A COMPLETE CLASSIFICATION FOR PAIRS OF REAL ANALYTIC CURVES IN THE COMPLEX PLANE WITH TANGENTIAL INTERSECTION

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1. Introduction and results

Let $\gamma = (\gamma_1, \gamma_2)$ be a pair of real analytic curves in $\mathbf{C}$, intersecting only at the origin. Two such pairs $\gamma = (\gamma_1, \gamma_2)$ and $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$ are said to be equivalent, if there is a (local) biholomorphic mapping h defined near $0 = h(0)$ such that $h(\gamma_j) = \tilde{\gamma}_j$ for $j = 1, 2$. In this paper we shall give a complete classification for pairs of smooth real analytic curves in $\mathbf{C}$ with tangential intersection under a change of local holomorphic coordinates of the complex plane.

For a biholomorphic map f of the complex plane defined near $0 = f(0)$, let $\hat{\mathcal{C}}(f)$ be the group of centralizers of f, i.e. biholomorphic maps g satisfying $fg = gf$, and let $\mathcal{C}(f)$ be the set of the centralizers of f that are tangent to the identity. We shall determine the structures of the centralizer group $\hat{\mathcal{C}}(f)$. We shall also determine all composition *roots* of f, i.e. holomorphic maps g whose n-th iterate $g^n = g \circ \cdots \circ g$, for some integer n, is f.

We shall classify, under a change of local holomorphic coordinates, the abelian subgroups F of $\mathcal{A}$, where $\mathcal{A}$ is the group of local biholomorphisms $g(z)$, $g(0) = 0$, for which $g'(0)$ is a *root of unity*. We shall also classify the abelian subgroups $F \subset \mathcal{A}$ which are invariant by an anti-holomorphic involution σ (i.e. $\{\sigma g \sigma \colon g \in F\} = F$) under a change of local holomorphic coordinates that preserves σ.

We now give some details for our main results.

1.1. **Pairs of real analytic smooth curves.** We first consider two smooth real analytic curves that are tangent at the origin. All real analytic curves in this paper are germs of smooth real analytic curves at the origin. By a local change of holomorphic coordinates, one of curves in the pair is the y-axis, and the other is of the form

$$x = ay^{p+1} + O(|y|^{p+2}), \quad a \neq 0,$$

where $p+1$ is the *order* of tangency of two curves. It turns out that by a formal change of coordinates the pair is defined by

$$\gamma_1 \colon x = 0, \quad \gamma_2 \colon x = ay^{p+1} + \lambda y^{2p+1},$$

where $\lambda \in \mathbf{R}$, and $a = 1$ when p is odd and $a = \pm 1$ when p is even. Moreover, a, λ, p form a complete set of formal invariants for the pairs of real analytic curves (see Kasner [11] and Pfeiffer [14]).

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Let $\mathcal{S}$ be the set of holomorphic functions h such that $h(\xi)-\xi$ is bounded, holomorphic, and of period one on some upper half-plane $\Pi^+(\epsilon)$: $\operatorname{Im}\xi > 1/\epsilon$. Denote by $\mathcal{M}_p$ the set of $2p$-tuples holomorphic functions $\Phi = (\Phi_1, \dots, \Phi_{2p})$ with $\Phi_j((-1)^{j-1}\xi) \in \mathcal{S}$. We say that $\Phi, \tilde{\Phi} \in \mathcal{M}_p$ are equivalent (written $\Phi \sim \tilde{\Phi}$), if there exist constants $c_j = c_{j+2p} \in \mathbf{C}$ and $k \in \mathbf{Z}$ such that for $L_c(\xi) = \xi + c$

$$\tilde{\Phi}_j = L_{c_{j+1}} \Phi_{j+2k} L_{-c_j}, \quad j = 1, \dots, 2p, \tag{1.1}$$

in which we put $\Phi_{j+2p} = \Phi_j$ and $\tilde{\Phi}_{j+2p} = \tilde{\Phi}_j$. Let $\mathcal{M}_{p,\sigma}$ be the set of $\Phi \in \mathcal{M}_p$ satisfying the additional condition

$$\Phi_{2p-j}^{-1} = \sigma \Phi_j \sigma, \quad j \in \mathbf{Z}, \quad \sigma(\xi) = -\overline{\xi}.$$

Denote $\tilde{\Phi} \overset{\sigma}{\sim} \Phi$, if they satisfy (1.1) for some $c_j \in \mathbf{C}$ with $c_{2p-j+1} = -\overline{c}_j$ and for $k = p/2 \in \mathbf{Z}$ or for $k = 0$. Denote by $\mathcal{M}_{p,\sigma}/\overset{\sigma}{\sim}$ the set of such equivalence classes.

The classification for the pair of smooth real analytic curves with tangential intersection is given by

Theorem 1.1. *There is a one-to-one correspondence between $\mathcal{M}_{p,\sigma}/\overset{\sigma}{\sim}$ and the set of local holomorphic equivalence classes of the pairs of real analytic curves that are tangent to order $p+1$, when p is odd; when p is even, the set of equivalence classes of the pairs of curves is one-to-one corresponding to the disjoint union of two copies of $\mathcal{M}_{p,\sigma}/\overset{\sigma}{\sim}$.*

The proof of the theorem is in section 4, where one can also find a moduli space for the holomorphic equivalence classes of pairs of real analytic curves intersecting with an angle that is a *rational* multiple of π.

The proof of Theorem 1.1 is based on the Ecalle-Voronin theory. Denote by $\mathcal{A}_p$ the set of holomorphic mappings

$$f(z) = z + f_{p+1} z^{p+1} + O(|z|^{p+2}), \quad f_{p+1} \neq 0. \tag{1.2}$$

Denote by $\mathcal{A}_p^*$ the set of holomorphic maps $f \in \mathcal{A}_p$ with $f_{p+1} = \mu_0$ (see (2.1) below). Let $\mathcal{A}_p/\sim$ be the equivalence classes of mappings in $\mathcal{A}_p$ under a change of holomorphic coordinates, and let $\mathcal{A}_p^*/\overset{*}{\sim}$ be the set of equivalence classes of mappings in $\mathcal{A}_p^*$ under a change of holomorphic coordinates that is tangent to the identity.

The Ecalle-Voronin theory says that there is a one-to-one correspondence between $\mathcal{A}_p^*/\overset{*}{\sim}$ and $\mathcal{M}_p/\overset{*}{\sim}$, where $\tilde{\Phi} \overset{*}{\sim} \Phi$ if they satisfy (1.1) with $k = 0$; see [6], [19], [12], and [9]. Note that $\mathcal{A}_p/\sim = \mathcal{A}_p^*/\overset{*}{\sim}$ when $p = 1$. In general, one has

Theorem 1.2. *There is a one-to-one correspondence between $\mathcal{A}_p/\sim$ and $\mathcal{M}_p/\sim$.*

The above theorem is likely not original with us. However, due to the lack of a precise reference and to its importance to other classification problems, a complete proof of the theorem is given in section 2. See [1] also for finitely smooth local normalization for entire functions that are tangent to the identity.

1.2. **Centralizers and roots.** In section 3 we shall give a complete description for the centralizer group $\hat{\mathcal{C}}(f)$ and roots of $f(z) = z + O(|z|^2)$. We assume that f is not the time-one map of a holomorphic vector field $A(z)\frac{\partial}{\partial z}$ with $A(z) = O(|z|^2)$, since the case that f is

in the flow is well-understood [7]. We shall see that $\hat{\mathcal{C}}(f)$ is an abelian group and $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$ is a finite group of order q. The elements of $\hat{\mathcal{C}}(f)$ of finite order form a cyclic group of finite order δ.

It is well-known that $\mathcal{C}(f)$ is generated by the root $f^{1/d}$ for some positive integer $d = d_f$ (see [2], [5], [10]). Throughout the paper $f^{1/d}$ stands for the (unique) d-th root of f that is tangent to the identity. Berthier, Cerveau and Meziani [3] also showed that $\hat{\mathcal{C}}(f)$ is an abelian group generated by two elements τ_0, ω where ω generated all periodic centralizers of f. The following theorem describes suitable generators of $\hat{\mathcal{C}}(f)$ which are relevant to other invariants of the group, which will be useful to our applications.

Theorem 1.3. *Let f be an element in $\mathcal{A}_p$ that is not embeddable into any flow of a holomorphic vector field $A(z)\frac{\partial}{\partial z}$ with $A(z) = O(|z|^{p+1})$. Then*

i) $\hat{\mathcal{C}}(f) = <\tau, f^{1/d}> = <\tau> \otimes <\omega>$ for some τ, ω satisfying $\tau^{q/\delta} = \omega f^{1/d}$ and $\omega^\delta = 1$. ($<x, y>$ stands for elements generated by x and y.)

ii) Let r be the largest factor of dq/δ with $(r, \delta)|d$. There exists a r-th root g of f and all t-th roots h of f have the property: t divides r and $h = \omega^j g^{r/t}$ for some j with δ dividing jt.

iii) In particular, when $\hat{\mathcal{C}}(f)$ has no periodic element, it is a cyclic group generated by the unique r-th root g.

Moreover, d, q and δ depend only on the holomorphic equivalence class of f.

Theorem 1.4. *For $f \in \mathcal{A}_p$ with $d_f \neq 0$, we have*

i) If $f^{-1} = HfH^{-1}$ and H is anti-holomorphic, then $H^2 = \mathrm{Id}$. Moreover $g^{-1} = HgH^{-1}$ for all $g \in \hat{\mathcal{C}}(f)$.

ii) The number of germs of smooth real analytic curves at the origin that are invariant under f is either 0, or δ_f. When the latter occurs, the curves are of the form $\gamma_j = \omega^{(j-1)/2}\gamma_1$, where ω generates all periodic elements in $\hat{\mathcal{C}}(f)$. In particular, f has no two invariant smooth real analytic curves tangent at the origin.

iii) Let g be a germ of holomorphic function at $0 \in \mathbf{C}$. Assume that f and gfg^{-1} are real-valued on $\mathbf{R}$. Then g is real-valued on $\mathbf{R}$, if and only if $g'(0)$ is real; in particular, g is real if $p = 1$ or 2.

iv) Assume further that $f = \sigma_1\sigma_2$, where σ_j are anti-holomorphic involutions generated by germs of smooth real analytic curves γ_j at the origin. There exist at most two biholomorphisms g with $g(\gamma_j) = \gamma_j$ for $j = 1, 2$. There exists $g \neq \mathrm{Id}$ with $g(\gamma_j) = \gamma_j$ for $j = 1, 2$ (and hence $g^2 = \mathrm{Id}$), if and only if δ_f is even.

1.3. **The σ-equivalence classifications of A_f and their group structures.** By the Schwarz reflection, a smooth real analytic curve determines a local anti-holomorphic involution fixing the curve pointwise, and vice versa. Thus a pair of smooth real analytic curves determines a pair of (local) anti-holomorphic involutions τ_1, τ_2. When one of real analytic curve is the y-axis, its corresponding anti-holomorphic involution is

$$\sigma\colon z \to -\overline{z}.$$

Then the composition $f = \tau_1\tau_2$ is *reversible* with respect to σ, i.e. $f^{-1} = \sigma f \sigma$. Thus the classification for pairs of smooth real analytic curves tangent to order $p + 1$ corresponds

to the classification of $\mathcal{A}_{p,\sigma}/\overset{\sigma}{\sim}$, where $\mathcal{A}_{p,\sigma}$ is the set of $f \in \mathcal{A}_p$ that are reversible by σ. Denote $\tilde{f} \overset{\sigma}{\sim} f$ when $\tilde{f} = gfg^{-1}$ for some biholomorphism $g = \sigma g \sigma$.

Given $f \in \mathcal{A}_{p,\sigma}$, let A_f be the set of elements in $\mathcal{A}_{p,\sigma}$ that are equivalent to f under some biholomorphism. In section 5 we classify A_f under various more restrictive biholomorphisms. It turns out the equivalence classes admit some group structures related to that of $\hat{\mathcal{C}}(f)$.

In section 5 we shall also classify reversible holomorphic maps in $\mathcal{A}_p$, i.e., maps that are reversible by some holomorphic involution, under general biholomorphic maps.

We shall then use the moduli functions of $\mathcal{A}_p$, described by Il'yashenko [9], to show the following.

Theorem 1.5. *There exists $f \in \mathcal{A}_p$ that is reversible by some formal anti-holomorphic involution, but not by any (convergent) anti-holomorphic map.*

For the existence of holomorphic maps f, with $f'(0)$ not a root of unity, such that f is not reversible by anti-holomorphic involution, see [8]. The method in [8], based on pairing periodic orbits of reversible maps, is completely different from the one in this paper, since maps in $\mathcal{A}_p$ have no periodic orbits (remaining in small neighborhoods of the origin).

1.4. **Real analytic maps on R.** In section 6 we classify (local) real analytic maps on $\mathbf{R}$ that is tangent to the identity, under real analytic maps of $\mathbf{R}$. We shall also show that Theorem 1.4 iii) implies that

Corollary 1.6. *Let $f, \tilde{f} \in \mathcal{A}_p$ be real valued on $\mathbf{R}$ with $p = 1$ or 2. Then $\tilde{f}, f$ are equivalent under a germ of real analytic diffeomorphism of $\mathbf{R}$ at the origin, if and only if they are equivalent under both a formal real analytic map of $\mathbf{R}$ and a holomorphic map defined near the origin.*

In contrast to the above, we also have

Theorem 1.7. *For $p \geq 3$, there exist $f, \tilde{f} \in \mathbf{R}_p$ which are equivalent under a formal real map and a holomorphic map, but not under any germ of real analytic diffeomorphism of $\mathbf{R}$ at the origin.*

Dropping the condition $d_f \neq 0$ in Theorem 1.4, we shall prove

Theorem 1.8. *Let $f \neq \mathrm{Id}$ be a biholomorphic mapping defined near $0 \in \mathbf{C}$ that admits two smooth real analytic curves tangent at the origin. Then in suitable holomorphic coordinates one of the followings holds*

- *i) $f(z) = -z$. The set of invariant smooth real analytic curves of f consists of $y = A(x)$, or $x = B(y)$ with $B'(0) = 0$, where A, B are odd real analytic functions.*
- *ii) f is the time-one map of $v_{\epsilon_p,\lambda} = \epsilon_p \frac{z^{p+1}}{1+\lambda z^p} \frac{\partial}{\partial z}$ with $\lambda \in \mathbf{R}$ and $\epsilon_p = 1$ for p odd and $\epsilon_p = \pm 1$ for p even. The invariant smooth real analytic curves of f are fixed-point sets of $z \to v^t_{\epsilon_p,\lambda}(e^{2\pi ji/p}\overline{z})$, where $j = 1, \ldots, p$ and $t \in i\mathbf{R}$.*
- *iii) f is $-v^{1/2}_{\epsilon_p,\lambda}$ with p even, while the invariant smooth real analytic curves of f are as the same as above.*

iv) Let f be as in i), ii) or iii). Let g be a germ of biholomorphic map defining at $0 \in \mathbf{C}$. Then $g^{-1}fg$ remains real on the real axis, if and only if there is a germ of real diffeomorphism h of $\mathbf{R}$ at the origin so that $g(z) = h(z) + iA(h(z))$ or $g(z) = B(h(z)) + ih(z)$ for case i), or $g(z) = v_{\epsilon_p,\lambda}^{t/2}(e^{\pi ji/p}h(z))$ with $j = 1, \ldots, p$ and $t \in i\mathbf{R}$ for ii) and iii).

1.5. **Abelian subgroups of $\mathcal{A}$.** In section 7 we turn to the classification of abelian subgroups F of $\mathcal{A}$ under a change of holomorphic coordinates, where we emphasize the relationship between group structures and invariants of generators of $F \cap \mathcal{A}_0$. Here we would like to mention that a moduli space for the classification of certain generators of F under holomorphic maps tangent to the identity is given by Elizarov, Il′yashenko, Shcherbakov, and Voronin [7], and that the moduli space can be easily adapted to the classification for the groups.

We restrict ourselves to the case that $F \cap \mathcal{A}_p$ is non-empty; consequently, F is a subgroup of $\hat{\mathcal{C}}(f)$ for some $f \in \mathcal{A}_p$. We shall see that if f is reversible or invariant by σ and f is not in the flow of any holomorphic vector field $A(z)\frac{\partial}{\partial z}$ with $A(z) = O(|z|^2)$, then $\hat{\mathcal{C}}(f)$ is σ-invariant. We shall also use holomorphic maps that commute with σ to classify the abelian subgroups F of $\mathcal{A}$ that are invariant under σ. For the above two classification problems about abelian subgroups, we shall use both moduli spaces for σ-reversible holomorphic maps on $\mathbf{C}$ and real analytic maps on $\mathbf{R}$. We shall also use the decomposition of $\hat{\mathcal{C}}(f)$ given by Theorem 1.3.

In section 8, we shall also classify pairs of holomorphic involutions and abelian subgroups of $\mathcal{A}$ that are invariant under a holomorphic involution.

1.6. **Exceptional non-abelian subgroups of Cerveau and Moussu.** In section 9, we shall find a moduli space for the exceptional non-abelian subgroups of Cerveau and Moussu [4] via the Ecalle-Voronin theory. The connection between the Ecalle-Voronin theory and the moduli space of the exceptional groups was already observed by Cerveau and Moussu [4]. We shall also determine the relationship between the classifications of the groups and of their generators.

1.7. **Comments.** The problem on classification for pairs of real analytic curves goes back to works of Poincaré [16], Kasner [11], and Pfeiffer [14]. In [15], Pfeiffer showed the existence of pairs of real analytic curves in $\mathbf{C}$ intersecting at some angle of irrational multiple of π that are formally but not holomorphically equivalent to a pair of real lines.

Nakai [13] studied the problems about pairs of real analytic curves, via the Ecalle-Voronin theory. Theorem 5.1 and Theorem 3 in [13] gave two moduli spaces for the classification of the *composition* maps of pairs of anti-holomorphic involutions generated by two real analytic curves intersecting tangentially to order 2. However, both theorems do not agree with Theorem 5.8 below (Theorem 5.8 gives a moduli space for the classification of the composition map of the pair of involutions). In fact, the moduli space given in Theorem 5.1 in [13] is the same as the one in Theorem 1.1 above, and it would thus seem to be the one for the classification of the *pairs* of real analytic curves.

Section 5 also contains more complete results for some questions about the composition of pairs of anti-holomorphic involutions studied by Nakai [13]. For recent work about pairs of real analytic curves in the complex plane, see Trépreau [18].

2. the Ecalle-Voronin theory: the complete set of invariants

In this section we shall find the moduli space for germs of holomorphic maps of the form

$$z \to a_1 z + a_{p+1} z^{p+1} + O(|z|^{p+2}), \quad a_{p+1} \neq 0, \quad p > 0$$

under a holomorphic change of coordinates, where a_1 is a root of unity. This moduli space was obtained by Ecalle [6] and Voronin [19], for $p = 1$ and $a_1 = 1$. The moduli space for $p > 1$ under biholomorphisms that are tangent to the identity is given by Il'yashenko [9].

The moduli functions constructed in this section will describe precisely when two holomorphic maps $z \to z + a_{p+1} z^{p+1} + O(|z|^{p+2})$ are holomorphically equivalent under a biholomorphism g with a given linear part. This result will play a role for the other classification problems in this paper.

Fix a positive integer p. Recall that $\mathcal{A}_p$ is the set of holomorphic mappings

$$f(z) = z + a_{p+1} z^{p+1} + O(|z|^{p+2}), \quad a_{p+1} \neq 0.$$

We say that $f, \tilde{f} \in \mathcal{A}_p$ are holomorphically equivalent, if there is a biholomorphic mapping g such that $\tilde{f} = g^{-1} f g$. Let $\mathcal{A}_p^*$ be the set of above mappings f with

$$a_{p+1} = \mu_0 = \begin{cases} 1 & p \text{ odd}, \\ i & p \text{ even}. \end{cases} \tag{2.1}$$

Each $f \in \mathcal{A}_p$ is equivalent to some $\tilde{f} \in \mathcal{A}_p^*$ by a linear change of coordinates. Thus there is a (canonical) one-to-one correspondence between the holomorphic equivalence classes in $\mathcal{A}_p$ and ones in $\mathcal{A}_p^*$. Note that if $f, g^{-1} f g \in \mathcal{A}_p^*$, then

$$g(z) = \mu^k z + O(|z|^2),$$

where μ is a primitive p-th root of unity. For our purpose, we choose

$$\sqrt{\mu} = \begin{cases} e^{i\frac{\pi}{p}}, & p = 0, 1 \mod 4, \\ e^{-i\frac{\pi}{p}}, & p = 2, 3 \mod 4. \end{cases} \tag{2.2}$$

We say that $\Omega_j, A_j, j = 1, \ldots, 2p$ are a canonical collection for $f \in \mathcal{A}_p^*$, if the following I i-v hold

- I i) $\Omega_j, j = 1, \ldots, 2p$, are connected and simply connected domains of $\mathbf{C}$, whose union is a punctured neighborhood of $0 \in \mathbf{C}$. $\Omega_{2j+1} \cap \Omega_{2k+1} = \Omega_{2j} \cap \Omega_{2k} = \emptyset$ for $k \neq j \mod p$.
- I ii) Denote by y_+ the germ of sectors at the origin, centered along the positive imaginary axis, i.e., a sector of the form $\{z \colon O < |z| < r, |\arg z - \frac{\pi}{2}| < \epsilon\}$ for some $r, \epsilon > 0$. Then $\Omega_j \cap \Omega_{j+1}$ contains $\sqrt{\mu}^j y_+$. (Combining with Ii yields $\sqrt{\mu}^k y_+ \cap \Omega_j = \emptyset$ for $k \neq j, j+1 \mod 2p$.)
- I iii) $A_j \colon \Omega_j \to \tilde{\Omega}_j = A_j(\Omega_j)$ are conformal and satisfy

$$\begin{array}{ccc} \Omega_{2j+1} & \xrightarrow{f} & \Omega_{2j+1} \\ \downarrow{\scriptstyle A_{2j+1}} & & \downarrow{\scriptstyle A_{2j+1}} \\ \tilde{\Omega}_{2j+1} & \xrightarrow{F} & \tilde{\Omega}_{2j+1} \end{array} \qquad \begin{array}{ccc} \Omega_{2j} & \xrightarrow{f^{-1}} & \Omega_{2j} \\ \downarrow{\scriptstyle A_{2j}} & & \downarrow{\scriptstyle A_{2j}} \\ \tilde{\Omega}_{2j} & \xrightarrow{F^{-1}} & \tilde{\Omega}_{2j} \end{array}$$

for $F(z) = z + 1$. In particular, $f(\Omega_{2j+1}) \subset \Omega_{2j+1}$ and $f^{-1}(\Omega_{2j}) \subset \Omega_{2j}$.

Iiv) There exists $\epsilon > 0$ such that

$$(2.3)\qquad \begin{gathered}(-1)^{j-1}R(\epsilon) \subset \tilde{\Omega}_j, \quad \Pi_{p,j}(\epsilon) \subset A_j(\Omega_j \cap \Omega_{j+1}) \subset \Pi_{p,j}(-\epsilon),\\ \Pi_{p,j}(\epsilon) \subset A_{j+1}(\Omega_j \cap \Omega_{j+1}) \subset \Pi_{p,j}(-\epsilon),\end{gathered}$$

for

$$R(\epsilon) = \{|\operatorname{Im}\xi| > 1/\epsilon\} \bigcup \{\operatorname{Re}\xi > 1/\epsilon\}, \quad L(\epsilon) = -R(\epsilon),$$

$$\Pi_p(\epsilon) = \begin{cases} \{\xi \mid \operatorname{Im}\xi > 1/\epsilon\}, & p = 2, 3 \mod 4, \\ \{\xi \mid \operatorname{Im}\xi < -1/\epsilon\}, & p = 0, 1 \mod 4, \end{cases}$$

$$\Pi_{p,j}(\epsilon) = (-1)^{j-1}\Pi_p(\epsilon).$$

For any $\epsilon' > 0$ and any punctured neighborhood ω' of origin, the union

$$\bigcup \{A_{2j+1}^{-1}(R(\epsilon')) \cup A_{2j}^{-1}(L(\epsilon'))\}$$

is a punctured neighborhood of the origin, and is contained in ω' if ϵ' is small.

Iv) If ω (resp. ω') is a subset of $\bigcup_j \Omega_j$ satisfying $f(\omega) \subset \omega$ (resp. $f^{-1}(\omega') \subset \omega'$) then $\omega \subset \bigcup_j \Omega_{2j+1}$ (resp. $\omega' \subset \bigcup_j \Omega_{2j}$).

In the above, we already take $A_{j+2p} = A_j$ and $\Omega_{j+2p} = \Omega_j$ for $k \in \mathbf{Z}$. This convention will be used throughout our discussion. Our construction of the canonical collection is still based on Voronin's approach for the case $p = 1$. In fact we do not exclude the case $p = 1$ in our construction. However, the second relation in (2.3) should be replaced with $\Pi^{\pm}(\epsilon) \subset A_1(\Omega_1 \cap \Omega_2 \cap \Pi^{\pm}(0)) \subset \Pi^{\pm}(-\epsilon)$, when $p = 1$.

