) \tag{1}$$

In eq.1, v is the velocity of groundwater, α is the longitudinal diffusion, D_y is the horizontal diffusion coefficient, Ω is the calculating regionas shown in fig.1.

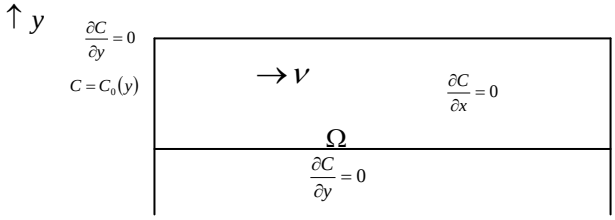


Fig.1 Sketch map of computing region Ω

1.2 The Initial and Boundary Conditions

(1) The initial conditions

$$C(x,y,0)=0 \tag{2}$$

(2) The boundary conditions

$$\begin{cases} C(0,y,t)=C_0(y) \\ \left.\dfrac{\partial C}{\partial x}\right|_{x=L_x}=0 \\ \left.\dfrac{\partial C}{\partial y}\right|_{\substack{y=0\\ y=L_y}}=0 \end{cases} \tag{3}$$

Where $C_0(y)$ is the known density distribution at $x=0$. L_x, L_y is the total length in the direction of x and y for region Ω.

2. The Numeric Model[5]

2.1 The dividing of computing region

As shown in fig.2,the computing region will be divided into N_e isosceles right triangle units, the apex of triangle is named node .Each triangle units $e_n\,(n=1,2,\ldots,N_e)$is corresponding with three nodes. The three nodes are prescribed as i,j,k according to the counterclockwise.So each node is corresponding with a coordinate value.

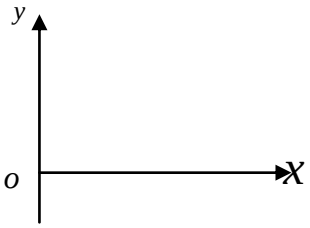
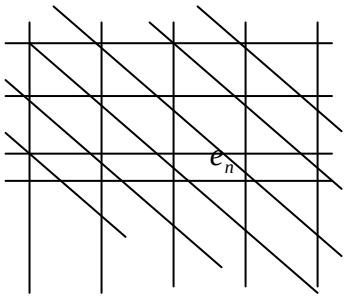


Fig.2 Sketch map of trianglation ofcalaulated



2.2 The establishment of numeric model[6-10]

By using the numerical simulation method of galerkin finite element, eq.4 is the equation of galerkin corresponding with two-D groundwater solute transport eq.1

$$\iint_{\Omega}\left[\frac{\partial C}{\partial t}-\frac{\partial}{\partial x}\left(\alpha v\frac{\partial C}{\partial x}\right)-\frac{\partial}{\partial y}\left(D_y\frac{\partial \theta}{\partial y}\right)+v\frac{\partial C}{\partial x}\right]N_m dxdy=0,\quad m=1,2,3,\ldots E \tag{4}$$

Where Ω is calculating region, E is node number,the serial numbers of node are$(1,2,3,\cdots,\ E)$ according to from down to up and left to right, N_m is shape function.So

So $C=(N_1(x,z)\quad N_2(x,z)\quad\ldots\quad N_{E-1}(x,z)\,N_E(x,z))\,(C_1(t)\quad C_2(t)\cdots C_{E-1}(t)\quad C_2(t))^T$,

$C_1(t),C_2(t),\cdots,C_{E-1}(t),C_E(t)$ are the values of the functions for C in every node.

By using Green formula , Eq.4 can be Simplified as eq.5.

$$\iint_{\Omega}\frac{\partial C}{\partial t}N_m dxdy+\iint_{\Omega}\alpha v\frac{\partial C}{\partial x}\cdot\frac{\partial N_m}{\partial x}dxdy+\iint_{\Omega}D_y\frac{\partial C}{\partial y}\cdot\frac{\partial N_m}{\partial y}dxdy+\iint_{\Omega}v\frac{\partial C}{\partial x}\cdot N_m dxdy=0 \tag{5}$$

By using the nature of integral , eq.5 can be turned into eq.6.