To see a concrete example of a canonical collection, we start with the vector field

$$(2.4)\qquad v\colon \frac{dz}{dt} = \mu_0 z^{p+1},$$

where μ_0 is given by (2.1). Denote by v^t the flow of v. Put

$$\xi = \hat{A}(z) = \frac{1}{-p\mu_0 z^p}.$$

Then $\hat{A}$ transforms v into the vector field

$$\hat{A}_* v\colon \frac{d\xi}{dt} = 1,$$

of which the flow is the translation $L_t\colon \xi \to \xi + t$. Define the p-th root function by

$$\sqrt[p]{z} = e^{\frac{1}{p}(\log|z| + i\arg z)},$$

$$2\pi[\frac{p}{4}] \le \arg z < 2\pi + 2\pi[\frac{p}{4}], \quad p = 1, 3 \mod 4$$

$$2\pi[\frac{p}{4}] - \frac{\pi}{2} \le \arg z < 2\pi[\frac{p}{4}] + \frac{3\pi}{2}, \quad p = 0, 2 \mod 4,$$

in which $[\frac{p}{4}]$ is the largest integer $\leq p/4$. Put $\hat{A}_j = \hat{A}|_{\hat{\Omega}_j(\rho)}$ and

$$\hat{A}^{-1}(\xi) = \sqrt[p]{\frac{1}{-p\mu_0\xi}}, \quad \hat{A}_j^{-1}(\xi) = \sqrt{\mu}^{j-1}\hat{A}^{-1}(\xi), \tag{2.5}$$
$$\hat{\Omega}_j(\rho) = \hat{A}_j^{-1}((-1)^{j-1}R(\rho)), \quad j = 1, \ldots, 2p.$$

We have chosen μ_0 so that both $\hat{\Omega}_1(\rho)$ and $\hat{\Omega}_{2p}(\rho)$ intersect with the positive y-axis, which forces $\hat{\Omega}_1(\rho), \ldots, \hat{\Omega}_{2p}(\rho)$ to be arranged counter-clockwise when $p = 1$, or 4 mod 4, and clockwise when $p = 2$, or 3 mod 4. We have also chosen μ so that $\hat{\Omega}_{j+1}(\rho) = \sqrt{\mu}\hat{\Omega}_j(\rho)$.

One can verify that the time-one mapping v^1 is in $\mathcal{A}_p^*$, and that $\hat{A}_j, \hat{\Omega}_j(\rho)$ form a canonical collection for v^1. Notice that, by our choices of μ_0 and μ, we also have

$$v^{-1} = \sigma v^1 \sigma, \quad \hat{A}_{2p-j+1} = \sigma\hat{A}_j\sigma, \quad \sigma(\hat{\Omega}_j(\rho)) = \hat{\Omega}_{2p-j+1}(\rho), \quad \sigma(z) = -\overline{z}.$$

We now redefine $\mathcal{M}_p$, by choosing $2p$ half-planes, which are possibly different from the ones in section one, on which $\Phi = (\Phi_1, \ldots, \Phi_{2p})$ are defined.

Let $\mathcal{M}_p$ be the set of holomorphic functions $\Phi_1, \ldots, \Phi_{2p}$, for which $\Phi_j(\xi) - \xi$ is bounded and of period 1 on the half-plane $\Pi_{p,j}(\epsilon)$. We say that $\Phi = (\Phi_1, \ldots, \Phi_{2p}), \tilde{\Phi} = (\tilde{\Phi}_1, \ldots, \tilde{\Phi}_{2p})$ are *equivalent*, if there are constants $c_1, \ldots, c_{2p}$, an integer k, and $\epsilon > 0$ such that

$$\tilde{\Phi}_j(\xi) = L_{c_{j+1}}\Phi_{j+2k}L_{-c_j}(\xi), \quad \xi \in \Pi_{p,j}(\epsilon) \quad j = 1, \ldots, 2p.$$

We first need to construct the moduli function for a holomorphic mapping $f \in \mathcal{A}_p^*$. We start with the following basic lemma of Voronin

Lemma 2.1. *Let* $G\colon R(\delta_0) \to \Omega$ *be a quasi-conformal mapping, and let* $f\colon \Omega \to \Omega$ *be holomorphic. Let* $F(z) = z + 1$. *Assume that*

$$\begin{array}{ccc} R(\delta_0) & \xrightarrow{F} & R(\delta_0) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G} \\ \Omega & \xrightarrow{f} & \Omega \end{array}$$

is a commutative diagram, and that $\Omega \supset R(\epsilon_0)$. *There exists a conformal* $A\colon \Omega \to \tilde{\Omega}$ *with* $\tilde{\Omega}$ *containing some* $R(\epsilon_1)$, *such that* $FA = Af$. *Moreover, if* $A'\colon \Omega \to \tilde{\Omega}'$ *is another conformal satisfying* $R(\epsilon') \subset \tilde{\Omega}'$ *and* $A'f = FA'$, *then* $A' - A$ *is constant.*

Let D, D' be two open subsets of $\mathbf{C}$. Recall that $f\colon \Omega \to \Omega'$ is a quasi-conformal mapping, if f is a homeomorphism, $f(x + iy)$ is locally absolutely continuous on almost each horizontal or vertical line, and for some constant $K \geq 1$

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq K(|f_z(z)| - |f_{\overline{z}}(z)|)$$

holds almost everywhere on D. Note that f is orientation-preserving, and belongs to the Hölder class H^{1/K_f}, where K_f is the least constant K satisfying the above inequality. Also f^{-1} is quasi-conformal, with $K_{f^{-1}} = K_f$.

We also need the following

Lemma 2.2. *Let* $\Pi^+(\epsilon) = \{\xi \mid \operatorname{Im}\xi > 1/\epsilon\}$ *and* $\Pi^-(\epsilon) = -\Pi^+(\epsilon)$. *Let* $B\colon R(\delta_0) \to \Omega$ *be an orientation-preserving homeomorphism, and let*

$$\begin{array}{ccc} R(\delta_0) & \xrightarrow{F} & R(\delta_0) \\ {\scriptstyle B}\big\downarrow & & \big\downarrow{\scriptstyle B} \\ \Omega & \xrightarrow{F} & \Omega \end{array}$$

be a commutative diagram. Assume that $\Omega \supset R(\epsilon_0)$. *Then for each* $\epsilon > 0$ *there is* $\delta > 0$ *such that*

$$B^{-1}(\Pi^+(\epsilon)) \supset \Pi^+(\delta), \quad B^{-1}(\Pi^-(\epsilon)) \supset \Pi^-(\delta), \quad B^{-1}(R(\epsilon)) \supset R(\delta),$$
$$B(\Pi^+(\epsilon)) \supset \Pi^+(\delta), \quad B(\Pi^-(\epsilon)) \supset \Pi^-(\delta), \quad B(R(\epsilon)) \supset R(\delta).$$

Proof. Let $0 < \epsilon < \epsilon_0$. Put $\gamma^{\pm}\colon \operatorname{Im}\xi = \pm 1/\epsilon$ and $\tilde{\gamma}^{\pm} = B^{-1}(\gamma^{\pm})$. Both $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$ are invariant under F and F^{-1}. Let $\omega^{\pm} = B^{-1}(\Pi^{\pm}(\epsilon))$ and $\omega^0 = R(\delta_0) \setminus (\overline{\omega}^+ \cup \overline{\omega}^-)$.

Let $\tilde{B}$ be the unique extension of B satisfying $F\tilde{B} = \tilde{B}F$, defined by $\tilde{B}(\xi) = F^{-n}BF^n$ for some large $n = n(\xi) \in \mathbf{Z}$. Notice that $\tilde{B}$ is a homeomorphism from $\mathbf{C}$ onto itself, and that $\tilde{B}$ still preserves the orientation. It is clear that $\operatorname{Im}\tilde{B}(x \pm iy)$ is bounded on any strip $\{\xi, |\operatorname{Im}\xi| < M\}$ for finite M. Thus ω^0 is the only connected component of $R(\delta_0)\setminus(\tilde{\gamma}^+ \bigcup \tilde{\gamma}^-)$, on which $|\operatorname{Im}\xi|$ is bounded. Now, ω^+ contains an upper half plane, and ω^- contains a lower half-plane. Otherwise, ω^+ contains a lower half-plane, and ω^- contains an upper half-plane. Let C be the boundary of the rectangle defined by $|\operatorname{Re}\xi| \le M$ and $|\operatorname{Im}\xi| \le 1/\epsilon$, oriented counter clockwise. Then the winding number, $n(C, 0)$, of C about the origin is one. Since $\operatorname{Re}\tilde{B}^{-1}(z)$ tends to $+\infty$ (resp. $-\infty$) as $\operatorname{Re}z \to +\infty$ (resp. $-\infty$), and since $\operatorname{Re}\tilde{B}(x+iy)$ tends to $+\infty$ uniformly for $|y| < 1/\epsilon$ as $x \to +\infty$, the winding $n(\tilde{B}^{-1}(C), \tilde{B}^{-1}(0)) = -1$ for large M, which contradicts the orientation-preserving assumption. This shows that $B^{-1}(\Pi^+(\epsilon))$ (resp. $B^{-1}(\Pi^-(\epsilon))$ contains some $\Pi^+(\delta)$ (resp. $\Pi^-(\delta)$).

Since $\operatorname{Re}B(z)$ tends to $+\infty$ as $\operatorname{Re}z \to +\infty$, we can choose $0 < \delta' < \delta$ such that $B(\{\xi\colon |\operatorname{Im}\xi| < 1/\epsilon, \operatorname{Re}\xi > 1/\delta'\})$ is contained in $\{\xi\colon |\operatorname{Re}\xi > 1/\epsilon\}$. Then $B^{-1}(R(\epsilon))$ contains $R(\delta')$.

Applying the above result to $B^{-1}\colon R(\epsilon) \to B^{-1}(R(\epsilon)) \supset R(\delta')$ yields the remaining assertion of the lemma. □

We have

Theorem 2.3. *For each* $f \in \mathcal{A}_p^*$ *and each punctured neighborhood* ω *of the origin, there exists a canonical collection* $A_j, \Omega_j, j = 1, \dots, 2p$ *satisfying Ii-v and* $\cup\Omega_j \subset \omega$.

Proof. Given $f \in \mathcal{A}_p^*$ and an open set ω containing $0 \in \mathbf{C}$, we want to find a canonical collection Ω_j, A_j for f with $\bigcup\Omega_j \subset \omega$.

By a theorem of Shcherbakov [17], for each $\epsilon_0 > 0$ there exist $\delta_0 > 0$ and a quasi-conformal mapping $H(z) = z + h(z)$ such that $H^{-1}fH(z)$ is the time-one map $v^1(z)$ of vector field (2.4), and

$$|h(z)| \le |z|^{3/2}, \quad |z| < \delta_0. \tag{2.6}$$

Let $\Omega_j(\rho) = H(\hat{\Omega}_j(\rho))$. Obviously, I$i$) and I$v$) hold. From (2.6), we can see that H preserves each germ of sector. For we first have $|z|/2 \le |H(z)| \le 2|z|$ for $z \in \Delta(\delta_0)$. Since H is a

homeomorphism, we write $H^{-1}(z) = z + \tilde{h}(z)$. Now (2.6) implies that

$$|\tilde{h}(z)| \le 8|z|^{3/2}, \quad z \in \Delta(\delta_1),$$
$$|\arg H^{-1}(z) - \arg z| \le C_1\sqrt{|z|}, \quad \arg z \in (2\delta_1, 2\pi - 2\delta_1), \quad z \in \Delta(\delta_1),$$

for some $C_1, \delta_1 > 0$. Thus for all small and positive ϵ, H^{-1} sends the sector $\{z\colon |\arg z - \pi/2| < \sqrt{\epsilon}, |z| < \epsilon\}$ into $\{z\colon |\arg z - \pi/2| < 2\sqrt{\epsilon}, |z| < 2\epsilon\}$, which implies that H sends $\{z\colon |\arg z - \pi/2| < 2\sqrt{\epsilon}, |z| < 2\epsilon\}$ into $\{z\colon |\arg z - \pi/2| < \sqrt{\epsilon}, |z| < \epsilon\}$. This shows that H fixes the germ of sector y_+ (and in fact each germ of sector by the same argument). Therefore, $H(y_+)$, and hence Ω_1, contains the germ of sector y_+. By a similar argument, we can show that $\Omega_k \cap \Omega_{k+1}$ contains $\sqrt{\mu}^k y_+$. Thus, I*ii*) holds.

Let $\hat{A}_j^{-1}$ be given by (2.5). The existence of A_j is given by Voronin, by applying Lemma 2.1 to the quasi-conformal mappings $G_j = H\hat{A}_j^{-1}$, sending $(-1)^{j-1}R(\rho)$ into $\Omega_j(\rho)$, and to each of the commutative diagrams

$$\begin{array}{ccc} R(\rho) & \xrightarrow{F} & R(\rho) \\ {\scriptstyle G_{2j+1}}\downarrow & & \downarrow{\scriptstyle G_{2j+1}} \\ \Omega_{2j+1}(\rho) & \xrightarrow{f} & \Omega_{2j+1}(\rho) \end{array} \qquad \begin{array}{ccc} L(\rho) & \xrightarrow{F^{-1}} & L(\rho) \\ {\scriptstyle G_{2j}}\downarrow & & \downarrow{\scriptstyle G_{2j}} \\ \Omega_{2j}(\rho) & \xrightarrow{f^{-1}} & \Omega_{2j}(\rho) \end{array}$$

To verify I*iii*), we apply Lemma 2.2 to $B = B_j = A_j H \hat{A}_j^{-1}$ for $j = 1, \dots, 2p$. Hence for each $\epsilon > 0$, there is $\delta > 0$ such that $B_j^{-1}((-1)^{j-1}R(\epsilon))$ contains $(-1)^{j-1}R(\delta)$ for all j. Now

$$\begin{aligned} \bigcup_j A_j^{-1}((-1)^{j-1}R(\epsilon)) &= \bigcup_j H\hat{A}_j^{-1}B_j^{-1}((-1)^{j-1}R(\epsilon)) \\ &\supset \bigcup H\hat{A}_j^{-1}((-1)^{j-1}R(\delta)). \end{aligned}$$

The latter is $H(\cup\hat{\Omega}_j(\delta))$, a punctured neighborhood of the origin. □

If f, Ω_j, A_j satisfy I*i-v*), we say that Ω_j, A_j are a canonical collection of f. We also need the following

Lemma 2.4. *If $\{\Omega_j, A_j\}$, $\{\Omega_j', A_j'\}$ are two canonical collections of f and ω in a neighborhood of the origin, there exists a canonical collection Ω_j'', A_j'' of f such that $\Omega_j'' \subset \Omega_j \cap \Omega_j' \cap \omega$ and $A_j'' = A_j|_{\Omega_j''} = A_j'|_{\Omega_j''} + c_j$ for some constants $c_j \in \mathbf{C}$.*

Proof. Take a canonical collection Ω_j'', A_j'' such that $\cup\Omega_j''$ is contained in $(\cup_j\Omega_j) \cap (\cup\Omega_j')$. Since Ω_{2k+1}'' is connected and invariant under f, then Ω_{2k+1}'' is contained in some Ω_{2j+1} for $1 \le j \le p$. Note that $j = k \mod p$, since $\Omega_{2j+1} \cap \Omega_{2k+1} = \emptyset$ for $k \ne j \mod p$. Therefore, we have $\Omega_{2k+1}'' \subset \Omega_{2k+1}$. By the same argument, we obtain $\Omega_j'' \subset \Omega_j \cap \Omega_j'$ for all j.

Now the uniqueness assertion in Lemma 2.1 implies that $A_j - A_j'', A_j' - A_j''$ are constants. The lemma is proved by replacing A_j'' with $A_j|_{\Omega_j}$. □

For each $f \in \mathcal{A}_p^*$ and for a canonical collection Ω_j, A_j of f, we denote $\Phi_{f,j} = A_{j+1}A_j^{-1}$. Then for some $\delta > 0$, $\Phi_{f,j}$ is a conformal mapping from the half-plane $\Pi_{p,j}(\delta)$ into $\Pi_{p,j}(-\delta)$,

which commutes with F. Thus Φ_f is an element in $\mathcal{M}_p$. Note that A_j are uniquely determined up to the form $L_{c_j}A_j$ for $c_j \in \mathbf{C}$, where

$$L_c\colon z \to z + c.$$

We now restate Theorem 1.2 in a detailed form

Theorem 2.5. *Let $f, \tilde{f} \in \mathcal{A}_p^*$, and μ be given by (2.2). If $\tilde{f} = g^{-1}fg$ for some holomorphic mapping g, then $g'(0) = \mu^k$ and*

$$\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_{f,j+2k}L_{-c_j}, \quad j = 1, \dots, 2p \tag{2.7}$$

for some $c_j \in \mathbf{C}$. Conversely, if (2.7) holds, $\Phi_{f,j} = A_{j+1}A_j$ and $\Phi_{\tilde{f},j} = \tilde{A}_{j+1}\tilde{A}_j^{-1}$, then $g = A_{j+2k}^{-1}L_{-c_j}\tilde{A}_j$ extends to a holomorphic mapping with $g'(0) = \mu^k$ and $\tilde{f} = g^{-1}fg$. For each $\Phi \in \mathcal{M}_p$, there exists $f \in \mathcal{A}_p^*$ such that $\Phi_{f,j} = \Phi_j$ for $j = 1, \dots, 2p$.

Proof. We first prove that if $f, \tilde{f}$ are equivalent, then their moduli functions are equivalent.

Assume that $f, \tilde{f} \in \mathcal{A}_p^*$ satisfy $\tilde{f} = g^{-1}fg$. Put $g(z) = cz + O(|z|^2)$. Then $g^{-1}fg(z) = z + \mu_0 c^p z^{p+1} + O(|z|^{p+2})$. Hence $c = \mu^k$. Let Ω_j, A_j be a canonical collection of f. We have

$$\begin{array}{ccc} \Omega_{2j+1} & \xrightarrow{f} & \Omega_{2j+1} \\ {\scriptstyle A_{2j+1}}\downarrow & & \downarrow{\scriptstyle A_{2j+1}} \\ \tilde{\Omega}_{2j+1} & \xrightarrow{F} & \tilde{\Omega}_{2j+1} \end{array} \qquad \begin{array}{ccc} \Omega_{2j} & \xrightarrow{f^{-1}} & \Omega_{2j} \\ {\scriptstyle A_{2j}}\downarrow & & \downarrow{\scriptstyle A_{2j}} \\ \tilde{\Omega}_{2j} & \xrightarrow{F^{-1}} & \tilde{\Omega}_{2j} \end{array}$$

Thus, when $\cup\Omega_j$ is contained in the domain of g^{-1}, we get

$$\begin{array}{ccc} g^{-1}(\Omega_{2j+1}) & \xrightarrow{g^{-1}fg} & g^{-1}(\Omega_{2j+1}) \\ {\scriptstyle A_{2j+1}g}\downarrow & & \downarrow{\scriptstyle A_{2j+1}g} \\ \tilde{\Omega}_{2j+1} & \xrightarrow{F} & \tilde{\Omega}_{2j+1} \end{array} \qquad \begin{array}{ccc} g^{-1}(\Omega_{2j}) & \xrightarrow{g^{-1}f^{-1}g} & g^{-1}(\Omega_{2j}) \\ {\scriptstyle A_{2j}g}\downarrow & & \downarrow{\scriptstyle A_{2j}g} \\ \tilde{\Omega}_{2j} & \xrightarrow{F^{-1}} & \tilde{\Omega}_{2j} \end{array}$$

Since Ω_j contains the germ of sector $\sqrt{\mu}^j y_+$. Thus $g^{-1}(\Omega_{j+2k})$ contains the germ of sector $\sqrt{\mu}^j y_+$. Thus, we can take $A_j' = A_{j+2k}g$ and $\Omega_j' = g^{-1}(\Omega_{j+2k})$ as another canonical collection of $\tilde{f}$. By Lemma 2.4, we have $\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_{f,j+2k}L_{-c_j}$.

Next we shall prove that $f, \tilde{f}$ are equivalent, if their moduli functions are equivalent.

Let $\{\Omega_j, A_j\}$ and $\{\tilde{\Omega}_j, \tilde{A}_j\}$ be canonical collections of $f, \tilde{f} \in \mathcal{A}_p^*$, respectively. Assume now that $\tilde{A}_{j+1}\tilde{A}_j^{-1} = L_{c_{j+1}}A_{j+1+2k}A_{j+2k}^{-1}L_{-c_j}$. Hence $A_{j+1+2k}^{-1}L_{-c_{j+1}}\tilde{A}_{j+1} = A_{j+2k}^{-1}L_{-c_j}\tilde{A}_j$ on

$$\begin{aligned} \tilde{A}_j^{-1}(\Pi_{p,j}(\epsilon_0)) &\supset \tilde{A}_j^{-1}(\Pi_{p,j}(\epsilon_0)) \cap \tilde{A}_{j+1}^{-1}(\Pi_{p,j}(\epsilon_0)) \\ &= \tilde{A}_j^{-1}((-1)^{j-1}R(\epsilon_0)) \cap \tilde{A}_{j+1}^{-1}((-1)^j R(\epsilon_0)). \end{aligned}$$

(Of course, when $p = 1$, we have to replace the above with $\tilde{A}_j^{-1}(\Pi^+(\epsilon_0) \cup \Pi^-(\epsilon_0)) \supset \tilde{A}_1^{-1}(R(\epsilon_0)) \cap \tilde{A}_2^{-1}(L(\epsilon_0))$.) Put $\omega_j = \tilde{A}_j^{-1}((-1)^{j-1}R(\epsilon_0))$. We also have $\omega_j \cap \omega_k = \emptyset$ for $k - j \neq 0, \pm 1 \mod 2p$. Thus $g = A_{j+2k}^{-1}L_{-c_j}\tilde{A}_j$ is well defined on $\cup\omega_j$. The latter is a punctured neighborhood of the origin, and g extends to a holomorphic mapping defined

near the origin. By the same argument, we know that $h = \tilde{A}_{j-2k}^{-1} L_{c_{j-2k}} A_j$ is also well-defined near the origin. Since $hg = \mathrm{Id}$, then g is biholomorphic. Now

$$g^{-1} f g = \tilde{A}_1^{-1} L_{c_1} A_{2k+1} f A_{2k+1}^{-1} L_{-c_1} \tilde{A}_1 = \tilde{f}$$

on ω_1, and hence in a neighborhood of the origin. Since $f, \tilde{f} \in \mathcal{A}_p^*$, then $g'(0) = \mu^l$ for some integer l. Since $g(y_+)$ is not contained in Ω_{2l+1} for $l \neq k \mod 2p$, we see that $g'(0) = \mu^k$.

Finally, we shall realize each $\Phi \in \mathcal{M}_p$ as the moduli function of some $f \in \mathcal{A}_p^*$.

Given $\Phi \in \mathcal{M}_p$, we need to find $f \in \mathcal{A}_p^*$ such that $\Phi_{f,j} = \Phi_j$ for all j. Note that $\Phi_j(\xi) - \xi$ is bounded and of period one on the half-plane $\Pi_{p,j}$. Write

$$\Phi_j(\xi) = \xi + \sum_{k \geq 0} c_{j,k} e^{\epsilon_j 2\pi i k \xi}, \quad j = 1, \ldots, 2p,$$

where

$$\epsilon_j = \begin{cases} (-1)^{j-1}, & p = 2, 3 \mod 4, \\ (-1)^j, & p = 0, 1 \mod 4. \end{cases}$$

Put

$$\tilde{\Pi}_{p,j+1}(\rho) = \Phi_j(\Pi_{p,j}(\rho)).$$

Note that $\Phi_j \colon \Pi_{p,j}(\rho) \to \tilde{\Pi}_{p,j+1}(\rho)$ is conformal, for small and positive ρ. Choose $\rho > 0$ so small that $|\Phi_j' - 1| < 1/4$ on $\Pi_{p,j}(\rho)$. Put

$$\tilde{\Omega}_j(\rho) = \Pi_{p,j}(\rho) \cup \tilde{\Pi}_{p,j}(\rho) \cup \{\xi \colon (-1)^{j-1} \operatorname{Re} \xi > 1/\rho\}, \quad j = 1, \ldots, 2p.$$

Denote by $\bigsqcup_{j=1}^{2p} \tilde{\Omega}_j(\rho)$ the disjoint union of $\tilde{\Omega}_j(\rho)$. Let S be obtained from the disjoint union by identifying $\xi_j \in \Pi_{p,j}(\rho)$ with $\xi_{j+1} = \Phi_j(\xi_j) \in \tilde{\Pi}_{p,j+1}(\rho)$. Let $\pi \colon \bigsqcup_{j=1}^{2p} \tilde{\Omega}_j(\rho) \to S$ be the canonical projection. By definition a subset ω of S is open, if and only if $\pi^{-1}(\omega) \cap \tilde{\Omega}_j(\rho)$ are open for all j. For $\rho > 0$ sufficiently small, the Riemann surface S is Hausdorff, with $\xi_j \colon \xi_j^{-1} \tilde{\Omega}_j(\rho) \to \tilde{\Omega}_j(\rho)$ being the holomorphic coordinates, and with $\Phi_j = \xi_{j+1} \xi_j^{-1}$ the transition functions.

Let $\tilde{\Omega}_j'(\rho) = \tilde{\Omega}_j(\rho) \cap F^{-1}(\tilde{\Omega}_j(\rho))$. Since Φ_j commutes with F, then

$$\tilde{\Omega}_{2j+1}'(\rho) = \tilde{\Omega}_{2j+1}(\rho), \quad \tilde{\Omega}_{2j}'(\rho) = \Pi_{p,2j}(\rho) \cup \tilde{\Pi}_{p,2j}(\rho) \cup \{\xi \colon \operatorname{Re} \xi < -1/\rho - 1\}.$$

Let $S' = \bigsqcup_{j=1}^{2p} \tilde{\Omega}_j'(\rho) / \sim$ be obtained by the same identification $\xi_{j+1} = \Phi_j(\xi_j), j = 1, \ldots, 2p$. Define a holomorphic mapping $\tilde{f} \colon S' \subset S \to S$ by $\tilde{f} = \xi_j^{-1} F \xi_j$. Then $\tilde{f}$ is well-defined since Φ_j commutes with F. Note that $\tilde{f}$ is injective also.

We shall now use a holomorphic map to identify S with a punctured neighborhood of the origin in $\mathbf{C}$, and to identify $\tilde{f}$ with a holomorphic mapping defined near the origin.

We shall first identify S with a punctured neighborhood of the origin by a quasi-conformal mapping. Replacing ρ by a smaller number, one may assume that there exists a smooth function $\chi \colon \mathbf{R} \to [0, 1]$ such that $\chi(t) = 1$ for $|t| > \frac{1}{2\rho}$, $\chi(t) = 0$ for $|t| < \frac{1}{4\rho}$ and $|\chi'| < 8\rho$ on $\mathbf{R}$. Define mappings K_j from $\mathbf{C}$ into itself by

$$K_j(\xi) = \begin{cases} \xi, & \xi \in \Pi_{p,j-1}(-4\rho), \\ \xi + \chi(\operatorname{Im} \xi)(\Phi_j^{-1}(\xi) - \xi), & \text{otherwise.} \end{cases}$$

For $\rho > 0$ sufficiently small, we have $|K_j(\xi) - \xi| < C_1$ and $|K_{j,\xi}(\xi) - 1| + |K_{j,\bar{\xi}}(\xi)| < C_1\rho$ for some constant C_1. Thus

$$\left|\frac{\partial_{\bar{\xi}} K_j(\xi)}{\partial_\xi K_j(\xi)}\right| < C_2\rho < 1, \quad \xi \in \mathbf{C}.$$

Therefore, K_j is a quasi-conformal mapping from $\mathbf{C}$ onto itself. Recall that $\hat{A}_j$, given by (2.5), sends $\hat{\Omega}_j(\rho')$ onto $(-1)^{j-1}R(\rho')$ for all $\rho' > 0$. Define $K\colon \bigcup \hat{A}_j^{-1}K_j^{-1}\tilde{\Omega}_j(\rho) \to S$ by

$$K(z) = \xi_j^{-1}K_j\hat{A}(z), \quad z \in \hat{A}_j^{-1}K_j^{-1}\tilde{\Omega}_j(\rho).$$

We need to verify that K is well-defined. For $z \in \hat{A}_j^{-1}K_j^{-1}\tilde{\Omega}_j(\rho) \cap \hat{A}_{j+1}^{-1}K_{j+1}^{-1}\tilde{\Omega}_{j+1}(\rho)$ we have $\hat{A}(z) \in \Pi_{p,j}(2\rho)$. Hence, $K_j\hat{A}(z) = \Phi_j^{-1}\hat{A}(z)$ and $K_{j+1}\hat{A}(z) = \hat{A}(z)$, since $K_j = \Phi_j^{-1}$ and $K_{j+1} = \mathrm{Id}$ on $\Pi_{p,j}(2\rho)$.

Thus $K^{-1}\colon S \to \bigcup \hat{A}_j^{-1}K_j^{-1}\tilde{\Omega}_j(\rho)$ is a quasi-conformal mapping, and the latter contains $\bigcup \hat{\Omega}_j(\rho/2)$. By a lemma of Voronin [19], there exists a quasi-conformal mapping B from $\mathbf{C}$ onto itself, sending $\bigcup \hat{A}_j^{-1}K_j^{-1}\tilde{\Omega}_j(\rho)$ onto ω, a punctured neighborhood of the origin, such that $B(0) = 0$, $D = BK^{-1}\colon S \to \omega$ is conformal, and

$$\left|\frac{\partial_{\bar{\xi}} B(\xi)}{\partial_\xi B(\xi)}\right| < C_2\rho, \quad \xi \in \mathbf{C}. \tag{2.8}$$

Let $S'' = K(\bigcup \hat{\Omega}_j(\rho/2))$. Then S'' is a subset of S', for ρ sufficiently small. Recall that $\tilde{f} = \xi_j^{-1}F\xi_j$. Define $f = D\tilde{f}D^{-1}$ on $\omega'' = D(S'')$, which extends to a biholomorphic mapping defined near the origin. Note that $f = BK^{-1}\xi_j^{-1}F\xi_jKB^{-1} = B\hat{A}_j^{-1}K_j^{-1}FK_j\hat{A}_jB^{-1} = B\hat{A}_j^{-1}F\hat{A}_jB^{-1} = Bv^1B^{-1}$ holds on ω'', since $v^1 = \hat{A}_j^{-1}F\hat{A}_j$.

Since B is quasi-conformal and satisfies estimate (2.8), then

$$\frac{1}{C_4}|z'' - z'|^{\frac{1+c_2\rho}{1-c_2\rho}} \le |B(z'') - B(z')| \le C_4|z'' - z'|^{\frac{1-c_2\rho}{1+c_2\rho}}.$$

From $f = Bv^1B^{-1}$ and $|z|^{p+1}/2 \le |v^1(z) - z| \le 2|z|^{p+1}$, we see

$$\frac{1}{C_5}|z|^{\left(\frac{1+c_2\rho}{1-c_2\rho}\right)^2(p+1)} \le |f(z) - z| \le C_5|z|^{\left(\frac{1-c_2\rho}{1+c_2\rho}\right)^2(p+1)}.$$

Thus $f(z) = z + a_{p+1}z^{p+1} + O(|z|^{p+2})$ with $a_{p+1} \neq 0$, when ρ is sufficiently small.

Let R_b be the mapping $z \to bz$. Choose a constant $b \in \mathbf{C}^*$ such that $f_1 = R_b^{-1}fR_b \in \mathcal{A}_p^*$. Put

$$A_j = \xi_jD^{-1}R_b|_{\Omega_j}\colon \Omega_j \equiv R_b^{-1}B(\hat{\Omega}_j(\rho/2)) \to K_j((-1)^{j-1}R(\rho/2)) \equiv \tilde{\Omega}_j.$$

Now $\Phi_j = \xi_{j+1}\xi_j^{-1} = A_{j+1}A_j^{-1}$. Moreover, $A_{2j+1}f_1A_{2j+1}^{-1} = F$ and $A_{2j}f_1^{-1}A_{2j}^{-1} = F^{-1}$. Since $f_1 = R_b^{-1}Bv^1B^{-1}R_b$, we see that $f_1\Omega_{2j+1} \subset \Omega_{2j+1}$ and $f_1^{-1}\Omega_{2j} \subset \Omega_{2j}$. Moreover, any connected open subset of $\cup\Omega_j$, invariant under f_1 (resp f_1^{-1}), is contained in some Ω_{2k} (resp. Ω_{2l+1}). We have achieved Ii-v, except for Iii.

Note that $\tilde{\omega} = \bigcup \Omega_j$ is a punctured neighborhood of the origin. Let Ω_j', A_j' be a canonical collection of f_1 with $\bigcup \Omega_j' \subset \tilde{\omega}$, for which the existence has already been established. We know that each Ω_j' is contained in some $\Omega_{j'}$. Also there is no Ω_{j_0} that contains two distinct Ω_{j_1}' and Ω_{j_2}'; otherwise, there is a proper subset I of $\{1, \ldots, 2p\}$ such that $\omega_1 = \cup_{j=1}^{2p}\Omega_j'$

is contained in $\omega_2 = \cup_{j\in I}\Omega_j$, which contradicts that ω_1 is a punctured neighborhood of the origin, while $\cup_{j\in I}\hat{\Omega}_j$ (and hence ω_2) is not. Now Ω_1' is a connected subset of $\tilde{\omega}$ and satisfies $f_1(\Omega_1') \subset \Omega_1'$. We know that Ω_{2p}' is contained in Ω_{2j_0} for some unique $j_0 \mod p$. Since $\Omega_1' \cap \Omega_{2p}'$ is contained in $\Omega_{2k+1} \cap \Omega_{2j_0}$, we have only two cases: either $j_0 = k \mod p$ or $j_0 = k+1 \mod p$. If the latter occurs, then Ω_j' is contained in Ω_{2k-j+2} for all j. We take a point z_j in $\Omega_j' \cap \Omega_{j+1}'$, and connect z_{j-1} to z_j by an oriented curve γ_j' in Ω_j'. Let $\gamma' = \cup_{j=1}^{2p}\gamma_j'$. Then $\gamma_j = B^{-1}R_b\gamma_j'$ is an oriented curve in $\hat{\Omega}_{2k+2-j}(\rho/2)$. Thus the winding numbers of γ' and $\gamma = B^{-1}R_b\gamma'$ with respect to the origin have opposite sign, contradicting to the orientation-preserving property of the quasi-conformal mapping B. Therefore, we have $\Omega_j' \subset \Omega_{2k+j}$ for $j = 0, 1$, and hence for all j.

Finally, let $f_2 = R_\mu^k f_1 R_\mu^{-k}$, $A_j^* = A_j R_\mu^{-k}$, and $\Omega_j^* = R_\mu^k\Omega_j$. Then $A_{j+1}^* A_j^{*-1} = \Phi_j$, $f_2^{(-1)^j} = A_{j+1}^{*-1}F^{(-1)^j}A_{j+1}^*$, and $\Omega_j^* \supset R_\mu^k\Omega_{j-2k}' \supset \sqrt{\mu}^j y_+$. One can verify the remaining properties in Ii-v for f_2, A_j^* and Ω_j^*. This shows that f_2 has a canonical collection realizing Φ. □

The above construction of moduli functions of $\mathcal{A}_p$ follows Voronin's approach [19] ($p = 1$) in the sprit. Voronin's construction yields a decomposition of a punctured neighborhood $\cup\Omega_j$ of the origin. The domains Ω_j admit the maximality property that if ω is a connected open subset of $\cup_j\Omega_j$ and if $f(\omega) \subset \omega$ (resp. $f^{-1}\omega \subset \omega$), then ω is a subset of some Ω_{2j+1} (resp. Ω_{2j}). Malgrange [12] and Il'yashenko [9] gave another construction of moduli functions of $\mathcal{A}_p$, based on Fatou's sectorial normalization, where Ω_j are replaced with suitable sectors S_j with vertex at the origin. It is likely, the proof of Theorem 2.5 (in the most general form of allowing changes of coordinates that are not necessary to be tangent to the identity) can be obtained through that line of approach, except the maximality property on the domains Ω_j. Combining the above two approaches, one has both the maximality property on the domains Ω_j and asymptotic expansion about A_j in some sectoral domains contained in Ω_j. There is one advantage about the construction through sectorial normalization (see [9]). Namely, it links the coefficients f_{p+1} and f_{2p+1} in $f(z) = z + f_{p+1}z^{p+1} + f_{2p+1}z^{2p+1} + O(|z|^{2p+2})$ to c_{j0} in $\Phi_j(\xi) = \xi + \sum_{k\geq 0} c_{j,k}e^{(-1)^{j-1}\epsilon_p 2\pi i k\xi}$. Specifically, $c_{1,0} + \cdots + c_{2p,0} = 2\pi i\epsilon(f_{2p+1}/f_{p+1}^2 - (p+1)/2)$ ($\epsilon = 1$ for $p = 0, 1 \mod 4$ and $\epsilon = -1$ for $p = 1, 2 \mod 4$), which is needed for the proof of Theorem 1.5.

3. centralizers and composition roots

Given $f \in \mathcal{A}_p$, denote by $\hat{\mathcal{C}}(f)$ be germs of biholomorphic mappings g commuting with f, and by $\mathcal{C}(f) \subset \hat{\mathcal{C}}(f)$ the ones that are tangent to the identity. Note that $\hat{\mathcal{C}}(f)$ is a group under the composition.

The main result in this section is to give a complete description for the group of centralizers $\hat{\mathcal{C}}(f) = <\tau, f^{1/d}> = <\tau> \otimes <\omega>$, while $\tau, f^{1/d}$ and ω are linked by $\tau^{q/\delta} = \omega f^{1/d}$.

The group $\mathcal{C}(f)$ is studied in [2], [10], and [19] for $p = 1$, and in [5], [9] for arbitrary p. We shall now describe some facts in [9]. Given $\Phi \in \mathcal{M}_p$, denote by d_Φ the greatest common divisor of all $k \neq 0$ with $|c_{1,k}| + \cdots + |c_{2p,k}| \neq 0$, where $c_{j,k}$ are the coefficients of Φ_j. Put $d_\Phi = 0$, if all Φ_j are polynomials of degree one. Then $\mathcal{C}(f)$ consists of all holomorphic

mappings

$$f^t = g v_\lambda^t g^{-1}, \quad t \in \mathbf{C},$$

where g is a holomorphic mapping satisfying $g^{-1} f g = v_\lambda^1$, and v_λ^t is the flow of the vector field

$$v_\lambda = \frac{\mu_0 z^{p+1}}{1+\lambda z^p} \frac{\partial}{\partial z}. \tag{3.1}$$

A simple computation shows that the flow v^t of $v(z) = az^{p+1} + bz^{2p+1} + O(|z|^{2p+2})$ has the expansion

$$v^t(z) = z + atz^{p+1} + (bt + \frac{pat^2}{2}) z^{2p+1} + O(|z|^{2p+2}). \tag{3.2}$$

When $f \in \mathcal{A}_p$ is not holomorphic equivalent to any v_λ^1, one knows that there is a unique positive integer $d = d_f$ and a unique holomorphic mapping g, denoted by $f^{1/d}$, such that $g'(0) = 1$ and $g^d = f$. Moreover, $\mathcal{C}(f)$ consists of all mappings of the form $f^{l/d}$ with $l \in \mathbf{Z}$. Note that when $f \in \mathcal{A}_p^*$, we have $d_f = d_{\Phi_f}$ and $f^{l/d} = A_j^{-1} L_{l/d} A_j$, where $A_1, \ldots, A_{2p}$, together with some domains $\Omega_1, \ldots, \Omega_{2p}$, form a canonical collection of f.

We shall now describe $\hat{\mathcal{C}}(f)$ when f is not embeddable in a holomorphic flow. Let $\mathcal{R}_p$ be the group of p-th roots of unity. We first need the following

Lemma 3.1. *For $f \in \mathcal{A}_p$ with $d_f \neq 0$, $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$ is isomorphic to a subgroup of $\mathcal{R}_p$, and $\hat{\mathcal{C}}(f)$ is an abelian group given by $\{\tau^k h \colon h \in \mathcal{C}(f), k = 1, \ldots, q\}$ for some $\tau \in \hat{\mathcal{C}}(f)$, where q is the order of $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$. Moreover, one has*

i) For each $g \in \hat{\mathcal{C}}(f)$, we have $\hat{\mathcal{C}}(f) \subseteq \hat{\mathcal{C}}(g)$ and the equality holds if and only if g is non-periodic.

ii) If $f^{-1} = \sigma f \sigma$ and $d_f \neq 0$, then each $h \in \hat{\mathcal{C}}(f)$ satisfies $h^{-1} = \sigma h \sigma$.

iii) If $f = \sigma f \sigma$, then $h\sigma h\sigma \in \mathcal{C}(f)$ for $h \in \hat{\mathcal{C}}(f)$. If $d_f \neq 0$, then $h = \sigma h \sigma$ for $h \in \mathcal{C}(f)$.

Proof. We know that if g commutes with f, then $g'(0)$ is a p-th root of unity. Now one sees that $g \mapsto g'(0)$ is a group homomorphism from $\hat{\mathcal{C}}(f)$ into $\mathcal{R}_p$, and that $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$ is isomorphic to a subgroup of $\mathcal{R}_p$. In particular $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$ is cyclic. This implies that $\hat{\mathcal{C}}(f) = \{\tau^k h \colon h \in \mathcal{C}(f), k \in \mathbf{Z}, 0 \leq k < p\}$. We first consider the case that $d_f \neq 0$. Since $(\tau f^{1/d} \tau^{-1})^d = f$, the uniqueness of roots of f in $\mathcal{C}(f)$ implies that $\tau f^{1/d} \tau^{-1} = f^{1/d}$. Hence τ commutes with each centralizer in $\mathcal{C}(f)$; consequently, $\hat{\mathcal{C}}(f)$ is abelian.

When $d_f = 0$, we may assume that $f = v_\lambda^1$ is the time-one map of (3.1). Since $(R_\mu)_* v_\lambda = v_\lambda$, then R_μ commutes with v_λ^t. Since $\left|\hat{\mathcal{C}}(f)/\mathcal{C}(f)\right|$ divides p, one sees that $\hat{\mathcal{C}}(f)$ is the union of $R_\mu^k \mathcal{C}(f), k = 1, \ldots, p$. In particular, $\hat{\mathcal{C}}(f)$ is abelian.

To prove (i), we take $g \in \hat{\mathcal{C}}(f)$. Since $\hat{\mathcal{C}}(f)$ is abelian, then $\hat{\mathcal{C}}(f)$ is a subset of $\hat{\mathcal{C}}(g)$. Now take $h \in \hat{\mathcal{C}}(g)$. Since $g = h^{-1} g h$, then $g^q = h^{-1} g^q h$. We know that $g^q = f^{l/d}$. Assume that $l \neq 0$ first. Then h and f are centralizers of $f^{l/d} \in \mathcal{A}_p$. Thus h, f commute, i.e., $h \in \hat{\mathcal{C}}(f)$. Assume now that g is periodic. By a change of coordinates, one may assume that $g(z) = \mu^k z$. Now it is straightforward that $\hat{\mathcal{C}}(f)$ is a proper subset of $\hat{\mathcal{C}}(g)$, since the latter is obviously non-abelian.

For (ii), we assume that $f^{-1} = \sigma f \sigma$ for some anti-holomorphic involution and we want to show that all elements in $\hat{\mathcal{C}}(f)$ are reversible by σ. (Nakai [13] showed that, when $d_f \neq 0$, each element of $\mathcal{C}(f)$ is reversible by σ.) In general, we take $h \in \hat{\mathcal{C}}(f)$. Since $f^{-1} = \sigma f \sigma$ and $h^{-1} f h = f = \sigma h^{-1} f^{-1} h \sigma$, then $\sigma h \sigma h^{-1} \in \hat{\mathcal{C}}(f)$, i.e. $h\sigma h\sigma \in \mathcal{C}(f)$. Write $h\sigma h\sigma = f^{l/d}$. Using $h^q = f^{s/d}$, we get $f^{ql/d} = \mathrm{Id}$, i.e., $l = 0$.

For (iii), we assume that $f = hfh^{-1}$ and $\sigma f = f\sigma$. Then $h^{-1}\sigma h\sigma$ is a centralizer of f, and hence $h\sigma h\sigma \in \mathcal{C}(f)$. Assume now that $d_f \neq 0$ and $h = f^{k/d}$. We have $f^k = (\sigma f^{k/d}\sigma)^d$. Since $\hat{\mathcal{C}}(f)$ is abelian, then $f^{k/d} = \sigma f^{k/d}\sigma$. □

We have

Lemma 3.2. *Let $f \in \mathcal{A}_p$ with $d = d_f \neq 0$, and let ν be a primitive q-th root of unity for $q = |\hat{\mathcal{C}}(f)/\mathcal{C}(f)|$. Then there is a unique $\tau_\nu \in \hat{\mathcal{C}}(f)$ with*

$$\tau_\nu^q = f^{\frac{q'}{d}}, \quad \tau_\nu'(0) = \nu, \quad 0 \leq q' < q.$$

Moreover, $\hat{\mathcal{C}}(f)$ consists of $\tau_\nu^k f^{l/d}$ with $l \in \mathbf{Z}$ and $0 \leq k < q$, and q_f, q_f' are holomorphic invariants.

Proof. Since $\{\nu^j : j = 0, \ldots, q-1\}$ is the unique subgroup of $\mathcal{R}_p$ of order q, there is $\tau_0 \in \hat{\mathcal{C}}(f)$ such that $\tau_0'(0) = \nu$. We have $\tau_0^q = f^{l/d} \in \mathcal{C}(f)$. Write $l = uq + q'$ with $0 \leq q' < q$. Set $\tau_\nu = \tau_0 f^{-u/d}$. Then we have $\tau_\nu'(0) = \nu$ and $\tau_\nu^q = f^{q'/d}$. Assume now that τ_1 also satisfies $\tau_1^q = f^{q''/d}$ for some $0 \leq q'' < q$ and $\tau_1'(0) = \nu$. Since τ_ν and τ_1 commute and have the same linear part, then $(\tau_1^{-1}\tau_\nu)^q = f^{(q'-q'')/d}$. On the other hand $\tau_1^{-1}\tau_\nu = f^{s/d}$ for some $s \in \mathbf{Z}$. Hence $sq = q' - q''$, which implies that $q' = q''$ and $s = 0$, i.e., $\tau_1 = \tau_\nu$.

Finally, assume $f, \tilde{f} \in \mathcal{A}_p$ and $\tilde{f} = g^{-1} f g$ for some holomorphic g. Then

$$\mathcal{C}(\tilde{f}) = \{g^{-1} h g : h \in \mathcal{C}(f)\}, \quad \hat{\mathcal{C}}(\tilde{f}) = \{g^{-1} h g : h \in \hat{\mathcal{C}}(f)\}.$$

In particular, $q(f) = q(\tilde{f})$. Assume now that $\tau_\nu'(0) = \nu$ and $\tau_\nu^q = f^{q'/d}$. Put $\tilde{\tau}_\nu = g^{-1}\tau_\nu g$. Then we still have $\tilde{\tau}_\nu'(0) = \nu$ and $\tilde{\tau}_\nu^q = g^{-1} f^{q'/d} g = \tilde{f}^{q'/d}$. By the uniqueness assertion proved above, we know that $q_{\tilde{f}}' = q_f'$. □

Conversely, we have the following realization

Proposition 3.3. *Let p, q, d be positive integers such that q divides p. Let q' be an integer satisfying $0 \leq q' < q$, and let ν be a primitive q-th root of unity. There exist $f \in \mathcal{A}_p^*$ and $\tau_\nu \in \hat{\mathcal{C}}(f)$ such that $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$ is isomorphic to the subgroup of $\mathcal{R}_p$ of order q and $\tau_\nu^q = f^{q'/d}$ with $\tau_\nu'(0) = \nu$.*

Proof. Put $q^* = p/q$. Then $\nu = \mu^{q^* s}$ with $(q, s) = 1$, where (q, s) denotes the greatest common divisor of q, s. Define

$$\Phi_j(\xi) = \xi, \quad j \neq 1 \mod 2q^*,$$
$$\Phi_1(\xi) = \xi + e^{\epsilon_1 2\pi i d \xi}, \quad \Phi_{1+2sq^*l} = L_{\frac{lq'}{qd}} \Phi_1 L_{-\frac{lq'}{qd}}, \quad 1 \leq l \leq q-1.$$

The above is well-defined, since $sq^*l \mod 2p$ are distinct for $0 \leq l < q$. Let $f \in \mathcal{A}_p^*$ be a realization of the above Φ. Since

$$\Phi_j = L_{-\frac{q'}{qd}} \Phi_{j+2sq^*} L_{\frac{q'}{qd}},$$

Theorem 2.5 implies that $\tau_\nu = A_{j+2sq^*}^{-1} L_{\frac{q'}{qd}} A_j$ satisfies $\tau_\nu f \tau_\nu^{-1} = f$, $\tau_\nu'(0) = \mu^{sq^*} = \nu$. We also have $\tau_\nu^q = f^{q'/d}$, and $\tau_\nu^j \notin \mathcal{C}(f)$ for $0 < j < q$. On the other hand, if h is in $\hat{\mathcal{C}}(f)$ with $h'(0) = \mu^l$, Theorem 2.5 says that $\Phi_j = L_{c'_{j+1}} \Phi_{j+2l} L_{-c'_j}$ for some $c'_j \in \mathbf{C}$. It it clear that q^* divides l. This shows that $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$ is realized as the subgroup of $\mathcal{R}_p$ of order q. □

Theorem 3.4. *Let $f \in \mathcal{A}_p$ with $d = d_f \neq 0$ and $q = \left|\hat{\mathcal{C}}(f)/\mathcal{C}(f)\right|$. Let $\tau_\nu^q = f^{q'/d}$ with $\tau_\nu'(0) = \nu$ a primitive q-th root of unity and $0 \leq q' < q$. We have*

i) There exists τ such that $\tau^{q/\delta} = \omega f^{1/d}$ for $\delta = (q', q)$, and $\hat{\mathcal{C}}(f) = <\tau, f^{1/d}> = <\omega> \otimes <\tau>$, for some ω of order δ (here $<x, y>$ denotes the subgroup generated by x and y). In particular, $\delta_f = (q, q')$ depends only on f. Moreover $\tilde{\tau}^q = f^{\delta/d}$ holds if and only if $\tilde{\tau} = \omega^j \tau$ for some $1 \leq j \leq \delta$.

ii) There exists a holomorphic mapping g, tangent to the identity, such that $f^*(z) = (g^{-1} f g(\sqrt[\delta]{z}))^{\delta_f} \in \mathcal{A}_{p/\delta_f}$, $q_{f^*} = q_f/\delta$, $\delta_{f^*} = 1$, and $d_f = d_{f^*}$. Moreover, the holomorphic equivalence class of f^* depends only on that of f.

iii) Conversely, if k is a positive integer and $h \in \mathcal{A}_p$ with $d_h \neq 0$ and $\delta_h = 1$, the map $f(z) = \sqrt[k]{h(z^k)} = z + O(|z|^2)$ satisfies $\delta_f = k$. Moreover, the holomorphic equivalence class of f depends on that of h.