$$\sum_{n=1}^{N_e}\iint_{e_n}\frac{\partial C}{\partial t}N_m dxdy+\sum_{n=1}^{N_e}\alpha v\frac{\partial C}{\partial x}\cdot\frac{\partial N_m}{\partial x}dxdy+\sum_{n=1}^{N_e}D_y\frac{\partial C}{\partial y}\cdot\frac{\partial N_m}{\partial y}dxdy+\sum_{n=1}^{N_e}v\frac{\partial C}{\partial x}\cdot N_m dxdy=0 \tag{6}$$

Considering the first n triangle unit, the nodes of according to counterclockwise are desigated i,j,k, the coordinate of the three points followe (x_i,y_i), (x_j,y_j), (x_k,y_k), there are the following eq.7 and eq.8 in e_n.

$$C=C(x,y,t)=a_1(t)+a_2(t)x+a_3(t)y \tag{7}$$

$$C(x,y,t)=N_i(x,y)C_i(t)+N_j(x,y)C_j(t)+N_k(x,y)C_k(t) \tag{8}$$

Where ($N_i(x,y)=\dfrac{1}{2\Delta_e}\begin{vmatrix}x & y & 1\\ x_j & y_j & 1\\ x_k & y_k & 1\end{vmatrix}$, $N_j(x,y)=\dfrac{1}{2\Delta_e}\begin{vmatrix}x & y & 1\\ x_k & y_k & 1\\ x_i & y_i & 1\end{vmatrix}$, $N_k(x,y)=\dfrac{1}{2\Delta_e}\begin{vmatrix}x & y & 1\\ x_i & y_i & 1\\ x_j & y_j & 1\end{vmatrix}$)

is linear basis function in e_n.

Order

$$N=(\ N_i(x,y),N_j(x,y),N_k(x,y)\ ),\quad C_e=(C_i\quad C_j\quad C_k)^T \tag{9}$$

The gradient vector of the $C(x,y,t)$ can be denoted by $\nabla C = \left(\frac{\partial C}{\partial x} \quad \frac{\partial C}{\partial y} \right)^T = BC_e$.

Where $B = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} \end{bmatrix}$.

By $\frac{\partial C}{\partial t} = N \cdot \frac{\partial C_e}{\partial t}$

To $\iint_{e_n} \frac{\partial C}{\partial t} N dxdy = \frac{dC_e}{dt} \iint_{e_n} N^T N dxdy$, $d_n = \iint_{e_n} N^T N dxdy$

Similarly, the matrixs d_i $(i = 1,2,3,...,n-1,n+1,...,N_e)$ of other units e_i can be gotten.

Set $a_n = \iint_{e_n} \alpha \nu \frac{\partial C}{\partial x} \cdot \frac{\partial N}{\partial x} dxdy = C \iint_{e_n} \alpha \nu B^T B dxdy$, $b_n = \iint_{e_n} D_y \frac{\partial C}{\partial y} \cdot \frac{\partial N}{\partial y} dxdy = C \iint_{e_n} D_y B^T B dxdy$

$$f_n = \iint_{e_n} \nu \frac{\partial C}{\partial x} N dxdy = C \iint_{e_n} \nu \frac{\partial N}{\partial x} N dxdy$$

Order

$$g_n = a_n + b_n + f_n = \begin{pmatrix} C_{ii} & C_{ij} & C_{ik} \\ C_{ji} & C_{jj} & C_{jk} \\ C_{ki} & C_{kj} & C_{kk} \end{pmatrix} \tag{10}$$

Similarly, the matrixs g_i $(i = 1,2,3,...,n-1,n+1,...,N_e)$ of other units e_i can be gotten,and then the d_i, $(i = 1,2,3,...,n-1,n,n+1,...,N_e)$ are expanded and overlaid .The total stiffness matrix $[D]$ can be gotten. The g_i $(i = 1,2,3,...,n-1,n,n+1,...,N_e)$ are expanded and overlaid .The total stiffness matrix $[G]$ can be gotten. $[D]$ and $[G]$ are carried out transformations of the rows or columns of a matrix. Such, we can get a set of equations about C. They can be Written the following matrix form.

$$[D] \times \left[\frac{dC}{dt} \right] + [C] \times [G] = [O] \tag{11}$$

Where $[C] = [C_1 \quad C_2 \cdots C_G \cdots C_E]^T$, $\left[\frac{dC}{dt} \right] = \left[\frac{dC_1}{dt} \quad \frac{dC_2}{dt} \cdots \frac{dC_G}{dt} \cdots \frac{dC_E}{dt} \right]^T$, E is the total node number, G is the first boundary node numbers.