Proof. Lemma 3.1 says that $\hat{\mathcal{C}}(f)$ is generated by τ_ν and $f^{1/d}$.

To prove i) we first look for an element of finite order in $\hat{\mathcal{C}}(f)$. Since $\delta = (q, q')$ then $q = \delta n$ and $q' = m\delta$ with $(m, n) = 1$. Hence we have $\lambda_0 n + \mu_0 m = 1$ for some λ_0 and μ_0. Notice that we have $(\mu_0, n) = 1$. Next we want to find λ, μ such that $\lambda n + \mu m = 1$ and $(q, \mu) = 1$. Take $\mu = \mu_0 + jn$ and $\lambda = \lambda_0 - jm$, where j is the largest divisor of q that is prime to μ_0. Then $(\mu, q) = 1$. For if k is prime and $k|(\mu, q)$. We have two cases: $k|\mu_0$ or $(k, \mu_0) = 1$. The former implies that $(k, j) = 1$, and hence $k|n$. Since $(\mu_0, n) = 1$, then $k = \pm 1$. For the latter we have $k|j$, and hence $k|\mu_0$ also. This shows that $k = \pm 1$. Now define $\omega = \tau_\nu^n f^{-m/d}$ and note that $\omega^\delta = \tau_\nu^{n\delta} f^{-m\delta/d} = f^{(q'-m\delta)/d} = 1$ and that $\omega'(0)$ is a primitive δ-th root of unity.

Since $\hat{\mathcal{C}}(f)$ is a finitely generated abelian group it is a direct product of cyclic groups. We want to find such a decomposition. To that end we define $\tau = \tau_\nu^\mu f^{\lambda/d}$ (with $(\mu, q) = 1$). Then $\hat{\mathcal{C}}(f)$ is still generated by τ and $f^{1/d}$. Note that $\tau^n \omega^{-\mu} = f^{1/d}$ and that $\tau^m \omega^\lambda = \tau_\nu$. This shows that $\hat{\mathcal{C}}(f)$ is generated by τ and ω. Now τ must have infinite order or else $\hat{\mathcal{C}}(f)$ would have finite order. From this it follows that $<\omega> \cap <\tau> = \{1\}$. So we see that $\hat{\mathcal{C}}(f) = <\omega> \otimes <\tau>$. We have achieved $\tau^n = \omega^\mu f^{1/d}$. Using $(\mu, q) = 1$ again, we replace ω with ω^μ and keep τ. We get all desired properties.

Note that $\tau^q = f^{(q'\mu + \lambda q)/d} = f^{\delta/d}$. Let $\tilde{\tau}$ satisfy $\tilde{\tau}^q = f^{\delta/d}$ also. Since $(\tau^{-1}\tilde{\tau})^q = \mathrm{Id}$ then $\tilde{\tau} = \omega^j \tau$.

To prove ii) we choose a holomorphic mapping g tangent to the identity such that $g^{-1}\omega g(z) = c_1 z$, with c_1 a primitive δ-th root of unity. Still denote $g^{-1}\tau g, g^{-1}\omega g$ and $g^{-1} f g$ by τ, ω, f, respectively. Now $f^*(z) = [f(\sqrt[\delta]{z})]^\delta = z + O(|z|^2) \in \mathcal{A}_{p/\delta}$.

Assume that h is a centralizer of f^*. Let P_δ^{-1} be the multi-valued inverse of $P_\delta \colon z \to z^\delta$. Let $\tilde{h}$ be a single-valued branch of $P_\delta^{-1} h P_\delta$ with $\tilde{h}'(0)$ a δ-th root of $h'(0)$. Then

$$\tilde{h} f = P_\delta^{-1} h P_\delta f = P_\delta^{-1} h f^* P_\delta = P_\delta^{-1} f^* h P_\delta = f \tilde{h}.$$

(In the above, all quantities, except two ends, are equal as multiple-valued functions, and the two ends are equal as single-valued branches.) Consequently, $\tilde{h}$ is a centralizer of f. In particular, all roots $[h'(0)]^{\frac{1}{\delta}}$ have the form $\mu^{lp/q}$, which implies that $h'(0)^{q/\delta} = 1$ and that $\hat{\mathcal{C}}(f^*)/\mathcal{C}(f^*)$ has order at most q/δ. On the other hand τ and ω commute and $P_\delta \tau P_\delta^{-1}$ has a single-valued branch τ^*. Now $f^*\tau^* = P_\delta f \tau P_\delta^{-1} = P_\delta \tau f P_\delta^{-1} = \tau^* f^*$. Hence τ^* is a centralizer of f^*. Also, $\tau^{*\prime}(0)$ is a primitive (q/δ)-th root of unity. So $\hat{\mathcal{C}}(f^*)/\mathcal{C}(f^*)$ is of order q/δ. To verify $\delta(f^*) = 1$ we take a periodic centralizer h of f^* with period k. As shown above single-valued branches of $\tilde{h} = P_\delta^{-1} h P_\delta$ are centralizers of f. Also $\tilde{h}^k$ is a single-valued branch of $P_\delta^{-1} h^k P_\delta = P_\delta^{-1} P_\delta$. Hence all $\tilde{h}$ are periodic centralizer of f. Hence $\tilde{h} = \omega^s$ for some s, which implies that $h = \mathrm{Id}$.

To show $d_f = d_{f^*}$, take $f^t \in \mathcal{C}(f)$. Since $\mathcal{C}(f)$ is abelian, then $\omega f^t = f^t \omega$. Thus $f_t^*(z) \equiv [f^t(\sqrt[\delta]{z})]^\delta = z + O(|z|^2)$ is holomorphic for all $f^t \in \mathcal{C}(f)$. We have

$$f_s^* f_t^* = P_\delta f^s f^t P_\delta^{-1} = f_t^* f_s^*.$$

Thus $(f^*_{1/d_f})^{d_f} = f_1^* = f^*$, which implies that $d_{f^*} = d_f$.

To show that the holomorphic equivalence class of f^* depends only on that of f, we assume that $\tilde{g}^{-1} f \tilde{g}$ commutes with ω also and let $\tilde{f}^* = P_\delta \tilde{g}^{-1} f \tilde{g} P_\delta^{-1}$. Put $f_1 = g^{-1} f g$. We have

$$\tilde{g}^{-1} g f_1 g^{-1} \tilde{g} = \tilde{g}^{-1} f \tilde{g} = \omega \tilde{g}^{-1} f \tilde{g} \omega^{-1} = \omega \tilde{g}^{-1} g f_1 g^{-1} \tilde{g} \omega^{-1}.$$

Hence $g^{-1}\tilde{g}\omega\tilde{g}^{-1}g$ is a periodic centralizer of f_1; consequently it is ω. Thus $g^{-1}\tilde{g}$ and ω commute. Let g_1 be a single-valued branch of $P_\delta g^{-1} \tilde{g} P_\delta^{-1}$. We now have

$$g_1 \tilde{f}^* g_1^{-1} = P_\delta g^{-1} \tilde{g} P_\delta^{-1} \tilde{f}^* P_\delta \tilde{g}^{-1} g P_\delta^{-1} = P_\delta g^{-1} f g P_\delta^{-1} = f^*.$$

For the proof of iii), let $g(z) = e^{2\pi i/k} z$. Then g is a centralizer of f and δ_f is a multiple of k. Let $\tilde{g} \in \hat{\mathcal{C}}(f)$ be of period δ_f. Since $g, \tilde{g}$ commute then $\tilde{g}^*(z) = (\tilde{g}(\sqrt[k]{z}))^k$ is holomorphic. Now $\tilde{g}^* h = h \tilde{g}^*$ and $\tilde{g}^*$ is periodic. Thus $\tilde{g}^* = \mathrm{Id}$. In particular, $\tilde{g}'(0)^k = 1$. This shows that $\delta_f = k$.

If $\tilde{h} = ghg^{-1}$ and $\tilde{f}(z) = \sqrt[k]{\tilde{h}(z^\delta)} = z + O(|z|^2)$, we put $g_1(z) = \sqrt[k]{g(z^k)}$. Then

$$g_1 f g_1^{-1} = P_k^{-1} g P_k f P_k^{-1} g P_k = P_k^{-1} g h g^{-1} P_k = \tilde{f}.$$

Therefore, $f, \tilde{f}$ are holomorphically equivalent. □

Theorem 3.4 provides a decomposition via τ and ω, which are in general not uniquely determined by the relations

$$\hat{\mathcal{C}}(f) = <\tau, f^{1/d}> = <\tau> \otimes <\omega>, \quad \omega = \tau^{q/\delta} f^{-1/d}. \tag{3.3}$$

In fact, if τ_1 satisfies $\tau_1^q = f^{\delta/d}$, and if $\tau_1'(0)$ is a primitive q-th root of unity, then τ_1, in place of τ (and $\omega_1 = \tau^{q/\delta} f^{-1/d}$ in place of ω), satisfies (3.3) also, and vice versa. Therefore, when $\delta_f = 1$, the τ in Theorem 3.4 is uniquely determined by the condition that $\tau^q = f^{1/d}$. In general we decide τ to be the map satisfying (3.3), for which $\tau'(0) = \mu^{pq_0/q}$ has the *smallest* possible integer $q_0 \geq 0$ so that $(q_0, q) = 1$ and

$$\tau^q = f^{\delta/d}. \tag{3.4}$$

We now have

Proposition 3.5. *Let μ be defined by (2.2). Let $f \in \mathcal{A}_p$ with $d_f = d$ and $q_f = q$, and let $q_{0f} = q_0$ be the smallest positive integer such that $\tau = \tau_f$ with $\tau'(0) = \mu^{pq_0/q}$, satisfy (3.4). Put $\omega_f = \tau^{q/\delta} f^{-1/d}$. Then (3.3) holds. Moreover, if $\tilde{f} = gfg^{-1}$, then $\tau_{\tilde{f}} = g\tau_f g^{-1}, \omega_{\tilde{f}} = g\omega_f g^{-1}$, $q_{\tilde{f}} = q_f$, and $q_{0\tilde{f}} = q_{0f}$.*

Proof. We already see the existence of τ, ω, q, q_0 for the given f. Assume that $\tilde{f} = gfg^{-1}$. Then $\tilde{f}, \tilde{\omega} = g\omega g^{-1}, \tilde{\tau} = g\tau g^{-1}$ satisfies (3.3). We also have $\tilde{\tau}'(0) = \mu^{pq_0/q}$. Hence $q_0(\tilde{f}) \le q_0(f)$; analogously, $q_0(f) \le q_0(\tilde{f})$. We get $q_0(f) = q_0(\tilde{f})$. □

Let d, p, q, δ be positive integers with $\delta | q$ and $q | p$. Let q_0 be an integer with $0 \le q_0 < q$ and $(q_0, q) = 1$ such that $q_0 + jq/\delta$ is not prime to q when $0 \le q_0 + jq/\delta < q_0$.

Let $\mathcal{A}_{p,q,q_0,\delta,d}$ be the set of functions $f \in \mathcal{A}_p$ with $(d_f, q_f, q_{0f}, \delta_f) = (d, q, q_0, \delta)$. We shall now describe a moduli space for $\mathcal{A}_{p,q,\delta,q_0,d}$, and we shall see that the latter is non-empty.

Let $\mathcal{M}_{p,q,\delta,q_0,d}$ be the set of functions $\Phi \in \mathcal{M}_p$ with $(d_\Phi, \delta_\Phi, q_\Phi, q_{0\Phi}) = (d, \delta, q, q_0)$, where q_Φ is the largest integer q satisfying

$$\Phi_j = L_{c_{j+1}} \Phi_{j+2p/q} L_{-c_j}, \quad c_{j+2p} = c_j, \tag{3.5}$$

δ_Φ is the largest δ satisfying

$$\Phi_j = L_{c_{j+1}} \Phi_{j+2p/\delta} L_{-c_j}, \quad \sum_{k=1}^{\delta} c_{j+2kp/\delta} = 0, \quad c_{j+2p} = c_j, \tag{3.6}$$

and $q_{0\Phi} = q_0$ is the smallest positive integer that is prime to q and satisfies

$$\Phi_j = L_{-c'_{j+1}} \Phi_{j+2q_0p/q} L_{c'_j}, \quad \sum_{k=1}^{q} c'_{j+2kp/q} = \delta/d, \quad c'_{j+2p} = c'_j, \tag{3.7}$$

where $(d, q, \delta) = (d_\Phi, q_\phi, \delta_\Phi)$. We say that $\Phi, \tilde{\Phi} \in \mathcal{M}_{p,q,\delta,q_0,d}$ are equivalent, if $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_{j+2k} L_{-c_j}$ for some $k \in \mathbf{Z}$ and $c_j = c_{j+2p} \in \mathbf{C}$.

Theorem 3.6. *Let p, q, and δ be positive integers with $\delta | q$ and $q | p$, and let $q_0 \ge 0$ be an integer prime to q such that $q_0 + jq/\delta \in [0, q)$ for $j \in \mathbf{Z}$ is not prime to q. There is a one-to-one correspondence between holomorphic equivalence classes of $\mathcal{A}_{p,q,\delta,q_0,d}$ and ones of $\mathcal{M}_{p,q,\delta,q_0,d}$ for each positive integer d. Moreover, the latter is non-empty.*

Proof. Assume that $\Phi_j = \Phi_{f,j} = A_{j+1} A_j^{-1}$. If q satisfies (3.5), then $g = A^{-1}_{j+2p/q} L_{-c_j} A_j$, with $g \in \hat{\mathcal{C}}(f)$ and $g'(0) = \mu^{p/q}$, is well-defined. Thus $q_\Phi = q_f$. Note that if δ satisfies (3.6), then $\omega = A^{-1}_{j+2p/\delta} L_{-c_j} A_j$ is well-defined, $\omega^\delta = \mathrm{Id}$, and ω commutes with f. Thus $\delta_\Phi = \delta_f$. Also if q_0 satisfies (3.7), then $\tau = A^{-1}_{j+2pq_0/q} L_{c'_j} A_j$ is well-defined, $\tau^q = f^{\delta/d}$, and $\tau'(0) = \mu^{pq_0/q}$ is a primitive q-th root of unity. Thus $q_{0\Phi} = q_{0f}$.

To show that $\mathcal{M}_{p,q,\delta,q_0,d}$ is non-empty, we put

$$\Phi_{1+2jq_0p/q} = L_{\frac{j\delta}{qd}} \Phi_1 L_{-\frac{j\delta}{qd}}, \quad j = 1, \dots, q,$$

$$\Phi_1(\xi) = c_0 + \xi + e^{\epsilon_p 2\pi\xi d i}, \quad \Phi_j = c_0 + \xi, \quad j \ne 1 \mod 2p/q.$$

Then $\Phi_j = L_{-\delta/(qd)} \Phi_{j+2pq_0/q} L_{\delta/(qd)}$ for all j. To show that $q_\Phi = q$, we first assume that

$$\Phi_j = L_{c_{j+1}} \Phi_{j+2p/s} L_{-c_j}, \quad c_{j+2p} = c_j \tag{3.8}$$

for some positive integer s. Then $p/s = jpq_0/q \mod p$, which implies that $s \le q$. Next, choose integers a, b so that $aq_0 + bq = 1$. Then $\Phi_j = L_{-a\delta/(qd)}\Phi_{j+2p/q}L_{a\delta/(qd)}$. Thus $q_\Phi = q$. To show that $\delta_\Phi = \delta$, we first note that $\Phi_j = L_{-a/d}\Phi_{j+2p/\delta}L_{a/d} = \Phi_{j+2p/\delta}$. Assume now that (3.8) holds with $\sum_{k=1}^{s} c_{j+2kp/s} = 0$. Obviously $c_1 = \cdots = c_{2p} = 0$. We know that δ divides s and s divides q. Thus $\Phi_1 = L_{-a\delta/(sd)}\Phi_{1+2p/s}L_{a\delta/(sd)}$. Now (3.8) implies that s divides $a\delta$. Since $(a, d) = 1$, then s divides δ. Thus $\delta_\Phi = \delta$. By the definition we know that there is τ satisfying $\tau^q = f^{\delta/d}$, for which $\tau'(0) = \mu^{pq_0/q}$. Since no $q_0 + jq/\delta$ is both in $[0, q_0)$ and prime to q, this shows that $q_{0\Phi} = q_0$. □

We now turn to the (composition) roots of f.

Theorem 3.7. *Let $f \in \mathcal{A}_p$ with $d = d_f \neq 0$. Let q, δ, and ω be as in Proposition* 3.5. *Let r be the largest factor of dq/δ with $(r, \delta)|d$. There exists $g \in \mathcal{A}_p$ such that $g^r = f$. Moreover, $h^t = f$ for some integer t, if and only if t divides r and $h = \omega^j g^{r/t}$ for some j with δ dividing jt.*

Proof. Now suppose that $h^t = f$. Write $h = \tau^a\omega^b$, where τ, ω are as in Theorem 3.4 i). (The ω before it is replaced by a suitable ω^k with $(k, \delta) = 1$.) Recall from the proof of Theorem 3.4 i) we have $f^{1/d} = \tau^n\omega^{-\mu}$ for $q = n\delta, q' = m\delta$ and $\lambda n + \mu m = 1$. Thus we get that $ta = dn$ and $tb = -d\mu + c\delta$, i.e., $t|dn$ and $(t, \delta)|d\mu$. Since $d = \lambda dn + md\mu$, we know that $(t, \delta)|d$. Conversely, if $ta = dn$ and $d = -tb' + c'\delta$, we put $b = \mu b'$ and $c = \mu c'$. Then $(\tau^a\omega^b)^t = f\omega^{\mu d - d\mu + c\delta} = f$.

Now consider all t satisfying $t|dn$ and $(t, \delta)|d$ (by definition r is the largest such t). we first need to show that $t|r$. To this end suppose that $t_j|dn$ and $(t_j, \delta)|d\mu$ for $j = 1, 2$. Let t_3 be the least common multiple of t_1 and t_2. A simple computation via the prime factorization shows that $(t_3, \delta)|d$ and $t_3|dn$. Therefore, r is in fact the lest common multiple of all such t.

Take g such that $g^r = f$. Return to equation $h^t = f$. We know from above that $r = tt_1$ and that $(g^{t_1})^t = h^t = f$. Since g^{t_1} and h commute we get $(h^{-1}g^{t_1})^t = \mathrm{Id}$, i.e. $h^{-1}g^{t_1} = \omega^j$. Therefore $\delta|jt$. □

We now describe the moduli space for holomorphic equivalence classes of holomorphic mappings $f(z)$ with $f'(0)$ a root of unity. Of course, the only interesting maps are the non-periodic ones. For coprime integers r and s with $0 \le r < s$, let $\mathcal{A}_{p,r,s}$ be the set of holomorphic mappings

$$f(z) = e^{2\pi i\frac{r}{s}}z + O(|z|^2), \quad f^s \in \mathcal{A}_p.$$

By Lemma 3.1, if $\mathcal{A}_{p,r,s}$ is non-empty then s divides p.

For a positive integer s, we know that $f^s \in \mathcal{A}_p$ if $f \in \mathcal{A}_p$; also if $f_1^s = f_2^s \in \mathcal{A}_p$ and $f_1'(0) = f_2'(0)$ then $f_1 = f_2$, since f_j are in $\hat{\mathcal{C}}(f_1^s)$ and $\hat{\mathcal{C}}(f_1^s)$ is abelian. Therefore, the mapping $g \to g^s$ yields a one-to-one correspondence between the holomorphic equivalence classes of $\mathcal{A}_{p,r,s}$ and ones of $f \in A_p$ with $f = g^s$ and $g'(0) = e^{2\pi i\frac{r}{s}}$.

Fix positive integers s and p with s dividing p, and fix an integer r with $0 \le r < s$ and $(r,s)=1$. Let $\mathcal{M}_{p,r,s}$ be the set of $\Phi \in \mathcal{M}_p$ satisfying

$$\Phi_j = L_{c_{j+1}} \Phi_{j+2\epsilon pr/s} L_{-c_j}, \quad j = 1, \dots 2p, \tag{3.9}$$

$$\sum_{l=1}^{s} c_{j+2lpr/s} = -1, \tag{3.10}$$

where $\epsilon = 1$ for $p = 0, 1 \mod 4$, and $\epsilon = -1$ otherwise. We say that $\tilde{\Phi}, \Phi \in \mathcal{M}_{p,r,s}$ are equivalent, if

$$\tilde{\Phi}_j = L_{c_{j+1}} \Phi_{j+2k} L_{-c_j}, \quad j = 1, \dots, 2p$$

for some $k \in \mathbf{Z}$ and $c_j \in \mathbf{C}$.

Theorem 3.8. *Let r, s be integers with $(r,s)=1$, $0 \le r < s$, and let p be a positive integer divisible by s. There is a one-to-one correspondence between the holomorphic equivalence classes of mappings in $\mathcal{A}_{p,r,s}$ and ones in $\mathcal{M}_{p,r,s}$.*

Proof. We have already seen that for $g \in \mathcal{A}_{p,r,s}$ the holomorphic equivalence class of g depends only on that of g^s. For each $g \in \mathcal{A}_{p,r,s}$, we choose a linear transformation h such that $f = h^{-1} g^s h \in \mathcal{A}_p^*$. Denote by $\mathcal{A}_{p,r,s}^*$ the set of all such $h^{-1}gh$ as g varies in $\mathcal{A}_{p,r,s}$. Then $g \to \Phi_{g^s}$ is a map from $\mathcal{A}_{p,r,s}^*$ into $\mathcal{M}_{p,r,s}$, which is well-defined up to the equivalence relations.

Put $k = pr/s$. Let A_j, Ω_j be a canonical collection of $f = g^s$ with $g \in \mathcal{A}_{p,r,s}^*$. Since g commutes with f, and $g'(0) = \mu^{\epsilon k}$, Theorem 2.5 says that $A_{j+2\epsilon k} g = L_{-c_j} A_j$ for some constants c_j, which yields (3.9). From $f = A_j^{-1} F A_j = g^s$ we obtain (3.10). Therefore, Φ_{g^s} is actually in $\mathcal{M}_{p,r,s}$.

Conversely, if Φ_f satisfies (3.9), Theorem 2.5 says that $A_{j+2\epsilon k}^{-1} L_{-c_j} A_j = g$ extends to a holomorphic map with $g'(0) = \mu^{\epsilon k} = e^{2\pi i r/s}$. Now (3.10) implies that $g^s = f$. □

We conclude the section with the following.

Proof of Theorem 1.4. Let $f \in \mathcal{A}_p$ with $d_f \neq 0$.

For (i), assume that $H(z) = c\overline{z} + O(|z|^2)$ is an anti-holomorphic map satisfying $f^{-1} = HfH^{-1}$. We need to show that $H^2 = \mathrm{Id}$. One has $H^2(z) = z + O(|z|^2)$. So $H^2 = f^{l/d}$. Now $f^{-l} = (f^{-l/d})^d = H^{2d} = (Hf^{l/d}H^{-1})^d = f^l$. Hence $l = 0$, i.e., $H^2 = \mathrm{Id}$. By Lemma 3.1, we know that $HgH^{-1} = g^{-1}$ for all $g \in \hat{\mathcal{C}}(f)$.

For (ii), we assume that $\sigma_1 f = f \sigma_1$ and σ_1 is an anti-holomorphic involution. Let ω be a generator of periodic elements in $\hat{\mathcal{C}}(f)$. Then $\sigma\omega\sigma$ and ω^{-1} are in $\hat{\mathcal{C}}(f)$. So they commute. Since $\sigma\omega\sigma$ and ω^{-1} have the same linear part, then they are the same.