According to the boundary conditions, the nodes in the first typeis boundary is known, So the items of involving these nodes is moved to the right of the equation, as given items. Eq.11 in a matrix can be showed using the following form:

$$[M] \left[\frac{dC}{dt} \right] + [H][C] = [F] \tag{12}$$

Where in the eq.12, $[C]$ is a vector being made of the concentration $(C_{G+1}, C_{G+2}, \cdots, C_E)$ of Unknown nodes, $[H]$ is the coefficient matrix of the coefficient before $(C_{G+1}, C_{G+2}, \cdots, C_E)$, $\left[\frac{dC}{dt} \right]$ is a vector being made of the $(\frac{dC_{G+1}}{dt}, \frac{dC_{G+2}}{dt}, \cdots, \frac{dC_E}{dt})$, $[M]$ is the coefficient matrix of the coefficient before $(\frac{dC_{G+1}}{dt}, \frac{dC_{G+2}}{dt}, \cdots, \frac{dC_E}{dt})$, $[F]$ is a vector being made of the known items. Thus, the original mathematical model has been discreted in the space. Remainder problem is how to make calculating time discretization.In this paper,the discretization of the time uses discrete forward method.

So

$$[M]\left[\frac{C(t_{k+1})-C(t_k)}{\Delta t}\right]+[H][C]=[F] \tag{13}$$

In this formula, t_{k+1} is the time at the end of the t_{k+1}, t_k is the time at the early stages of t_k .Geting different time for the water head of the first item of the left of the equation also can forms different calculations format. In this paper, the central differencing scheme is used.So, eq.13 can be changed into eq.14.

$$\left(\frac{1}{\Delta t}[M]+[H]\right)[C^{k+1}]=[F]+\frac{1}{\Delta t}[M][C^{k}] \tag{14}$$

By using the iterative method solves the $(E-G)$ elements system of quations. First, to set the control standard e, then to adopt the initial values $C^{(0)}$ as the first iterative values , $C^{(1)}$ can be gotten. So, and so on,until iteration results meet $\max\left|\frac{C_j^{(p)}-C_j^{(p-1)}}{C_j^{(p-1)}}\right|\le e$. p is number of iterations.So can enter the next time calculating.

3. Simulation example [11-16]

For an insance of groundwater solute transport is carried on finite element numerical simulation by using the finite element numeric model of the text. Its physical parameters is: $\alpha=1.0m, \nu=0.1m\cdot d^{-1}, D_y=8\times10^{-5}m^2\cdot d^{-1}$, $L_x=200m, L_y=2m$ .The inflowcurrent boundary conditions is: $C(0,y,t)=C_0(y)=\begin{cases}1 & y\le 0.4m\\ (0.8-y)/0.4 & 0.4<y\le 0.8m\\ 0 & 0.8<y\le 2.0m\end{cases}$. The discrete steps is $\Delta x=2.0m, \Delta y=0.2m, \Delta t=20(d)$.From the longitudinal section of density distribution of the fig.3,we can see the result simulations and analytical solution with good agreement.

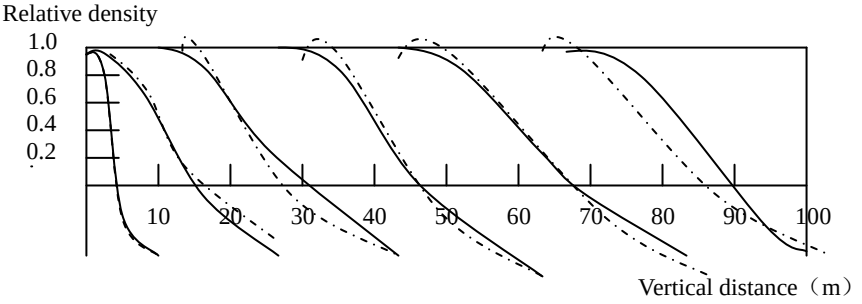


Fig.3Distributing chart of deepness of vertical section

4. The Conclusion

In this paper, first,the fundamental equations of two-dimensional groundwater solute transport with variable C was established.Then, by using the finite element method to discretize it and to get the finite element model of groundwater solute transport. Finally, by using the gotten numerical model carried on numerical simulation for an insance. By contrast, we can see have a good consistency between the simuiated result and the analytical solution.

Acknowledgmentsclusion

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