Assume now that there are two germs of real analytic curves γ_j with $f(\gamma_j) = \gamma_j$. Let σ_j be the anti-holomorphic involution generated by γ_j through the Schwarz reflection. Then $\sigma_j = f\sigma_j f^{-1}$. Thus $g = \sigma_1\sigma_2 \in \hat{\mathcal{C}}(f)$ and $g^q = f^{l/d}$ for $q = q_f$. Now $f^{-l} = \sigma_1 g^{dq} \sigma_1 = (\sigma_1 f \sigma_1)^l = f^l$. Hence $f^{2l} = 1$, which implies that $l = 0$ since $d_f \neq 0$. Now $g = \omega^j$. This shows that there are exactly δ anti-holomorphic involutions that are preserved by f, i.e, that there are exactly δ germs of real analytic curves at the origin invariant under f. Next, we need to show that $\sigma_1 \omega^{1/2} = (\omega^{1/2})^{-1} \sigma_1$ for square roots of ω. It suffices to find a holomorphic change of coordinates so that ω and τ_1 are linear. In suitable

holomorphic coordinates, $\sigma_1(z) = c\overline{z}$. Then ω and its linear part $D\omega$ are reversed by σ_1. Take $g = \sum_{k=1}^{\delta} (D\omega)^{-1} \circ \omega$. Then $g\omega = D\omega g$, and $\sigma_1 g = g\sigma_1$. Finally we know that $\sigma_1 \omega^j = \omega^{-j/2}\sigma_1\omega^{j/2}$. This shows that the fixed point of $\tilde{\sigma}_j$ is given by $\omega^{-j/2}\gamma_1$.

For (iii), we assume for the sake of contradiction that f and $\tilde{f} = gfg^{-1}$ are both real valued real analytic functions on $\mathbf{R}$, and that $g'(0) \in \mathbf{R}$ and $\tilde{\mathbf{R}} = g^{-1}(\mathbf{R}) \neq \mathbf{R}$. Then both $\mathbf{R}$ and $\tilde{\mathbf{R}}$, as germs at the origin, are invariant under f. On the other hand, $\tilde{\mathbf{R}}$ and $\mathbf{R}$ are tangent at the origin, which contradicts to (ii).

For (iv), we know $g\sigma_j g^{-1} = \sigma_j$. Thus $g \in \hat{\mathcal{C}}(f)$ for $f = \sigma_1\sigma_2$, where σ_j are anti-holomorphic involutions generated by γ_j. Lemma 3.1 says that $g^{-1} = \sigma_j g \sigma_j$ also. Hence $g^2 = \mathrm{Id}$. When $g \neq \mathrm{Id}$, this means that δ_f is even. Conversely, when δ_f is even, let ω be a generator for periodic centralizers of f. Then $\omega^{\delta/2}$ preserves both σ_j and hence both γ_j. □

4. pairs of smooth real analytic curves

In this section we shall give a complete classification for pairs of smooth real analytic curves intersecting at an angle that is a rational multiple of π. We shall first consider the case that two curves intersects tangentially and treat the general case later. (Recall that all real analytic curves in this paper are smooth.)

By the Schwarz reflection, a smooth real analytic curve determines a unique anti-holomorphic involution locally, and vice versa. Also, all smooth real analytic curves are locally holomorphically equivalent. Given a pair of smooth real analytic curves (γ_1, γ_2) intersecting at the origin, let σ_1, σ_2 be the anti-holomorphic involutions generated by γ_1, γ_2, respectively. Put $f = \sigma_1\sigma_2$. We choose local holomorphic coordinates such that γ_1 is the y-axis, so σ_1 becomes $\sigma\colon z \to -\overline{z}$. Under the new coordinates, f is reversible by σ, i.e., $f^{-1} = \sigma f \sigma$. Conversely each holomorphic map reversible by σ is the composition of anti-holomorphic involutions σ and σf. Thus, classifying the pairs of real analytic curves is equivalent to classifying holomorphic maps reversible by σ under biholomorphic maps g satisfying $g\sigma = \sigma g$.

Nakai [13] considered the classification of compositions $f = \sigma_1\sigma_2$ under biholomorphic maps g without the restriction $g\sigma = \sigma g$. We shall return to this classification in next section.

Denote by $\mathcal{A}_{p,\sigma}$ be the set of holomorphic maps $f \in \mathcal{A}_p$ satisfying

$$f^{-1} = \sigma \circ f \circ \sigma, \quad \sigma(z) = -\overline{z}.$$

Let $\mathcal{A}_{p,\sigma}/\overset{\sigma}{\sim}$ be holomorphic equivalence classes of $\mathcal{A}_{p,\sigma}$ under holomorphic changes of coordinates that preserve σ. For $f \in \mathcal{A}_{p,\sigma}$, write $f(z) = z + cz^{p+1} + O(|z|^{p+2})$ with $c \neq 0$. Then

$$\sigma f \sigma(z) = z + \overline{c}(-1)^p z^{p+1} + O(|z|^{p+2}), \quad f^{-1} = z - cz^{p+1} + O(|z|^{p+2}).$$

$$(c/|c|)^2 = (-1)^{p+1}. \tag{4.1}$$

Put $g(z) = (-1)^p \sqrt[p]{|c|}z$. Then $g\sigma g^{-1} = \sigma$, while $\tilde{f} = gfg^{-1}(z) = z + \mu_0^\sigma z^{p+1} + O(|z|^{p+2})$ with

$$\mu_0^\sigma = \begin{cases} 1 & p \text{ odd}, \\ \pm i & \text{otherwise}. \end{cases}$$

When $p > 0$ is even, let $\mathcal{A}^-_{p,\sigma}$ be the set of above $\tilde{f}$ with $\mu_0^\sigma = -\mu_0$. For a positive integer p, let $\mathcal{A}^*_{p,\sigma}$ be the set of above $\tilde{f}$ with $\mu_0^\sigma = \mu_0$. Thus $\mathcal{A}_{p,\sigma}/\overset{\sigma}{\sim} = \mathcal{A}^*_{p,\sigma}/\overset{\sigma}{\sim}$ when p is odd. When p is even, $\mathcal{A}_{p,\sigma}/\overset{\sigma}{\sim}$ is the disjoint union of $\mathcal{A}^*_{p,\sigma}/\overset{\sigma}{\sim}$ and $\mathcal{A}^-_{p,\sigma}/\overset{\sigma}{\sim}$. To see that, we assume that $f \in \mathcal{A}^*_{p,\sigma}$ and g commutes with σ. Then $g(z) = cz + O(|z|^2)$ with $c \in \mathbf{R}$. Then $g^{-1}fg(z) = z + \mu_0 c^p z^{p+1} + O(|z|^{p+2})$ is not in $\mathcal{A}^-_{p,\sigma}$ for $c^p > 0$.

We shall identify $\mathcal{A}^-_{p,\sigma}$ with $\mathcal{A}^*_{p,\sigma}$ by mapping f to f^{-1}, when p is even.

Lemma 4.1. *For each $f \in \mathcal{A}^*_{p,\sigma}$, there exists a canonical collection Ω_j, A_j such that A_j is conformal on $\Omega_j \bigcup \sigma(\Omega_{2p-j+1})$, and $A_{2p-j+1} = \sigma A_j \sigma$ on Ω_{2p-j+1}. Moreover, for any $\epsilon > 0$,*

$$\bigcup_{j=1}^{p} \{A_j^{-1}(R(\epsilon)) \cup \sigma A_j^{-1}(R(\epsilon))\}$$

is a punctured neighborhood of the origin, and is contained in any given neighborhood of the origin when ϵ is sufficiently small.

Proof. Let Ω_j, A_j be a canonical collection of f. From $f = A_{2j+1}^{-1} F A_{2j+1}$ and $f^{-1} = A_{2p-2j}^{-1} F^{-1} A_{2p-2j}$, we have

$$\begin{aligned} f &= \sigma A_{2p-2j}^{-1} \sigma F^{-1} \sigma A_{2p-2j} \sigma, \quad \text{on } \sigma(\Omega_{2p-2j}), \\ f^{-1} &= \sigma A_{2j+1}^{-1} \sigma F \sigma A_{2j+1} \sigma, \quad \text{on } \sigma(\Omega_{2j+1}). \end{aligned}$$

One readily sees that $\sigma(\Omega_{2p-j+1}) = \Omega'_j, \sigma A_{2p-j+1}\sigma = A'_j$ are a canonical collection of f also. By the uniqueness of the canonical collection, there exists a canonical collection Ω''_j, A''_j such that $A''_j = A_j$ for $1 \le j \le p$ and $A''_{2p-j+1} = \sigma A_j \sigma$ on Ω''_{2p-j+1} for $1 \le j \le p$. Since $A_{2p-j+1} = L_{c_{2p-j+1}} \sigma A_j \sigma$ for $1 \le j \le p$, then $\bigcup_{j=1}^p \{A_j^{-1}(R(\epsilon)) \cup \sigma A_j^{-1}(R(\epsilon))\}$ remains to be a punctured neighborhood. □

Let $f \in \mathcal{A}^*_{p,\sigma}$, and let Ω_j, A_j be a canonical collection of f satisfying I*i-v*. Based on Lemma 4.1, we shall always choose $A_{2p-j+1} = \sigma A_j \sigma$ for $j = 1, \dots, p$, unless specifying otherwise. Of course, $A_1, \dots, A_{2p}$ (together with some domains Ω_j) satisfy I*i-iv*. Now define $\Phi_{f,j} = A_{j+1} A_j^{-1}$, so $\Phi_{f,2p-j}^{-1} = \sigma \Phi_{f,j} \sigma$. Note that $\Phi_{f,j}$ are well-defined up to the form $L_{c_{j+1}} \Phi_{f,j} L_{c_j}$ with $c_{2p-j+1} = -\overline{c}_j$.

Let $\mathcal{M}_{p,\sigma}$ be the set of $\Phi \in \mathcal{M}_p$ satisfying $\Phi_{2p-j}^{-1} = \sigma \Phi_j \sigma, j = 1 \dots, 2p$. Denote $\tilde{\Phi} \overset{\sigma}{\sim} \Phi$ in $\mathcal{M}_p$, if $\Phi_j = L_{c_{j+1}} \Phi_{f,j+2k} L_{c_j}$ with $c_{2p-j+1} = -\overline{c}_j$ for $k = 0$ or $p/2$ (the latter occurs only for p even).

We now restate Theorem 1.1 as follows.

Theorem 4.2. *Let $f, \tilde{f} \in \mathcal{A}^*_{p,\sigma}$. If $\tilde{f} = g^{-1} f g$ for some holomorphic mapping g commuting with σ, there exist $c_j = -\overline{c}_{2p-j+1} \in \mathbf{C}$ such that one of the following holds*

i) $g'(0) = 1$ *and*

$$\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j} L_{-c_j}, \quad j = 1, \dots, 2p; \tag{4.2}$$

ii) $g'(0) = -1$, p *is even, and*

$$\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j+p} L_{-c_j}, \quad j = 1, \dots, 2p. \tag{4.3}$$

Conversely, if (4.2) *holds, then* $\tilde{f} = g^{-1} f g$ *for some* g *with* $g\sigma = \sigma g$ *and* $g'(0) = 1$*; if* (4.3) *holds then* $\tilde{f} = g^{-1} f g$ *for some* g *with* $g\sigma = \sigma g$ *and* $g'(0) = -1$*. For each* $\Phi \in \mathcal{M}_{p,\sigma}$*, there exists* $f \in \mathcal{A}^*_{p,\sigma}$ *with* $\Phi_{f,j} = \Phi_j$ *for all* j*.*

Proof. Let $\tilde{f}, f \in \mathcal{A}^*_{p,\sigma}$. Assume that $\tilde{f} = g^{-1} f g$ with $g\sigma = \sigma g$. Then $g'(0) = 1$ if p is odd, and $g'(0) = 1$ or -1 when p is even.

We first consider the case $g'(0) = 1$. Let Ω_j, A_j be a canonical collection of f with $A_{2p-j+1} = \sigma A_j \sigma$. Then $A'_j = A_j g$ and $g^{-1}(\Omega_j)$ are a canonical collection of $\tilde{f}$. Thus, $A'_1, \dots, A'_p, \sigma A'_p \sigma, \dots, \sigma A'_1 \sigma$, together with some domains Ω'_j, form a canonical collection of $\tilde{f}$. Of course we have $A'_{j+1} A_j^{'-1} = A_{j+1} A_j^{-1}$ for $j = 1, \dots, p-1$. From $\sigma g = g\sigma$, we get the remaining identities $A'_{j+1} A_j^{'-1} = A_{j+1} A_j^{-1}$ for $j = p, \dots, 2p$. In other words, $\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j} L_{-c_j}$ with $c_{2p-j+1} = -\overline{c}_j$.

Next, consider the case that $g'(0) = -1$ and p is even. Again, let Ω_j, A_j be a canonical collection of f. Now $\Omega'_j = g^{-1}(\Omega_{j+p})$ and $A'_j = A_{j+p} g$ are a canonical collection of $\tilde{f}$. Thus $A'_1, \dots, A'_p, \sigma A'_p \sigma, \dots, \sigma A'_1 \sigma$, together with some domains Ω''_j, form a canonical collection of $\tilde{f}$. From $g\sigma = \sigma g$, we see that $\Phi_{\tilde{f},j} = \Phi_{f,j+p}$ for $j = 1, \dots, 2p$.

Next, assume that $\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j} L_{-c_j}$ with $c_{2p-j+1} = \overline{c}_j$. Recall that $\Phi_{\tilde{f},j} = \tilde{A}_{j+1} \tilde{A}_j^{-1}$, $\Phi_{f,j} = A_{j+1} A_j^{-1}$ with $\tilde{A}_{2p-j+1} = \sigma \tilde{A}_j \sigma$ and $A_{2p-j+1} = \sigma A_j \sigma$. Replacing $\tilde{A}_j$ with $L_{-c_j} \tilde{A}_j$, we may assume that $\tilde{A}_{j+1} \tilde{A}_j^{-1} = A_{j+1} A_j^{-1}$. As in the proof of Theorem 2.5, we know that $g = A_j^{-1} \tilde{A}_j$ $(j = 1, \dots, 2p)$ extends to a holomorphic mapping defined near the origin and $\tilde{f} = g^{-1} f g$. Also $\sigma g \sigma = \sigma A_1^{-1} \tilde{A}_1 \sigma = A_{2p}^{-1} \tilde{A}_{2p} = g$ on $\tilde{\Omega}_{2p} \cap A_{2p}^{-1} \tilde{A}_{2p}(\tilde{\Omega}_{2p})$, and hence in a neighborhood of the origin.

Assume now that p is even, and that $\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j+p} L_{-c_j}$ with $c_{2p-j+1} = -\overline{c}_j$. Again, we may assume that all $c_j = 0$. Let $g = A_{j+p}^{-1} \tilde{A}_j$. Then g extends to a biholomorphic mapping defined near the origin. We have $\tilde{f} = g^{-1} f g$. Also $\sigma g \sigma = \sigma A_{1+p}^{-1} \tilde{A}_1 \sigma = A_p^{-1} \tilde{A}_{2p} = g$ on $\tilde{\Omega}_{2p} \cap A_p^{-1}(\tilde{A}_{2p}(\tilde{\Omega}_{2p}))$. Therefore, $g\sigma = \sigma g$.

Given $\Phi \in \mathcal{M}_{p,\sigma}$, we need to find $f \in \mathcal{A}^*_{p,\sigma}$ such that $A_{j+1} A_j^{-1} = \Phi_j$, where $A_j = \sigma A_{2p-j+1} \sigma$, $j = 1, \dots, 2p$, together with some domains $\Omega_1, \dots, \Omega_{2p}$, are a canonical collection of f. By Theorem 2.5, there exists $f \in \mathcal{A}^*_p$ such that $\Phi_j = A_{j+1} A_j^{-1} = \Phi_{f,j}$, where A_j, Ω_j form a canonical collection of f (here $A_{2p-j+1} = \sigma A_j \sigma$ does not necessarily hold). Since $\Phi_{2p-j}^{-1} = \sigma \Phi_j \sigma$, we have $\sigma A_{j+1} A_j^{-1} \sigma = A_{2p-j} A_{2p-j+1}^{-1}$ on some half-plane $\Pi_{p,j}(\rho_1)$. Hence,

$$A_{2p-j+1}^{-1} \sigma A_j = A_{2p-j}^{-1} \sigma A_{j+1}$$

holds on

$$A_j^{-1}(\Pi_{p,j}(\rho_1)) \supset A_j^{-1}(\Pi_{p,j}(\rho_1)) \cap A_{j+1}^{-1}(\Pi_{p,j}(\rho_1)) \\ = A_j^{-1}((-1)^{j-1}R(\rho_1)) \cap A_{j+1}^{-1}((-1)^j R(\rho_1)).$$

Put $\omega_j = A_j^{-1}((-1)^{j-1}R(\rho_1))$. We know that $\omega_j \cap \omega_k = \emptyset$ for $k \neq j-1, j+1 \mod 2p$. Hence, $\hat{\sigma} = A_{2p-j+1}^{-1}\sigma A_j$ is well-defined on $\cup\omega_j$. By removable singularities, $\hat{\sigma}$ extends to an anti-holomorphic mapping defined near the origin. On ω_1 we have $\hat{\sigma}^2 = \mathrm{Id}$. Hence, $\hat{\sigma}$ is an anti-holomorphic involution.

Note that

$$\hat{\sigma} f \hat{\sigma} = \hat{\sigma} A_1^{-1} F A_1 \hat{\sigma} = A_{2p}^{-1}\sigma F \sigma A_{2p} = f^{-1}.$$

Put $\hat{\sigma}(z) = c\overline{z} + O(|z|^2)$. Hence $|c| = 1$. We have

$$f(z) = z + \mu_0 z^{p+1} + O(|z|^{p+2}), \quad f^{-1}(z) = z - \mu_0 z^{p+1} + (|z|^{p+2}), \\ \hat{\sigma} f \hat{\sigma} = z + \overline{\mu}_0 \overline{c}^p z^{p+1} + O(|z|^{p+2}).$$

By the choice of μ_0, we have $\mu_0 = (-1)^{p+1}\overline{\mu}_0$; consequently, $c = -\mu^k$ and $\hat{\sigma}(y_+) = \mu^k y_+$ for some $0 \leq k < p$. One the other hand, $\hat{\sigma} = A_{2p}^{-1}\sigma A_1$ implies that $\hat{\sigma}(y_+)$ is contained in Ω_{2p}. Therefore, $k = 0$ by Iii).

Since $\hat{\sigma}(z) - \sigma(z) = O(|z|^2)$, then $g(z) = \frac{1}{2}(z + \sigma\hat{\sigma}(z)) = z + O(|z|^2)$. Now g is biholomorphic, and $g\hat{\sigma} = \sigma g$. Thus $\sigma\tilde{f}\sigma = \tilde{f}^{-1}$ for $\tilde{f} = gfg^{-1}$. We now have $\Phi_{\tilde{f},j} = \tilde{A}_{j+1}\tilde{A}_j^{-1}$ and $\Phi_{\tilde{f},2p-j}^{-1} = \sigma\Phi_{\tilde{f},j}\sigma$. Since $\tilde{f} = gfg^{-1}$ and $g'(0) = 1$, Theorem 2.5 says $\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_j L_{-c_j}$. Using $\sigma\Phi_j\sigma = \Phi_{2p-j}^{-1}$, we get

$$L_{c_{j+1}}\Phi_j L_{-c_j} = \Phi_{\tilde{f},j} = \sigma L_{c_{2p-j}}\Phi_{2p-j}^{-1} L_{-c_{2p-j+1}}\sigma = L_{-\overline{c}_{2p-j}}\Phi_j L_{\overline{c}_{2p-j+1}}.$$

Thus $c' = c_j + \overline{c}_{2p-j+1}$ for all j and $L_{-c'}\Phi_j L_{c'} = \Phi_j$. In particular, $c' = \frac{1}{p}\operatorname{Re}(c_1 + \cdots + c_{2p})$. When $d_\Phi = 0$, we put $A'_j = L_{-c_j + \frac{c'}{2}}\tilde{A}_j$ for $j = 1, \ldots, 2p$. Clearly, $A'_{j+1}A_j^{'-1} = \Phi_j$. Now $\overline{c}_j - \frac{c'}{2} = -c_{2p-j+1} + \frac{c'}{2}$ implies that $\sigma A'_j\sigma = \tilde{A}'_{2p-j+1} + \overline{c}_j - \frac{c'}{2} = A'_{2p-j+1}$, which gives us the desired realization. When $d = d_\Phi = d_{\Phi_f} \neq 0$, then $c' = \frac{k}{d}$, $k \in \mathbf{Z}$. When k is even, $A'_j = \tilde{A}_j - c_j + \frac{c'}{2}$ gives us the realization as above. Assume now that $k = 2l+1$. Put $A'_j = L_{-c_j + \frac{l}{d}}\tilde{A}_j$ for $j = 1, \ldots, p$ and $A'_j = L_{-c_j + \frac{l+1}{d}}\tilde{A}_j$ for $j = p+1, \ldots, 2p$. Then we have

$$\sigma A'_j\sigma = A'_{2p-j+1}; \quad A'_{j+1}A_j^{'-1} = \Phi_j, \quad j \neq p, 2p; \\ A'_{p+1}A_p^{'-1} = L_{\frac{1}{d}}\Phi_p, \quad A'_1 A_{2p}^{'-1} = L_{-\frac{1}{d}}\Phi_{2p}. \tag{4.4}$$

Take $\tilde{\sigma} = \tilde{f}^{\frac{1}{d}}\sigma$. Then $\tilde{\sigma}$ is an anti-holomorphic involution with $\tilde{\sigma}(z) = -\overline{z} + O(|z|^2)$. Put $h = \frac{1}{2}(\mathrm{Id} + \sigma\tilde{\sigma})$, so $h\tilde{\sigma} = \sigma h$. Put $f^* = h\tilde{f}h^{-1}$. Then $f^{*-1} = \sigma f^*\sigma$. Put $A_j^* = A'_j h^{-1} = \sigma A_{2p-j+1}^*\sigma$ for $j = 1, \ldots, p$. Obviously, $A_{j+1}^* A_j^{*-1} = A'_{j+1}A_j^{'-1} = \Phi_j$ for $j = 1, \ldots, p-1$. Now rewrite $h\tilde{\sigma} = \sigma h$ as

$$h^{-1}\sigma h = \tilde{\sigma} = A_p^{'-1}L_{\frac{1}{d}}A'_p\sigma.$$

We have

$$A_{p+1}^* A_p^{*-1} = \sigma A'_p h^{-1}\sigma h A_p^{'-1} = \sigma L_{\frac{1}{d}}A'_p\sigma A_p^{'-1} = \Phi_p.$$

For $j = p+1, \dots, 2p$, we obtain $\Phi_{f^*,j} = \sigma\Phi_{f^*,2p-j+1}\sigma = \sigma\Phi_{2p-j+1}\sigma = \Phi_j$, which gives us the desired realization. □

Remark 4.3. We shall see from Theorem 5.1 below that f and f^* appeared in the above proof are σ-equivalent, if and only if $f^{1/d}$ admits a square root.

We shall now give a complete classification of pairs of real analytic curves which form an angle that is a rational multiple of π.

Let r, s be coprime with $0 \le r < s$. Let $\mathcal{P}_{p,r,s}$ be the set of pairs of real analytic curves $\gamma = (\gamma_1, \gamma_2)$ satisfying the following: $f(z) = \tau_1\tau_2(z) = e^{2\pi ir/s}z + O(|z|^2)$, where τ_1, τ_2 are the anti-holomorphic reflections generated by γ_1 and γ_2, respectively; f^s is in $\mathcal{A}_p$. Notice that when the latter does not hold for all $p > 0$, f is periodic. Then γ is holomorphically equivalent to a pair of real lines, which follows from the following

Proposition 4.4. *Let f be a periodic holomorphic map which is reversible by σ. Then f is linearizable by some holomorphic g commuting with σ.*

Proof. Let Df be the linear part of f. Then Df is also reversible by σ. Take $g = \sum_{k=1}^{n}(Df)^{-k} \circ f^k$, assuming $f^n = \mathrm{Id}$. Then $g\sigma = \sigma g$ and $(Df) \circ g = gf$. □

Denote by $\mathcal{M}_{p,r,s,\sigma}$ the set of $\Phi \in \mathcal{M}_{p,r,s}$ such that $\sigma\Phi_j\sigma = \Phi_{2p-j}^{-1}$. We say that $\Phi, \tilde{\Phi} \in \mathcal{M}_{p,r,s,\sigma}$ are equivalent, if there exist constants $c_j \in \mathbf{C}$ and $k \in \mathbf{Z}$ such that

$$\tilde{\Phi}_j = L_{c_{j+1}}\Phi_{j+2k}L_{-c_j}, \quad c_{2p-j+1} = -\overline{c}_j, \quad j = 1, \dots, 2p,$$

where $k = 0$ or $k = p/2$ (and p is even). The set of equivalence classes of $\mathcal{M}_{p,r,s}$ is denoted by $\mathcal{M}_{p,r,s,\sigma}/\overset{\sigma}{\sim}$.

We have

Theorem 4.5. *Let r, s be coprime integers with $0 \le r < s$, and let p be a positive integer divisible by s. When p is odd there exists a one-to-one correspondence between the set of holomorphic equivalence classes of pairs of real analytic curves in $\mathcal{P}_{p,r,s}$ and $\mathcal{M}_{p,r,s,\sigma}/\overset{\sigma}{\sim}$; when p is even there is a one-to-one correspondence between the set of holomorphic equivalence classes of $\mathcal{P}_{p,r,s}$ and the disjoint union $\mathcal{M}_{p,r,s,\sigma}/\overset{\sigma}{\sim} \bigsqcup \mathcal{M}_{p,r',s,\sigma}/\overset{\sigma}{\sim}$, where $r' = s - r$ when $r > 0$, or $r' = 0$ when $r = 0$.*

Proof. Given a pair of real analytic curves $\gamma = (\gamma_1, \gamma_2) \in \mathcal{P}_{p,r,s}$, we choose local coordinates such that γ_1 is the y-axis. Let σ_j be the anti-holomorphic involutions generated by γ_j respectively, and put $h = \sigma_1\sigma_2$. Next we use a linear transformation which commutes with $\sigma(= \sigma_1)$ to transforms h into f_γ such that $f_\gamma \in \mathcal{A}^*_{p,r,s}$ when p is odd, or $f_\gamma \in \mathcal{A}^*_{p,r,s}$ or $f_\gamma^{-1} \in \mathcal{A}^*_{p,r,s}$ when p is even.

We first assume that p is odd. Then $\Phi_\gamma = \Phi_{f_\gamma^s} \in \mathcal{M}_{p,r,s,\sigma}$. It is also clear that the equivalence class of Φ_γ in $\mathcal{M}_{p,\sigma}$ depends only on that of γ.

Conversely, if $\Phi_\gamma, \Phi_{\tilde{\gamma}}$ are equivalent in $\mathcal{M}_{p,\sigma}$, then $f_{\tilde{\gamma}}^s = gf_\gamma^s g^{-1}$ for some holomorphic g commuting with σ. Thus $\tilde{f}_{\tilde{\gamma}} = gf_\gamma g^{-1}$; consequently, g sends the pair $\tilde{\gamma}$ into the pair γ.

For the realization, we take $\Phi \in \mathcal{M}_{p,r,s,\sigma}$. Then $\Phi = \Phi_g$ for some $g \in \mathcal{A}_{p,\sigma}$ and $g = f^s$ for $f \in \mathcal{A}_{p,r,s}$. Since g is reversible by σ we have $(\sigma f\sigma)^s = f^{-s}$. Since $\sigma f\sigma, f^{-1}$ have the same linear part, then $f^{-1} = \sigma f\sigma$ and hence f is in $\mathcal{A}_{p,r,s,\sigma}$.

Assume now that p is even. Let $\mathcal{P}^*_{p,r,s}$ be the set of $\gamma \in \mathcal{P}_{p,r,s}$ such that f^s_γ is in $\mathcal{A}^*_p$. Let $\mathcal{P}^-_{p,r,s} = \mathcal{P}_{p,r,s} \setminus \mathcal{P}^*_{p,r,s}$. Then $\mathcal{P}_{p,r,s}/\sim$ is the disjoint union of $\mathcal{P}^*_{p,r,s}/\sim$ and $\mathcal{P}^-_{p,r,s}/\sim$. As in the case that p is odd, we can verify that the former is identified with $\mathcal{M}_{p,r,s,\sigma}/\overset{\sigma}{\sim}$ and the latter is $\mathcal{M}_{p,r',s,\sigma}/\overset{\sigma}{\sim}$. □

5. classification of reversible maps

In this section we first turn to σ-equivalence classification for σ-reversible holomorphic maps g that are holomorphically equivalent to a given $f \in \mathcal{A}_{p,\sigma}$. An example of such maps g is f^*, appeared in the realization part of the proof of Theorem 4.2. In general, for $f \in \mathcal{A}_{p,\sigma}$, define

- $A_f = \{\tilde{f} \in \mathcal{A}_{p,\sigma} \colon \tilde{f} = gfg^{-1}\}$.
- $A^{\pm}_f = \{\tilde{f} \in \mathcal{A}_{p,\sigma} \colon \tilde{f} = gfg^{-1}, g'(0) = \pm 1\}$.
- $A^*_f = \{\tilde{f} \in \mathcal{A}_{p,\sigma} \colon \tilde{f} = gfg^{-1}, g'(0) = 1\}$.
- $\tilde{f} \overset{*}{\sim} f$, $\tilde{f} \overset{\pm}{\sim} f$ $\tilde{f} \sim f$, if $\tilde{f} = hfh^{-1}$ for some biholomorphism $h = \sigma h \sigma$ satisfying $h'(0) = 1, \pm 1$, and $h'(0) \neq 0$, respectively. (In the above notations g are holomorphic.)

The main purpose of this section is to identify $A_f/\overset{*}{\sim}$, $A_f/\overset{\pm}{\sim}$, and $A_f/\sim$ and their group structures. In this section we shall also classify reversible holomorphic maps and show the existence of non-reversible holomorphic maps.

We shall need the following notation. Given two biholomorphisms f and ν with $|\nu'(0)| = 1$, we put

$$f_\nu = gfg^{-1}, \quad g = \frac{1}{2\sqrt{\nu'(0)}}(\mathrm{Id} + \nu'(0)\sigma\nu\sigma). \tag{5.1}$$

Note that Lemma 4.1 says that if $\nu \in \hat{\mathcal{C}}(f)$, then $\nu\sigma$ is an anti-holomorphic involution, and hence $g\nu\sigma g^{-1} = \sigma$. Put $f^* = f_{f^{1/d}}$, when $d_f \neq 0$.

We start with

Theorem 5.1. *Let $f \in \mathcal{A}_{p,\sigma}$ with $d_f \neq 0$. The map $R\colon g \mapsto g^{-1}\sigma g\sigma$ sends $\{g\colon gfg^{-1} \in A_f\}$ onto $\hat{\mathcal{C}}(f)$, and induces a one-to-one correspondence between $A_f/\sim$ and the quotient group $\hat{\mathcal{C}}(f)/\hat{\mathcal{C}}^2(f) = <\tau>/<\tau^2> \otimes <\omega>/<\omega^2>$. Moreover, $f^* = gfg^{-1}$ for some holomorphic $g = \sigma g \sigma$, if and only if $f^{1/d}$ admits a square root (i.e., $r_{f^{1/d}}$, given by Theorem 3.7, is even).*

Proof. Let g be a biholomorphism. Note that $\sigma gfg^{-1}\sigma = \sigma g\sigma f^{-1}\sigma g^{-1}\sigma$. Thus gfg^{-1} remains reversible by σ if and only if $\nu = g^{-1}\sigma g\sigma$ is a centralizer of f^{-1}. Conversely, if ν is a centralizer of f, then $\nu^{-1} = \sigma\nu\sigma$, by Lemma 3.1(ii). Hence $\nu\sigma$ is an anti-holomorphic involution, and $\nu\sigma = g^{-1}\sigma g$ for some g. Now gfg^{-1} is still reversible by σ as we just proved. This shows that the map R, defined by $g \mapsto g^{-1}\sigma g\sigma$, sends

$$G = \{g\colon gfg^{-1} \in \mathcal{A}_{p,\sigma}, g \text{ holomorphic}\}$$

onto $\hat{\mathcal{C}}(f)$.

Next we need to show that for $\nu_j = g_j^{-1}\sigma g_j\sigma \in \hat{\mathcal{C}}(f)$ and $f_j = g_j f g_j^{-1}$, the identity $kf_1k^{-1} = f_2$ holds for some $k = \sigma k\sigma$, if and only if $v_2v_1^{-1} \in \hat{\mathcal{C}}^2(f)$. (In particular, the σ-equivalence class of gfg^{-1}, when $g^{-1}\sigma g = \nu\sigma$, depends only on $\nu \in \hat{\mathcal{C}}(f)$, i.e., on that of f_ν.)

We first assume that $kf_1k^{-1} = f_2$ with $k\sigma = \sigma k$. Then $\varphi = g_2^{-1}kg_1$ and f commute. Now

$$g_2\varphi g_1^{-1} = k = \sigma g_2\varphi g_1^{-1}\sigma = \sigma g_2\sigma\varphi^{-1}\sigma g_1^{-1}\sigma = g_2\nu_2\varphi^{-1}\sigma\nu_1\sigma g_1^{-1}.$$

By Lemma 3.1(ii) we have $\sigma\nu_1\sigma = \nu_1^{-1}$. Hence $\varphi^2 = \nu_1^{-1}\nu_2$. Conversely, assume that $\varphi^2 = \nu_1^{-1}\nu_2$ for some $\varphi \in \hat{\mathcal{C}}(f)$. Put $k = g_2\varphi g_1^{-1}$. Then $kf_1k^{-1} = f_2$ and $\sigma k\sigma = g_2\nu_2\sigma\varphi\nu_1\sigma g_1^{-1} = g_2\nu_2\varphi^{-1}\nu_1^{-1}g_1^{-1} = k$.

Since $\hat{\mathcal{C}}(f) = <\tau> \otimes <\omega>$, then

$$\hat{\mathcal{C}}(f)/\hat{\mathcal{C}}^2(f) = <\tau>/<\tau^2> \otimes <\omega>/<\omega^2>.$$

By definition $f^* = f_{f^{1/d}}$. Hence f^* and f are equivalent by some holomorphic k commuting with σ, if and only if $f^{1/d} \in \hat{\mathcal{C}}^2(f)$. □

Theorem 5.2. *Let f be in $\mathcal{A}_{p,\sigma}$ with $d_f \neq 0$. Then the map $R\colon g \mapsto g^{-1}\sigma g\sigma$ induces one-to-to correspondence between $A_f^*/\overset{*}{\sim}$ and the quotient group $\mathcal{C}(f)/\mathcal{C}^2(f)$, and between $A_f^{\pm}/\overset{\pm}{\sim}$ and the quotient group $\mathcal{C}(f)/\hat{\mathcal{C}}^2(f)\cap\mathcal{C}(f)$. The latter equals $\{1\}$ when $f^{1/d}$ admits a square root, and equals $\mathcal{C}(f)/\mathcal{C}^2(f)$ otherwise.*

Proof. We need to modify the proof of Theorem 5.1. We already know that $\tilde{f} = gfg^{-1}$ remains reversible by σ if and only if $g^{-1}\sigma g\sigma = \nu \in \hat{\mathcal{C}}(f)$. Notice that ν is actually in $\mathcal{C}(f)$ when $g'(0) = \pm 1$. Therefore, the map R sends both $A_f^{\pm}$ and A_f^* onto $\mathcal{C}(f)$.

For $f_j = g_j f g_j^{-1} \in A_f^*$ and $\nu_j = g_j^{-1}\sigma g_j\sigma$, we have $kf_1k^{-1} = f_2$ for some $k = \sigma k\sigma$, if and only if $\nu_1^{-1}\nu_2 = \varphi^2$ for some $\varphi \in \hat{\mathcal{C}}(f)$ (and $\varphi = g_2^{-1}kg_1$). Thus $f_1, f_2 \in A_f^*$ are equivalent by some $k = \sigma k\sigma$ with $k'(0) = 1$ if and only if $\nu_1^{-1}\nu_2 \in \mathcal{C}^2(f)$. The latter is a subgroup of $\mathcal{C}(f)$. Thus $A_f^*/\overset{*}{\sim}$ is identified with the quotient group $\mathcal{C}(f)/\mathcal{C}^2(f)$.

Analogously, $A_f^{\pm}/\overset{\pm}{\sim}$ is identified the quotient of $\mathcal{C}(f)$ by the subgroup $\mathcal{C}(f)\cap\hat{\mathcal{C}}^2(f)$. Note that $\mathcal{C}(f)\cap\hat{\mathcal{C}}^2(f) = \mathcal{C}^2(f)$, unless $f^{1/d} \in \hat{\mathcal{C}}^2(f)$. □

Remark. (a) Let $f \in \mathcal{A}_p$ with $d_f \neq 0$. Then $f^{1/d}$ admits a square root, if and only if q is even and δ is odd.

To see this, assume that $f^{1/d} = g^2$. Then $g'(0) = -1$, and $2d$ divides dq/δ by Theorem 3.7. The latter implies that q is even. Also δ is odd; otherwise $\omega^{\delta/2}g$ would be a square root of $f^{1/d}$ in $\mathcal{C}(f)$. Conversely, if q is even and δ is odd, then $\tau^q = f^{\delta/d}$ yields $(f^{-\frac{\delta-1}{2d}}g^{\frac{q}{2}})^2 = f^{1/d}$.

(b) In [13] Nakai introduced an *associativity relation* between pairs of real analytic curves of tangential intersection as follows: If γ and $\tilde{\gamma}$ be two pair of real analytic curves with tangential intersection, and if $f = \sigma_1\sigma_2$ and $\tilde{f} = \tilde{\sigma}_1\tilde{\sigma}_2$ are the composition of the pairs of anti-holomorphic involutions, then $\tilde{\gamma}$ and γ are *associated* if $gfg^{-1} = \tilde{f}$ or $\tilde{f}^*$ and $g\sigma_1g^{-1} = \tilde{\sigma}_1$ for some holomorphic mapping g (assuming $d_f \neq 0$). Nakai proved that f and f^* are not equivalent by holomorphic mappings preserving σ_1 when p is odd, and

Theorem 5.1 above gives a more complete answer. Namely, f and f^* are equivalent by holomorphic mappings commuting with σ_1, if and only if $f^{1/d}$ admits a square root.

Theorem 5.3. *Let f be a reversible map in $\mathcal{A}_p$ with $d_f \neq 0$. Let $\tilde{R}(g) = (g^{-1}\sigma g\sigma, g'(0))$ and $N(z) = -z$. Then $\tilde{R}$ induces a one-to-one correspondence between $A_f^{\pm}/\overset{*}{\sim}$ and the quotient group of $\mathcal{C}(f) \otimes \mathbf{Z}_2$ given by*

$$A_f^{\pm}/\overset{*}{\sim} = \begin{cases} \{f, f^*, NfN, Nf^*N\} = C(f)/\mathcal{C}^2(f) \otimes \mathbf{Z}_2, & q \text{ odd}, \\ \{f, f^*\} = \mathcal{C}(f)/\mathcal{C}^2(f) \otimes \{1\}, & q \text{ even and } \delta \text{ even}, \\ \{f, NfN\} = \{f, f^*\} \\ \quad = \mathcal{C}(f) \otimes \mathbf{Z}_2/<(f^{1/d}, -1)>, & q \text{ even and } \delta \text{ odd}. \end{cases}$$

Proof. We already know that for each $\nu \in \mathcal{C}(f)$ there is g such that $g^{-1}\sigma g\sigma = \nu$ and $g'(0) = 1$. Put $\tilde{f} = gfg^{-1} \in A_f^{\pm}$. Note that $Ngfg^{-1}N$ is also reversible by σ. Thus $\tilde{R}$ maps $\{g\colon gfg^{-1} \in A_f^{\pm}, g'(0) = \pm 1\}$ onto $\mathcal{C}(f) \otimes \mathbf{Z}_2$. Let $f_j = g_j f g_j^{-1}, j = 1, 2$ be reversible by σ with $g_j'(0) = \pm 1$, and let $\nu_j = g_j^{-1}\sigma g_j \sigma$. Assume that $kf_1k^{-1} = f_2$ for some $k = \sigma k \sigma$ with $k'(0) = 1$. As in the proof of Theorem 5.2, we still have $\varphi = g_2^{-1}kg_1$ is in $\hat{\mathcal{C}}(f)$ and $\nu_1^{-1}\nu_2 = \varphi^2$. Note that $k'(0) = 1$, if and only if $\varphi'(0) = g_2'(0)^{-1}g_1'(0)$. Thus $\tilde{R}$ induces a one-to-one and onto map between $A_f^{\pm}/\overset{*}{\sim}$ and $(\mathcal{C}(f) \otimes \mathbf{Z}^2)/\sim$, where $(\nu_1, s_1) \sim (\nu_2, s_2)$, if and only if $(\nu_1, s_1)^{-1}(\nu_2, s_2)$ is in the subgroup $\{(\varphi^2, \varphi'(0))\colon \varphi^2 \in \hat{\mathcal{C}}^2(f) \cap \mathcal{C}(f)\}$. Thus $(\mathcal{C}(f) \otimes \mathbf{Z}^2)/\sim$ is indeed a quotient group of $\mathcal{C}(f) \otimes \mathbf{Z}^2$.

We first consider the case that $\mathcal{C}(f) \cap \hat{\mathcal{C}}^2(f) = \mathcal{C}(f)$, i.e., that q is even and δ is odd. Given $(\nu_1, s_1) \sim (\nu_2, s_2)$ there is $\varphi \in \hat{\mathcal{C}}(f)$ with $\nu_1^{-1}\nu_2 = \varphi^2$. In fact φ is unique, since δ is odd. Therefore, we always have that $(\nu_2, s_2) \sim (\nu_1, s_1)$ or $(\nu_2, s_2) \sim (\nu_1, -s_1)$ and $(\nu_1, s_1) \not\sim (\nu_1, -s_1)$. In other words, either f_1, f_2, or Nf_1N, f_2 are equivalent in $A_f^{\pm}/\overset{*}{\sim}$, while f_1, Nf_1N are never equivalent in $A_f^{\pm}/\overset{*}{\sim}$. Therefore, $A_f^{\pm}/\overset{*}{\sim} = \{f, NfN\} = \mathcal{C}(f) \otimes \mathbf{Z}^2/<(f^{1/d}, -1)>$ is isomorphic to $\mathbf{Z}_2$. Since $(f^{1/d}, 1)$ is not in $<(f^{1/d}, -1)>$, we also know that f^* is not equivalent to f in $A_f^{\pm}/\overset{*}{\sim}$.

Next we consider the case that $\hat{\mathcal{C}}^2(f) \cap \mathcal{C}(f) = \mathcal{C}^2(f)$. There are only two cases: either q is odd, or both q, δ are even. For the former case, we know that δ is odd. Hence, $\{(\varphi^2, \varphi'(0))\colon \varphi \in \hat{\mathcal{C}}^2(f) \cap \mathcal{C}(f)\} = \mathcal{C}^2(f) \otimes \{1\}$, and $A_f^{\pm}/\overset{*}{\sim} = \mathcal{C}(f) \otimes \mathbf{Z}_2/(\mathcal{C}^2(f) \otimes \{1\}) \cong \mathbf{Z}_2 \otimes \mathbf{Z}_2$. When q and δ are even, we have $\{(\varphi^2, \varphi'(0))\colon \varphi^2 \in \hat{\mathcal{C}}^2(f) \cap \mathcal{C}(f)\} = <(f^{2/d}, 1), (1, -1)>$ and $A_f^{\pm}/\overset{*}{\sim} = \mathcal{C}(f)/\mathcal{C}^2(f) \otimes \{1\} \cong \mathbf{Z}_2$. □

Proposition 5.4. *Let $f \in \mathcal{A}_{p,\sigma}$ with $d_f \neq 0$. Let $f_\varphi = gfg^{-1}$ for $\varphi\sigma = g^{-1}\sigma g$, where g is defined by (5.1). Then for each $\psi \in \hat{\mathcal{C}}(f)$, we have $(f_\varphi)_{g\psi g^{-1}} = h^{-1} f_{\varphi\psi} h$ for some holomorphic map $h = \sigma h \sigma$. In particular, $(f^*)^*$ is equivalent to f by some holomorphic $h = \sigma h \sigma$.*

Proof. Notice that $\hat{\mathcal{C}}(f_\varphi) = g\hat{\mathcal{C}}(f)g^{-1}$. Hence $(f_\varphi)_{g\psi g^{-1}}$ is well-defined for all $\psi \in \hat{\mathcal{C}}(f)$. We have

$$f_\varphi = gfg^{-1}, \quad \varphi\sigma = g^{-1}\sigma g,$$
$$(f_\varphi)_{g\psi g^{-1}} = kgfg^{-1}k^{-1}, \quad g\psi g^{-1}\sigma = k^{-1}\sigma k.$$

Since $(f_\varphi)_{g\psi g^{-1}} \in A_f$, then Theorem 5.1 says that it equals $h^{-1}f_\nu h$ for some centralizer ν of f; in fact, we can take $\nu = (gk)^{-1}\sigma gk\sigma = \psi g^{-1}\sigma g\sigma = \psi\varphi$.

Now $(f^*)^*$ is σ-equivalent to $f_{f^{2/d}}$, and the later is σ-equivalent to f by Theorem 5.1. □

Theorem 5.5. *Let $f \in \mathcal{A}^*_{p,\sigma}$ with $d_f \neq 0$. The map $R\colon g \mapsto g^{-1}\sigma g\sigma$ sends $\{g\colon gfg^{-1} \in \mathcal{A}^*_{p,\sigma}\}$ onto a subgroup K of $\hat{\mathcal{C}}(f)$. Moreover R induces a one-to-one correspondence between $A_f \cap \mathcal{A}^*_{p,\sigma}/\sim$ and the quotient group $K/K \cap \hat{\mathcal{C}}^2(f)$, and*

$$A_f \cap \mathcal{A}^*_{p,\sigma}/\sim = \begin{cases} \hat{\mathcal{C}}(f)/\hat{\mathcal{C}}^2(f), & p \text{ odd, or } p/q \text{ even} \\ \{f\}, & p/q \text{ odd, } q/\delta \text{ even, and } \delta \text{ odd,} \\ \{f, f_\omega\}, & p/q \text{ odd, } q/\delta \text{ even, and } \delta \text{ even,} \\ \{f, f_{\tau\omega}\}, & p/q \text{ odd, } q/\delta \text{ odd, and } \delta \text{ even.} \end{cases}$$

Proof. Recall that $\mathcal{A}^*_{p,\sigma}$ consists of $\tilde{f} = \sigma\tilde{f}^{-1}\sigma$ of the form $z \to z + \mu_0 z^{p+1} + O(|z|^{p+2})$, with μ_0 being given by (2.1). Then $\mathcal{A}_f \cap \mathcal{A}^*_{p,\sigma}$ consists of maps $\tilde{f} = gfg^{-1}$ with

$$\nu = g^{-1}\sigma g\sigma \in \hat{\mathcal{C}}(f), \quad g'(0)^p = 1. \tag{5.2}$$

Let $G = \{g\colon gfg^{-1} \in A_f \cap \mathcal{A}^*_{p,\sigma}\}$ and $K = R(G)$. We need to show that K is a subgroup of $\hat{\mathcal{C}}(f)$. Take $g_j \in G$ and put $\nu_j = g_j^{-1}\sigma g_j\sigma$. Then $\nu_3 = \nu_1\nu_2$ is still reversible by σ. Hence there is g_3 such that $g_3\nu_1\nu_2\sigma g_3^{-1} = \sigma$. We may also take g_3 satisfying $g_3'(0) = g_1'(0)g_2'(0)$, since $\nu_3'(0) = g_1'(0)^{-2}g_2'(0)^{-2}$. Then $g_3fg_3^{-1}$ is reversible by σ and $g_3'(0)^p = 1$, which implies that $g_3 \in G$. Obviously, we have $R(g_3) = R(g_1)R(g_2)$, from which we conclude that K is a subgroup. Again we know that for $g_1, g_2 \in G$, $kg_1fg_1^{-1}k^{-1} = g_2fg_2^{-1}$ for some $k = \sigma k\sigma$, if and only if $\nu_1^{-1}\nu_2 = \varphi^2$ (and $g_2\varphi g_1^{-1} = k$). Thus $A_f \cap \mathcal{A}^*_{p,\sigma}/\sim$ is the quotient group $K/K \cap \hat{\mathcal{C}}^2(f)$.

We need to find K.

Write $\nu = \tau^a\omega^b$. Since $\tau'(0) = \mu^{pq_0/q}$ and $\omega'(0) = \mu^{pq_0/\delta}$, the derivatives of (5.2) at the origin yield $apq_0/q + bpq_0/\delta = 2l + jp$. Conversely, if a, b satisfy the identity, then $\nu'(0) = \mu^{2l}$. Hence $h\nu\sigma h^{-1} = \sigma$ for some h with $h'(0)^p = \mu^{-pl} = 1$. Therefore,

$$\{(a,b)\colon gfg^{-1} \in A_f \cap \mathcal{A}^*_{p,\sigma}, \tau^a\omega^b\sigma = g^{-1}\sigma g, [g'(0)]^p = 1\}$$

is precisely the set of $(a,b) \in \mathbf{Z}^2$ such that $(2,p)$ divides $(a + bq/\delta)pq_0/q$. Denote by I the set of all such (a,b). Obviously, when p is odd or p/q is even, $I = \mathbf{Z}^2$, i.e., $K = \hat{\mathcal{C}}(f)$, and $K/K \cap \hat{\mathcal{C}}^2(f) = \hat{\mathcal{C}}(f)/\hat{\mathcal{C}}^2(f)$.

Next, we assume that p is even and p/q is odd. Since $(q_0, q) = 1$ then q_0 is odd; consequently, I is the set of (a,b) such that $a + bq/\delta$ is even. When q/δ is even, $I = \{(a,b)|a \in 2\mathbf{Z}\}$, i.e., $K = <\tau^2, \omega>$. When q/δ is odd, $I = \{(a,b)\colon a + b \in 2\mathbf{Z}\}$, i.e., $K = <\tau^2, \tau\omega, \omega^2>$.

Finally, we compute $K/K \cap \hat{\mathcal{C}}^2(f)$. The case that p/q is odd and q/δ is even is further divided into two cases: δ is odd, which implies that $K/K \cap \hat{\mathcal{C}}^2(f) = \{f\}$; δ is even, which implies that $K/K \cap \hat{\mathcal{C}}^2(f) = \{f, f_\omega\}$. When p is even, and $p/q, q/\delta$ are odd (and hence δ is even), we have $K/K \cap \hat{\mathcal{C}}^2(f) = \{f, f_{\tau\omega}\}$. □

Let $\mathcal{A}_p^r$ be the set of $f \in \mathcal{A}_p$ such that f is reversible by some anti-holomorphic involution. We shall identify the moduli space for the equivalence classes of $\mathcal{A}_p^r$ under holomorphic maps, i.e. the moduli space for equivalence classes of $\mathcal{A}_{p,\sigma}$ under holomorphic maps (without preserving σ). As we mentioned in the introduction, the classification of $\mathcal{A}_{p,\sigma}$ for $p = 1$ under holomorphic maps was studied by Nakai [13].

Put $\mathcal{A}_{p,\sigma}^{*-} = \{f^{-1} \colon f \in \mathcal{A}_{p,\sigma}^*\}$ for p even, and put $\mathcal{A}_{p,\sigma}^{*-} = \emptyset$ for p odd. We start with the following

Lemma 5.6. *Let $f \in \mathcal{A}_{p,\sigma}^*$ and $\Phi_f \in \mathcal{M}_{p,\sigma}$. Then $g^{-1}fg = \tilde{f} \in \mathcal{A}_{p,\sigma}^{*-}$ for some holomorphic map g, if and only if*

$$\Phi_{\tilde{f}^{-1},j} = L_{c_{j+1}} \rho \Phi_{f,2k-j+1}^{-1} \rho L_{-c_j}, \quad \rho(\xi) = \overline{\xi}, \quad j = 1, \dots, 2p \tag{5.3}$$

for some $k \in \mathbf{Z}$ and $c_j \in \mathbf{C}$, in which case $g'(0) = \sqrt{\mu}^{2k+1}$ with $\sqrt{\mu}$ being given by (2.2).

Proof. Choose a canonical collection A_j, Ω_j for f with $A_{2p-j+1} = \sigma A_j \sigma$. Choose $\Omega_1, \dots, \Omega_{2p}$ so small that $g^{-1} \colon \Omega_j \to g^{-1}\Omega_j$ are well-defined. Since f and $\tilde{f}^{-1}$ are in $\mathcal{A}_{p,\sigma}^*$, we get $g'(0)^p = -1$, i.e., $g'(0) = \sqrt{\mu}^{2k+1}$. Put $\Omega_j' = \sigma g^{-1} \Omega_{2k+2-j}$ and $A_j' = \rho A_{2k+2-j} g \sigma$. Then A_j' sends Ω_j' onto $\tilde{\Omega}_j' = \rho A_j(\Omega_j)$. Note that $\Omega_{j+1}' \cap \Omega_j' = \sigma g^{-1}(\Omega_{2k-j+2} \cap \Omega_{2k-j+1})$ contains $\sqrt{\mu}^j y_+$. Since $\rho F = F\rho$, $A_{2j+1} f = F A_{2j+1}$, $A_{2j} f = F^{-1} A_{2j}$, and $\sigma \tilde{f} \sigma^{-1} = \tilde{f}^{-1}$, one sees that Ω_j', A_j' form a canonical collection for $\tilde{f}^{-1} \in A_p^*$. A simple computation shows that

$$\Phi_{\tilde{f}^{-1},j} = L_{c_{j+1}'} A_{j+1}' A_j'^{-1} L_{-c_j'} = L_{c_{j+1}'} \rho \Phi_{f,2k-j+1}^{-1} \rho L_{-c_j'},$$

which yields (5.3). Assume now that (5.3) holds. For $\Phi_{f,j} = A_{j+1} A_j^{-1}$ and $\Phi_{f^{-1},j} = \tilde{A}_{j+1} \tilde{A}_j^{-1}$, rewrite (5.3) as

$$A_{2k+1-j}^{-1} \rho L_{-c_{j+1}} \tilde{A}_{j+1} \sigma = g = A_{2k+2-j}^{-1} \rho L_{-c_j} \tilde{A}_j \sigma.$$

Then g extends to a biholomorphic map defined near the origin. One readily verifies that $g^{-1} f g = \sigma \tilde{f}^{-1} \sigma = \tilde{f}$. Obviously, $g(y_+)$ is contained in $\Omega_{2k+1} \cap \Omega_{2k+2}$. Since g transforms $f \in \mathcal{A}_{p,\sigma}^*$ to $\tilde{f} \in \mathcal{A}_{p,\sigma}^-$, then $g'(0) = \sqrt{\mu}^{2l+1}$. Now one sees that $k = l \mod 2p$. □

Proposition 5.7. *Let p be even. Then $f \in \mathcal{A}_{p,\sigma}^*$ is holomorphic equivalent to some element in $\mathcal{A}_{p,\sigma}^{*-}$, if and only if $\Phi_f \in \mathcal{M}_{p,\sigma}$ satisfies*

$$\Phi_{f,j} = L_{c_{j+1}} \Phi_{f,j+4k+2} L_{-c_j}, \quad j = 1, \dots, 2p \tag{5.4}$$

for some $k \in \mathbf{Z}$ and $c_j \in \mathbf{C}$.

Proof. The proposition holds for $d_f = 0$. We assume $d_f \neq 0$.

Assume that (5.3) holds. Then

$$\begin{aligned} L_{c_{2k+j+2}} \rho \Phi_{f,-j}^{-1} \rho L_{-c_{2k+j+1}} &= \Phi_{\tilde{f}^{-1},j+2k+1} = \sigma \Phi_{\tilde{f}^{-1},-j-2k-1}^{-1} \sigma \\ &= L_{-\overline{c}_{-2k-j-1}} \rho \Phi_{f,-4k-j-2}^{-1} \rho L_{\overline{c}_{-2k-j}}, \end{aligned}$$

from which (5.4) follows.

Assume now that (5.4) holds. We first notice that we may replace $\Phi_{f,j}$ with $L_{c_{j+1}} \Phi_{f,j} L_{-c_j}$ such that the resulted $\Phi_{f,j} = \xi_j + \sum_{k=0}^{\infty} c_{j,k} e^{\epsilon_k 2\pi i k}$ satisfies $c_{1,0} = \cdots = c_{2p,0}$. We still have

(5.4), in which we now have $c_1 = \cdots = c_{2p} = c$. We know that L_{pc} commutes with all $\Phi_{f,j}$. Hence c is real. Put $\tilde{\Phi}_{2k-j+1} = L_{\frac{c}{2}}\rho\Phi_{f,j}^{-1}\rho L_{-\frac{c}{2}}$. Then $\tilde{\Phi}$ is in $\mathcal{M}_{p,\sigma}$, since

$$\begin{aligned}\sigma\tilde{\Phi}_{2k-j+1}\sigma &= L_{-c/2}\rho\Phi_{f,-j}\rho L_{c/2}\\ &= L_{c/2}\rho\Phi_{f,-j+4k+2}\rho L_{-c/2} = \tilde{\Phi}_{-2k+j-1}^{-1}.\end{aligned}$$

□

Let $\mathcal{M}_p^r$ be the disjoint union $\mathcal{M}_{p,\sigma}^+ \sqcup \mathcal{M}_{p,\sigma}^-$, where $\mathcal{M}_{p,\sigma}^- = \mathcal{M}_{p,\sigma}$ when p is even and $\mathcal{M}_{p,\sigma}^- = \emptyset$ when p is odd, and $\mathcal{M}_{p,\sigma}^+ = \mathcal{M}_{p,\sigma}$. We say that $\Phi, \tilde{\Phi} \in \mathcal{M}_p^r$ are equivalent, if there are $k \in \mathbf{Z}$ and $c_j \in \mathbf{C}$ satisfying either $\tilde{\Phi}_j = L_{c_{j+1}}\Phi_{j+2k}L_{-c_j}$ when $\Phi, \tilde{\Phi}$ both are in $\mathcal{M}_{p,\sigma}^+$ or in $\mathcal{M}_{p,\sigma}^-$, or $\tilde{\Phi}_j = L_{c_{j+1}}\rho\Phi_{2k-j+1}^{-1}\rho L_{-c_j}$ otherwise.

Define

$$\Phi_f^r = \begin{cases}\Phi_f \in \mathcal{M}_{p,\sigma}^+, & f \in \mathcal{A}_{p,\sigma}^*,\\ \Phi_{f^{-1}} \in \mathcal{M}_{p,\sigma}^-, & f \in \mathcal{A}_{p,\sigma}^{*-}.\end{cases}$$

We have

Theorem 5.8. *There is a one-to-one correspondence between the holomorphic equivalence classes of reversible maps $\mathcal{A}_p^r$ and ones of $\mathcal{M}_p^r$.*

Proof. For $f \in \mathcal{A}_p^r$ with $\sigma_1 f\sigma_1 = f^{-1}$ for some anti-holomorphic involution σ_1, choose a biholomorphism g such that $f_1 = gfg^{-1} \in \mathcal{A}_{p,\sigma}^* \cup \mathcal{A}_{p,\sigma}^{*-}$, and $g\sigma_1 g^{-1} = \sigma$. Define $\Phi_f = \Phi_{f_1}^r$.

Assume now that $\tilde{f} = hfh^{-1}$ and $\tilde{\sigma}_1\tilde{f}\tilde{\sigma}_1 = \tilde{f}^{-1}$ for some anti-holomorphic involution $\tilde{\sigma}_1$. Let $\tilde{g}\tilde{\sigma}_1\tilde{g}^{-1} = \sigma$ and $\tilde{f}_1 = \tilde{g}\tilde{f}\tilde{g}^{-1} \in \mathcal{A}_{p,\sigma}^* \cup \mathcal{A}_{p,\sigma}^{*-}$. Then $\Phi_{\tilde{f}} = \Phi_{\tilde{f}_1}^r$.

We need to show that moduli functions Φ_f and $\Phi_{\tilde{f}}$ are equivalent. We have $\tilde{h}f_1\tilde{h}^{-1} = \tilde{f}_1$ for $\tilde{h} = \tilde{g}hg^{-1}$. There are two cases: if $f_1, \tilde{f}_1$ both are in $\mathcal{A}_{p,\sigma}^*$ or in $\mathcal{A}_{p,\sigma}^-$, then, say for the former, $\Phi_{\tilde{f}_1} = L_{c_{j+1}}\Phi_{f_1,j+2k}L_{-c_j}$ for some $k \in \mathbf{Z}$ and $c_j \in \mathbf{C}$; if $f_1 \in \mathcal{A}_{p,\sigma}^*$ and $\tilde{f}_1 \in \mathcal{A}_{p,\sigma}^{*-}$, or $f_1 \in \mathcal{A}_{p,\sigma}^{*-}$ and $\tilde{f}_1 \in \mathcal{A}_{p,\sigma}^*$, then Lemma 5.6 says that, say for the former, $\Phi_{\tilde{f}_1^{-1},j} = L_{c_{j+1}}\Phi_{f,2k-j+1}L_{-c_j}$ for some $k \in \mathbf{Z}$ and $c_j \in \mathbf{C}$.

Conversely, if $f, \tilde{f}$ are in $\mathcal{A}_p^r$, and if Φ_f and $\Phi_{\tilde{f}}$ are equivalent, then Theorem 2.5 and Lemma 5.6 imply that $f, \tilde{f}$ are holomorphically equivalent. The realization follows from Theorem 4.2. □

In this section and preceding one, we have seen mappings in $\mathcal{A}_p$ that are reversible either by σ (a fixed anti-holomorphic involution), or by some anti-holomorphic involution, and their classifications. By Theorem 1.4, we also know that when $d_f \neq 0$, f is reversible by some anti-holomorphic map H, then H is actually an involution.

Next we want to show the existence of $f \in \mathcal{A}_p^*$ that is reversible by some formal anti-holomorphic involution but not by any convergent holomorphic involution.

Proof of Theorem 1.5. We first describe the set of $f \in \mathcal{A}_p$ that are reversible by some anti-holomorphic map. Let $H(z) = c\overline{z} + O(|z|^2)$ be an anti-holomorphic map so that $f^{-1} = HfH^{-1}$. Write $f(z) = z + f_{p+1}z^{p+1} + O(|z|)^{p+2}$. Then $HfH^{-1}(z) = z + \overline{\mu}_0 c\overline{c}^{-p-1}z^{p+1} + O(|z|^{p+2})$. Thus $c^p = -\mu_0/\overline{\mu}_0 = (-1)^p$, i.e., $c = -\mu^k$. Let A_j, Ω_j be a canonical collection of f. Then $H\Omega_j$ contains $H(\sqrt{\mu}^j y_+) = \sqrt{\mu}^{2k-j} \subset \Omega_{2k-j} \cap \Omega_{2k-j+1}$. Thus $A'_{2k-j+1} = \sigma A_j H^{-1}$

and Ω'_{2k-j+1} form another canonical collection of f; consequently, $A_{2k-j+1} = L_{c_j}\sigma A_j H^{-1}$, i.e.,

$$\Phi^{-1}_{f,2k-j} = L_{c_{j+1}}\sigma\Phi_{f,j}\sigma L_{-c_j}. \tag{5.5}$$

Thus $\Phi^{-1}_{f,2k-j} = L_{c_{j+1}-\overline{c}_{2k-j}}\Phi^{-1}_{2k-j}L_{\overline{c}_{2k-j+1}-c_j}$ and $c_j - \overline{c}_{2k-j+1} = l/d$. Consequently, $l = 0$ and $H = A^{-1}_{2k-j+1}L_{c_j}\sigma A_j$ is an involution. (Notice that this give another proof via moduli functions for Theorem 1.4 i.)

Conversely, if Φ_f satisfies (5.5), then $H = A^{-1}_{2k-j+1}L_{c_j}\sigma A_j$ defines an anti-holomorphic map satisfying $f^{-1} = HfH^{-1}$.

Now take

$$\Phi_1(\xi) = \xi + \alpha + \beta e^{\epsilon_p 2\pi i\xi}, \quad \alpha \in \mathbf{R},\ \beta \neq 0,$$
$$\Phi_j = \xi + \alpha, \quad j \neq 1 \mod (2p).$$

Since $\beta \neq 0$, then $\Phi_1^{-1} \neq \xi - \alpha + \gamma e^{\epsilon_p 2\pi i\xi}$ for any γ. Thus (5.5) does not hold. Let $f \in \mathcal{A}^*_p$ be a realization of Φ. By [9], we know that $f(z) = v^1(z) + O(|z|^{2p+2})$ where $v^1(z)$ is the time-one map of $v = \frac{\mu_0 z^{p+1}}{1-i\lambda z^p}\frac{\partial}{\partial z}$ with $\lambda \in \mathbf{R}$. Since f is formally equivalent to v^1 and $\tau_* v = -v$, then f is reversible by some formal anti-holomorphic involution. □

6. real analytic maps on $\mathbf{R}$

Denote by $\mathbf{R}_p$ be the set of real analytic maps on $\mathbf{R}$ of the form

$$f(x) = x + f_{p+1}x^{p+1} + O(|x|^{p+2}), \quad f_{p+1} \neq 0.$$

Let $\mathbf{R}_p/\sim$ be real analytic equivalence classes of $\mathcal{A}_p$ under real analytic changes of coordinates on $\mathbf{R}$. For each $f \in \mathbf{R}_p$, there is a $\mathbf{R}$-linear map g such that $\tilde{f} = g^{-1}fg$ satisfies $\tilde{f}_{p+1} = 1$ for p odd and $\tilde{f}_{p+1} = \pm 1$ for p even.

To apply results of section two, we shall introduce new $\mathbf{C}$-linear coordinates such that all real analytic maps become elements in $\mathcal{A}^*_p$, which leads us to a new anti-holomorphic involution σ_1 which might be different from ρ.

When p is odd, let $\mathcal{I}^+_{p,\sigma_1}$ be the set of holomorphic maps $f \in \mathcal{A}^*_p$ with $f\sigma_1 = \sigma_1 f$ with $\sigma_1(z) = \overline{z}$. Denote $\tilde{f} \overset{\sigma_1}{\sim} f$ if $\tilde{f} = gfg^{-1}$ for some biholomorphic map g with $g\sigma_1 = \sigma_1 g$. When p is even, for each $f \in \mathbf{R}_p$ with $f_{p+1} = 1$, we see that $\tilde{f} = R^{-1}_{e^{\pi i/(2p)}}fR_{e^{\pi i/(2p)}}$ is in $\mathcal{A}^*_p$ and satisfies $\tilde{f}\sigma_1 = \sigma_1\tilde{f}$ for $\sigma_1(z) = e^{-\pi i/p}\overline{z}$. In general, for $f \in \mathbf{R}_p$, we have

$$R^{-1}_{e^{\pi i/(2p)}}fR_{e^{\pi i/(2p)}}(z) = z + a\mu_0 z^{p+1} + O(|z|^{p+2}), \quad a \in \mathbf{R}. \tag{6.1}$$

Let $\mathcal{I}^+_{p,\sigma_1}$ be the set of $f \in \mathcal{A}^*_p$ with $f\sigma_1 = \sigma_1 f$. Let $\mathcal{I}^-_{p,\sigma_1}$ be the set of $f \in \mathcal{A}_p$ with $f^{-1} \in \mathcal{A}^*_p$ and $f\sigma_1 = \sigma_1 f$. Thus $\mathbf{R}_p/\sim = \mathcal{I}^+_{p,\sigma_1}/\overset{\sigma_1}{\sim}$, when p is odd. When p is even, $\mathbf{R}_p/\sim$ is the disjoint union of $\mathcal{I}^+_{p,\sigma_1}/\overset{\sigma_1}{\sim}$ and $\mathcal{I}^-_{p,\sigma_1}/\overset{\sigma_1}{\sim}$. Note that we can identify $\mathcal{I}^-_{p,\sigma}$ with $\mathcal{I}^+_{p,\sigma_1}$ by mapping f to f^{-1}, when p is even.

Lemma 6.1. *Let $\rho(z) = \overline{z}$. Let*

$$m = \begin{cases} (p+1)/2 & p \text{ odd}, \\ p/2+1 & p = 2 \mod 4, \\ p/2 & p = 0 \mod 4. \end{cases} \tag{6.2}$$

For each $f \in \mathcal{I}^+_{p,\sigma_1}$, there exists a canonical collection Ω_j, A_j such that A_j is conformal on $\Omega_j \cup \sigma_1(\Omega_{2m-j})$, and $A_{2m-j} = \rho A_j \sigma_1$ on Ω_{2m-j}. Moreover, for any $\epsilon > 0$,

$$\bigcup_{j=1}^{p} \{A_{2j+1}^{-1}(R(\epsilon)) \cup \sigma A_{2j}^{-1}(L(\epsilon))\}$$

is a punctured neighborhood of the origin, and is contained in any given neighborhood of the origin when ϵ is sufficiently small.

Proof. Let Ω_j, A_j be a canonical collection of f satisfying Ii-v. One readily sees that for some m

$$\sigma_1(\Omega_{2m-j}) = \Omega'_j, \quad \rho A_{2m-j}\sigma_1 = A'_j, \quad j = 1, \dots, 2p$$

are a canonical collection of f also. By the uniqueness of the canonical collection, there exists a canonical collection $A''_j, \Omega''_j \subset \Omega_j \cap \Omega'_j$ such that $A''_j = A_j|_{\Omega''_j}$ and $A'_j = A''_j + c_j$ for $1 \le j \le 2p$. Thus $\rho A_{2m-j}\sigma_1 = A_j + c_j$. From $\rho^2 = \mathrm{Id}$ and $\sigma_1^2 = \mathrm{Id}$, we get $c_j = -\overline{c}_{2m-j}$. In particular, c_m and c_{m+p} are pure imaginary. Put $\tilde{A}_j = A_j$, $\tilde{A}_{2m-j} = \rho A_j \sigma_1$ for $j \neq m, m+p$ mod $2p$, and $\tilde{A}_j = A_j + c_j/2$ for $j = m, p+m$. Then $\tilde{A}_j, \Omega''_j$ are a canonical collection of f, satisfying $\rho \tilde{A}_j \sigma_1 = \tilde{A}_{2m-j}$.

To find m, we start with $\sigma_1 \Omega'_1 = \Omega_{2m-1}$. If $\sigma_1(z) = \overline{z}$ (and p is odd), then $\sigma_1(y_+) = \sqrt{\mu}^p y_+ \subset \Omega_{2m-1}$, which implies that $2m - 1 = p \mod 2p$. We take $m = (p+1)/2$. Recall that $\sqrt{\mu} = e^{-\pi i/p}$ for $p = 2 \mod 4$ and $\sqrt{\mu} = e^{\pi i/p}$ for $p = 0 \mod 4$. When $\sigma_1(z) = e^{-\pi i/p}\overline{z}$ (and p is even), $\sigma_1(y_+) = -e^{-\pi i/p} y_+ = \sqrt{\mu}^{p+1} y_+ \subset \Omega_{2m-1}$ for $p = 2 \mod 4$, and $\sigma_1(y_+) = \sqrt{\mu}^{p-1} y_+ \subset \Omega_{2m-1}$ for $p = 0 \mod 4$. Therefore, $p+1 = 2m-1$ when $p = 2 \mod 4$, and $p = m$ when $p = 0 \mod 4$. □

Let $f \in \mathcal{I}^+_{p,\sigma_1}$, and let Ω_j, A_j be a canonical collection of f satisfying I*i-v*. Based on Lemma 6.1, we shall always choose A_j satisfying $A_{2m-j} = \rho A_j \sigma_1$ for $j = 1, \dots, 2p$, unless specifying otherwise. In particular, if A'_j, Ω'_j are another canonical collection, there is a canonical collection A''_j, Ω''_j such that on $\Omega''_j \subset \Omega_j \cap \Omega'_j$, $A''_j = A_j$ and $A'_j = A_j + c_j$ with $c_j = \overline{c}_{2m-j}$. Now define $\Phi_{f,j} = A_{j+1} A_j^{-1}$, so $\Phi_{f,2m-j-1}^{-1} = \rho \Phi_{f,j} \rho$. Note that $\Phi_{f,j}$ are well-defined up to the form $L_{c_{j+1}} \Phi_{f,j} L_{-c_j}$ with $c_{2m-j} = \overline{c}_j$.

Let $\mathcal{N}_{p,\sigma_1}$ be the set of $\Phi \in \mathcal{M}_p$ satisfying $\Phi_{2m-j-1}^{-1} = \rho \Phi_j \rho, j = 1 \dots, 2p$, for m given by Lemma 6.1. Note that

$$\sigma_1 = -\sqrt{\mu}^{2m-1}\overline{z}.$$

We now have

Theorem 6.2. *Let $f, \tilde{f} \in \mathcal{I}^+_{p,\sigma_1}$. If $\tilde{f} = g^{-1} f g$ for some holomorphic mapping g commuting with σ_1, there exist $c_j = \overline{c}_{2m-j} \in \mathbf{C}$ such that one of the following holds*

i) $g'(0) = 1$ and

$$\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j} L_{-c_j}, \quad j = 1, \dots, 2p; \tag{6.3}$$

ii) $g'(0) = -1$, p is even, and

$$\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j+p} L_{-c_j}, \quad j = 1, \dots, 2p. \tag{6.4}$$

Conversely, if (6.3) *holds, then* $\tilde{f} = g^{-1}fg$ *for some* g *with* $g\sigma_1 = \sigma_1 g$ *and* $g'(0) = 1$*; if* (6.4) *holds then* $\tilde{f} = g^{-1}fg$ *for some* g *with* $g\sigma_1 = \sigma_1 g$ *and* $g'(0) = -1$*. For each* $\Phi \in \mathcal{N}_{p,\sigma_1}$*, there exists* $f \in \mathcal{I}^+_{p,\sigma_1}$ *with* $\Phi_{f,j} = \Phi_j$ *for all* j*.*

Proof. Let $\tilde{f}, f \in \mathcal{I}^+_{p,\sigma_1}$. Assume that $\tilde{f} = g^{-1}fg$ with $g\sigma_1 = \sigma_1 g$. Then $g'(0) = 1$ if p is odd, and $g'(0) = 1$ or -1 when p is even.

Let Ω_j, A_j be a canonical collection of f with $A_{2m-j} = \rho A_j \sigma_1$, and let $\tilde{\Omega}_j, \tilde{A}_j$ be a canonical collection of $\tilde{f}$ with $\tilde{A}_{2m-j} = \rho \tilde{A}_j \sigma_1$

We first consider the case $g'(0) = 1$. Then $A'_j = A_j g$ and $g^{-1}(\Omega_j)$ are a canonical collection of $\tilde{f}$ (satisfying $A'_{2m-j} = \rho A'_j \sigma_1$). Thus, A'_j, together with some domains $\Omega'_j \subset \tilde{\Omega}_j \cap g^{-1}(\Omega_j)$, form a canonical collection of $\tilde{f}$, and hence $\tilde{A}_j = A_j g + c_j$ for some $c_j = \overline{c}_{2m-j}$. In other words, $\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j} L_{-c_j}$ with $c_{2m-j} = \overline{c}_j$.

Next, consider the case that $g'(0) = -1$ and p is even. Now $\Omega'_j = g^{-1}(\Omega_{j+p})$ and $A'_j = A_{j+p} g$ are a canonical collection of $\tilde{f}$. Thus A'_j, together with some domains $\Omega'_j \subset \tilde{\Omega}_j \cap g^{-1}(\Omega_{j+p})$, form a canonical collection of $\tilde{f}$, and $\tilde{A}_j = A_{j+p} g + c_j$ for some $c_j = \overline{c}_{2m-j}$. We have $\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j+p} L_{-c_j}$ for $j = 1, \ldots, 2p$.

Next, assume that $\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j} L_{-c_j}$ with $c_{2m-j} = \overline{c}_j$. Recall that $\Phi_{\tilde{f},j} = \tilde{A}_{j+1} \tilde{A}_j^{-1}$, $\Phi_{f,j} = A_{j+1} A_j^{-1}$ with $\tilde{A}_{2m-j} = \rho \tilde{A}_j \sigma_1$ and $A_{2m-j} = \rho A_j \sigma_1$. Replacing $\tilde{A}_j$ with $L_{-c_j} \tilde{A}_j$, we may assume that $\tilde{A}_{j+1} \tilde{A}_j^{-1} = A_{j+1} A_j^{-1}$. We know that $g = A_j^{-1} \tilde{A}_j$ $(j = 1, \ldots, 2p)$ extends to a holomorphic mapping defined near the origin and $\tilde{f} = g^{-1} f g$. Also $\sigma_1 g \sigma_1 = \sigma_1 A_1^{-1} \tilde{A}_1 \sigma_1 = A_{2m-1}^{-1} \tilde{A}_{2m-1} = g$ on $\tilde{\Omega}_{2m-1} \cap A_{2m-1}^{-1} \tilde{A}_{2m-1}(\tilde{\Omega}_{2m-1})$, and hence in a neighborhood of the origin.

Assume now that p is even, and that $\Phi_{\tilde{f},j} = L_{c_{j+1}} \Phi_{f,j+p} L_{-c_j}$ with $c_{2m-j} = \overline{c}_j$. Again, we may assume that all $c_j = 0$. Let $g = A_{j+p}^{-1} \tilde{A}_j$. Then g extends to a biholomorphic mapping defined near the origin. We have $\tilde{f} = g^{-1} f g$. Also $\sigma_1 g \sigma_1 = \sigma A_{1+p}^{-1} \tilde{A}_1 \sigma_1 = A_{2m-p-1}^{-1} \tilde{A}_{2m-1} = g$ on $\tilde{\Omega}_{2m-1} \cap A_{2m-p-1}^{-1}(\tilde{A}_{2m-1}(\tilde{\Omega}_{2m-1}))$. Therefore, $g\sigma_1 = \sigma_1 g$.

Given $\Phi \in \mathcal{N}_{p,\sigma_1}$, we need to find $f \in \mathcal{I}^+_{p,\sigma_1}$ such that $A_{j+1} A_j^{-1} = \Phi_j$, where $A_j = \rho A_{2m-j} \sigma_1, j = 1, \ldots, 2p$, together with some domains $\Omega_1, \ldots, \Omega_{2p}$, are a canonical collection of f. By Theorem 2.5, there exists $f \in \mathcal{A}_p^*$ such that $\Phi_j = A_{j+1} A_j^{-1} = \Phi_{f,j}$, where A_j, Ω_j form a canonical collection of f. Since $\Phi_{2m-j-1}^{-1} = \rho \Phi_j \rho$, we have $\rho A_{j+1} A_j^{-1} = A_{2m-j-1} A_{2m-j}^{-1} \rho$ on some half-plane $\Pi_{p,j}(\rho_1)$. Hence,

$$A_{2m-j}^{-1} \rho A_j = A_{2m-j-1}^{-1} \rho A_{j+1}$$

holds on

$$\begin{aligned} A_j^{-1}(\Pi_{p,j}(\rho_1)) &\supset A_j^{-1}(\Pi_{p,j}(\rho_1)) \cap A_{j+1}^{-1}(\Pi_{p,j}(\rho_1)) \\ &= A_j^{-1}((-1)^{j-1} R(\rho_1)) \cap A_{j+1}^{-1}((-1)^j R(\rho_1)). \end{aligned}$$

Put $\omega_j = A_j^{-1}((-1)^{j-1} R(\rho_1))$. We know that $\omega_j \cap \omega_k = \emptyset$ for $k \neq j-1, j+1$. Hence, $\hat{\sigma}_1 = A_{2m-j}^{-1} \rho A_j$ is well-defined on $\cup \omega_j$. By removable singularities, $\hat{\sigma}_1$ extends to an anti-holomorphic mapping defined near the origin. On ω_1 we have $\hat{\sigma}_1^2 = \mathrm{Id}$. Hence, $\hat{\sigma}_1$ is an anti-holomorphic involution.

Note that

$$\hat{\sigma}_1 f \hat{\sigma}_1 = \hat{\sigma}_1 A_1^{-1} F A_1 \hat{\sigma}_1 = A_{2m-1}^{-1} \rho F \rho A_{2m-1} = f.$$

Put $\hat{\sigma}_1(z) = c\overline{z} + O(|z|^2)$. We have

$$f(z) = z + \mu_0 z^{p+1} + O(|z|^{p+2}),$$
$$\hat{\sigma}_1^{-1} f \hat{\sigma}_1 = z + \overline{\mu}_0 \overline{c}^p z^{p+1} + O(|z|^{p+2}).$$

By the choice of μ_0, we have $\mu_0 = (-1)^{p+1}\overline{\mu}_0$; consequently, $c = -\mu^{k+1/2}$ for some $0 \le k < p$ and $\hat{\sigma}_1(y_+) = \mu^{k+1/2} y_+$. One the other hand, $\hat{\sigma}_1 = A_{2m-1}^{-1}\rho A_1$ implies that $\hat{\sigma}_1(y_+)$ is contained in Ω_{2m-1}. Therefore, $k = m-1$ by Iii), and hence $\hat{\sigma}_1(z) = \sigma_1(z) + O(|z|^2)$.

Now $g(z) = \frac{1}{2}(z + \sigma_1\hat{\sigma}_1(z)) = z + O(|z|^2)$ is biholomorphic, and $g\hat{\sigma}_1 = \sigma_1 g$. Thus $\sigma_1 \tilde{f} \sigma_1 = \tilde{f}$ for $\tilde{f} = gfg^{-1}$. We now have $\Phi_{\tilde{f},j} = \tilde{A}_{j+1}\tilde{A}_j^{-1}$ and $\Phi_{\tilde{f},2m-j-1}^{-1} = \rho\Phi_{\tilde{f},j}\rho$. Since $\tilde{f} = gfg^{-1}$ and $g'(0) = 1$, Theorem 2.5 says $\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_j L_{-c_j}$. Using $\rho\Phi_j\rho = \Phi_{2m-j-1}^{-1}$, we get

$$L_{c_{j+1}}\Phi_j L_{-c_j} = \Phi_{\tilde{f},j} = \rho L_{c_{2m-1-j}} \Phi_{2m-1-j}^{-1} L_{-c_{2m-j}} \rho = L_{\overline{c}_{2m-1-j}} \Phi_j L_{-\overline{c}_{2m-j}}.$$

Thus $c' = c_j - \overline{c}_{2m-j}$ for all j and $L_{-c'}\Phi_j L_{c'} = \Phi_j$. In particular, $c' = \frac{i}{p}\operatorname{Im}(c_1 + \cdots + c_{2p})$ is pure imaginary. When $d_\Phi = 0$, we put $A'_j = L_{-c_j + \frac{c'}{2}}\tilde{A}_j$ for $j = 1, \ldots, 2p$. Clearly, $A'_{j+1}A_j^{'-1} = \Phi_j$. Now $c_j - \frac{c'}{2} = -\overline{c}_{2m-j} + \frac{c'}{2}$ implies that $\rho A'_j \sigma_1 = \tilde{A}'_{2m-j} - \overline{c}_j - \frac{c'}{2} = A'_{2m-j}$, which gives us the desired realization. When $d = d_\Phi = d_{\Phi_f} \neq 0$, then $c' = 0$, i.e., $c_j = \overline{c}_{2m-j}$. So $\Phi_{\tilde{f}}$ is already the desired realization. □

In contrast to Theorem 1.4 ii, we shall prove the following

Theorem 6.3. *There exists $f \in \mathcal{A}_p^*$ that admits no invariant smooth real analytic curve, while it admits invariant formal smooth curves.*

Proof. Let g be in $\mathcal{A}_p^*$. Assume that H is an anti-holomorphic involution satisfying $Hg = gH$. Then $H(z) = -\mu^{m+k}\overline{z} + (|z|^2)$, where m is given by (6.2). Let A_j, Ω_j be a canonical collection of g. Then $A'_{2m+2k-j} = \rho A_j H^{-1}$ and $\Omega_{2m+2k-j}$ are a canonical collection of g also. Thus $A_{2m+2k-j} = L_{c_j}\rho A_j H^{-1}$, and $c_{2m+2k-j} = \overline{c}_j$ since $H^2 = \mathrm{Id}$. Thus

$$\Phi_{g,2m+2k-j-1}^{-1} = L_{c_{j+1}}\rho\Phi_{g,j}\rho L_{-c_j}, \quad c_{2m+2k-j} = \overline{c}_j. \tag{6.5}$$

Conversely, if Φ_g satisfies the above, then $H = A_{2m+2k-j}^{-1}\rho A_j$ defines an anti-holomorphic involution and $Hg = gH$. Let f be a realization of

$$\Phi_1 = \xi + e^{\epsilon_p 2\pi i\xi}, \quad \Phi_j = \xi, \quad j = 2, \ldots, 2p.$$

Since $\Phi_1^{-1} \neq \xi + a + be^{\epsilon_p 2\pi\xi}$, then (6.5) does not hold for any k and c_j, which means that there is no anti-holomorphic H satisfying $Hf = fH$.

By a result of Il'yashenko [9], f is formally equivalent to $\hat{f}(z) = z + z^{p+1}$. It admits formal smooth curves. □

Proof of Corollary 1.6. Assume that $f, f \in \mathcal{A}_p$ with $d_f \neq 0$ are real valued on $\mathbf{R}$. Assume that $\tilde{f} = gfg^{-1}$. We need to show that g is real when $p = 1$ and or $f_3\tilde{f}_3 > 0$ when $p = 2$. The case $p = 1$ follow from Theorem 1.4 iii. When $p = 2$ and $f_3\tilde{f}_3 > 0$, we know that $g'(0)^2 = f_3/\tilde{f}_3$. Thus $g'(0)$ is real; consequently g is real. □

Proof of Theorem 1.7. Assume that $p \geq 3$. We need to find $f, \tilde{f} \in \mathcal{A}_p$ which are real on $\mathbf{R}$, such that $\tilde{f}, f$ are equivalent under a forma real map and a holomorphic map. But they are not equivalent under any real analytic maps of $\mathbf{R}$.

Recall from Theorem 6.2 that if Φ satisfies

$$\Phi_{2m-1-j} = \rho \Phi_j \rho$$

then there is a realization f for Φ satisfying $f = \sigma_1 f \sigma_1$.

Write $p = 2^s p'$ with p' odd. Put $k = 2^s$. Define

$$\rho \Phi^{-1}_{2m-4kj-2} \rho = \Phi_{1+4kj} = \Phi_1 = \xi + e^{\epsilon_1 2\pi i \xi},$$
$$\Phi_j = \xi, \quad j \neq 1 + 4kj, 2m - 4kj - 2.$$

Then $\rho \Phi_j \rho = \Phi^{-1}_{2m-1-j}$ for all j. Let $\tilde{\Phi}_j = \Phi_{j+2k}$. Then $\rho \tilde{\Phi}_j \rho = \rho \Phi_{j+2k} \rho = \Phi^{-1}_{2m-1-j-2k} = \tilde{\Phi}_{2m-1-j-4k} = \tilde{\Phi}_{2m-1-j}$. Thus there is $\tilde{f} \in \mathcal{I}_{p,\sigma_1}$ such that $\Phi_{\tilde{f}} = \tilde{\Phi}$. Obviously, $\tilde{f} = g^{-1} f g$ for some holomorphic g with $g'(0) = \mu^k$.

Assume now that there is a biholomorphism g commuting with σ_1 such that $g f g^{-1} = \tilde{f}$. By Theorem 6.2, there are two cases. The first case is $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_j L_{-c_j}$, which contradicts to that Φ_{1+2k} is a polynomial in ξ, while Φ_1 is not. The second case says that p is even and $\Phi_{1+4k} = L_{c_{2+2k}} \Phi_{1+2k+p} L_{-c_{1+2k}}$, which is a contradiction since $2k + p = (2 + p')k \neq 4jk \mod 2kp'$ and $1 + 2k + p \neq 2m - 4kj - 2 \mod 2p$ since p is even.

Since $\tilde{f} = g^{-1} f g$ and $g'(0) = \mu^k$. We know that $\tilde{f}_{p+1} = f_k$. Since $\tilde{f}_{2p+1}/\tilde{f}^2_{p+1} = f_{2p+1}/f^2_{p+1}$, then there exists a unique formal holomorphic map h with coefficients $h_1 = 1$ and $h_{p+1} = 0$ so that $\tilde{f} = h^{-1} f h$. Since σ_1 commutes with both f and $\tilde{f}$, then $\tilde{f} = \sigma_1 h^{-1} \sigma_1 f \sigma_1 h \sigma_1$. By the uniqueness of h, we obtain $h = \sigma_1 h \sigma_1$. This shows that $\tilde{f}$ and f are equivalent under a formal transformation commuting with σ_1. □

Proof of Theorem 1.8. Assume that f have two invariant smooth real analytic curves γ_j that are tangent at the origin of $\mathbf{C}$. Let σ_j be the corresponding anti-holomorphic involutions generated by γ_j. Without loss of generality, we may assume that f preserves the real axis, i.e. that $f\rho = \rho f$ for $\rho(z) = \overline{z}$. We need to identify f and their invariant curves.

Note that $\sigma_1 \sigma_2 = h \in \mathcal{A}_p$, and f is a centralizer of h. In particular $f'(0) = \mu^j$. Hence $f'(0) = 1$ when p is odd and $f'(0) = \pm 1$ when p is even.

Consider the case that $d_h \neq 0$ first. By Lemma 3.1, we know that $\sigma_1 f \sigma_1 = f^{-1}$. Since f commutes with σ_1. We get $f^2 = \mathrm{Id}$. Then $g(z) = (z - f(z))/2$ commutes with ρ and transforms f into $I(z) = -z$. Let γ be a smooth real analytic curve invariant by I. We know that γ is given by $y = A(x)$, or by $x = B(y)$, where A, B are real-valued analytic functions defined near the origin of $\mathbf{R}$ satisfying $A(0) = B(0) = 0 = B'(0)$. It is easy to see that A and B are odd. We obtain case i).

Assume now that $d_h = 0$ and that $f^2 \neq \mathrm{Id}$. Consider the case that $f'(0) = 1$ first. Then $f \in \mathcal{A}_p$ and $d_f = 0$. There exists a real analytic change of coordinates so that f becomes the time-one map of $v_{\epsilon_p,\lambda} = \frac{\epsilon_p z^{p+1}}{1+\lambda z^p}$, where $\epsilon_p = 1$ when p is odd and $\epsilon_p = \pm 1$ when p is even. Since $v^1_{\epsilon_p,\lambda} = \rho v^1_{\epsilon_p,\lambda} \rho = v^1_{\epsilon_p,\overline{\lambda}}$, then (3.2) implies that λ is real. Let γ be preserved by f and H be the corresponding anti-holomorphic involution. Then $H\rho$ is a centralizer of f, and hence $H(z) = \mu^j v^t_{\epsilon_p,\lambda}(\overline{z})$ for some integer j. Using (3.2) and $\lambda \in \mathbf{R}$, we get $H^2 = f^{t+\bar{t}}$, which implies that $t \in i\mathbf{R}$. Conversely, for $t \in i\mathbf{R}$ we have

$f\mu^j v^t_{\epsilon_p,\lambda}\rho = f\rho\mu^{-j}v^{-t}_{\epsilon_p,\lambda} = \rho\mu^{-j}v^{-t}_{\epsilon_p,\lambda}f = \mu^j v^t_{\epsilon_p,\lambda}\rho f$. So H and f commute. We obtain case ii).

Consider now the case that $f'(0) = -1$ (and hence p is even). By Theorem 6.2, there is a real analytic change of coordinates so that $f^2 = v^1_{\epsilon_p,\lambda}$. Since f is a centralizer of f^2, then $f = Iv^{1/2}_{\epsilon_p,\lambda}$. Since ρ and If commute, then (3.2) says that $\tilde{\lambda}$ is real. Assume now that H is an anti-holomorphic involution commuting with f. Then H commutes with f^2, and we get $H = \mu^j v^t_{\epsilon_p,\lambda}\rho$ with $t \in i\mathbf{R}$. Conversely, such H commutes with f. We obtain case iii).

For iv), assume that g is biholomorphic and $g^{-1}fg$ is real on the real axis. Then $g(\mathbf{R})$ is invariant under f. In case i), we know that $g(\mathbf{R})$ is a reparameterization of some $(\mathrm{Id} + iA)(\mathbf{R})$ or $(i\,\mathrm{Id} + B)(\mathbf{R})$, which means that $g = h + \mathrm{Id}\, A \circ h$ or $g = ih + B \circ h$. For case ii), the fixed points of $H(z) = \mu^j v^t_{\epsilon_p,\lambda}(\overline{z})$ (with $t \in i\mathbf{R}$ and $\lambda \in \mathbf{R}$) are $v^{t/2}_{\epsilon_p,\lambda}(\mu^{j/2}\mathbf{R})$. Thus $g(z) = v^{t/2}_{\epsilon_p,\lambda}(\mu^{j/2}h(z))$. The case iii) we get $g(z) = v^{t/2}_{\epsilon_p,\lambda}(\mu^{j/2}h(z))$. □

7. invariant abelian subgroups of germs of holomorphic maps

Let $\mathcal{A}$ be the set of biholomorphic maps f with $f'(0)$ a root of unity. Let $\mathcal{A}_0$ be the set of biholomorphic maps f which are tangent to the identity. Let $F, \tilde{F}$ be two subgroups of $\mathcal{A}$. We say that $F, \tilde{F}$ are holomorphically equivalent, denoting $F \sim \tilde{F}$, if there is a biholomorphic map g such that $g^{-1}Fg \equiv \{g^{-1}fg \colon f \in F\} = \tilde{F}$; when g is tangent to the identity additionally, we denote $F \overset{*}{\sim} \tilde{F}$. Denote by $\mathcal{F}_p$ be the set of abelian subgroups F of $\mathcal{A}$ such that $F \cap \mathcal{A}_p$ is non-empty. Let $\mathcal{F}_p/\sim$ be the set of holomorphic equivalence classes of abelian subgroups of $\mathcal{A}$, and let $\mathcal{F}_p/\overset{*}{\sim}$ be the set of equivalence classes of abelian subgroups of $\mathcal{A}$ under holomorphic maps that are tangent to the identity.

Theorem 1.3 in [7] describes a moduli space for the generators τ, f of groups F in $\mathcal{F}_p$ with $\tau^s = f^k$ and $\tau'(0) = e^{2\pi i/s}$, where $s = |F/F \cap \mathcal{A}_0|$. One can see that that moduli space is also a moduli space for the classification of groups in $\mathcal{F}_p$, if k is subject to $0 \le k < s$.

The main purpose of this section is to describe a moduli space for $F \in \mathcal{F}_p$ in terms of its generator $F \cap \mathcal{A}_0$ and other discrete invariants of the group.

For each $F \in \mathcal{F}_p$, one first notices that $F \cap \mathcal{A}_0$ is an infinite cyclic subgroup generated by some $f \in \mathcal{A}_p$ (f^{-1} is the other generator), and that F is a subgroup of $\hat{\mathcal{C}}(f)$. Therefore, $d_F = d_f$ is well-defined. The classification for $F \in \mathcal{F}_p$ with $d_f = 0$ is already given by [7]. Recall that F is holomorphically equivalent to one of

$$F_{u,t} = <R_{\mu^{p/u}}> \otimes <v^t_\lambda>, \qquad u|p, \quad \lambda \in \mathbf{C}, \quad t \in \mathbf{C},$$

where v^t_λ is the flow of $v_\lambda = \frac{\mu_0 z^{p+1}}{1+\lambda z^p}\frac{\partial}{\partial z}$.

We shall now consider $F \in \mathcal{F}_p$ with $d_F \neq 0$.

Lemma 7.1. *Fix $f \in \mathcal{A}_p$. Let $F \in \mathcal{F}_p$ with $F \cap \mathcal{A}_0 = <f>$. Assume that $d_f \neq 0$. Then $s = |F/F \cap \mathcal{A}_0|$ satisfies*

$$s|q_f, \quad (s, d_f)|\delta_f.$$

Conversely, if s is a positive integer satisfying the above condition, there are exact (s, d_f) groups $F_{s,j}(f) = F \in \mathcal{F}_p$ with $F \cap \mathcal{A}_0 = <f>$ and $s = |F/F \cap \mathcal{A}_0|$. Moreover, if g is biholomorphic, then $gF_{s,u}(f)g^{-1} = F_{s,u}(gfg^{-1})$.

Proof. Let s be the order of $F/(F\cap\mathcal{C}(f))$. We first need to show that s divides $q=q_f$. (Put $d=d_f,\delta=\delta_f$.) To that end note that the map $h\to h\mathcal{C}(f)$ is a homomorphism from F into $\hat{\mathcal{C}}(f)/\mathcal{C}(f)$ and the latter is isomorphic to $\mathcal{R}_q$, which induces an monomorphism from $F/(F\cap\mathcal{C}(f))$ into $\mathcal{R}_q$ by differentiation at the origin. Hence s divides q. Let τ be the unique element in $\hat{\mathcal{C}}(f)$ given by (3.3) for which $\tau'(0)$ has the smallest non-negative argument. Since there is a unique subgroup of R_q having order s, there exists a unique $\tau_1=\tau^{q/s}f^{d'/d}\in F$ with $0\le d'<d$. So $\tau_1^s=f^{s'}$. From $\tau_1^s=f^{s'}$ and $\tau^q=f^{\delta/d}$ we get $\delta+sd'=s'd$. Thus (s,d) divides δ.

To find all such groups F, we start with $\delta=s'd-sd'=s'_0d-sd'_0$ with $0\le d'_0<d/(d,s)$. Consequently, there is a unique $0\le u<(d,s)$ such that $d'=d'_0+\frac{ud}{(d,s)}$, i.e, such that $\tau_1=\tau^{q/s}f^{\frac{d'_0}{d}+\frac{u}{(d,s)}}\in F$. Therefore,

$$F_{s,j}=<\tau^{q/s}f^{\frac{d'_0}{d}+\frac{j}{(d,s)}},f>,\quad j=0,\dots,(d,s)-1 \tag{7.1}$$

are all distinct groups.

Let $\tilde f=gfg^{-1}$. By Proposition 3.5, we know that $\tilde\tau=g\tau g^{-1}$. Thus $g<\tau^{q/s}f^{\frac{d'}{d}},f>g^{-1}=<\tilde\tau^{q/s}\tilde f^{\frac{d_0}{d}},\tilde f>$; in particular, $gF_{s,u}(f)g^{-1}=F_{s,u}(\tilde f)$. □

Proposition 7.2. *Let $f\in\mathcal{A}_p$ with $d_f\ne0$. (d,s) is the number of equivalence classes in $F_{s,j}(f)$ under holomorphic mappings tangent to the identity. The number of the equivalence classes under general biholomorphic mappings is (d,s) if f and f^{-1} are not equivalent, or $1+[(d,s)/2]$. When the latter occurs there exists an involution κ of $\{0,1,\dots,(d,s)-1\}$, depending only on δ,s and d, so that $F_{s,j}(f)$ is equivalent to $F_{s,\tilde j}(f)$, if and only if $\tilde j=\kappa(j)$.*

Proof. Assume that $gF_{s,m}(f)g^{-1}=F_{s,n}(f)$. Then $gfg^{-1}=f$ or f^{-1}. For the former we have $m=n$. In particular the first assertion holds. Assume now that $gfg^{-1}=f^{-1}$. Then $gF_{s,m}(f)g^{-1}=F_{s,m'}(f)$. By Lemma 9.1 below, we know that $q=\delta$ and $g\tau g^{-1}=\tau f^{-2/d}$. Thus

$$gF_{s,m}(f)g^{-1}=<\tau^{\delta/s}f^{-\frac{d'_0}{d}-\frac{m}{(d,s)}-\frac{2\delta}{sd}},f>.$$

This implies that $-\frac{d'_0}{d}-\frac{m}{(d,s)}-\frac{2\delta}{sd}=\frac{d'_0}{d}+\frac{m'}{(d,s)}\mod 1$. Equivalently, $m'=-m-\frac{2d'_0(d,s)}{d}-\frac{2\delta(d,s)}{sd}\mod(d,s)=-m-2s'_0(d,s)/s\mod(d,s)$. This shows that $m\to m'$ is a non-trivial involution when $(d,s)>1$. Write $(d,s)=\alpha d+\beta s$. Then $s'_0=\alpha\delta/(d,s)\mod s/(d,s)$; consequently $s'_0(d,s)/s=\alpha\delta/s\mod 1=k\in\mathbf{Z}$. We get $m'=-m-2k\mod(d,s)$. One readily sees that $m\to m'$ is an involution on $0,1,\dots,(d,s)-1$, which has exactly one fixed point when (d,s) is odd and exactly two fixed points when (d,s) is even. In other words, the number of equivalence classes among $F_{s,j}(f)$ is $1+((d,s)-1)/2$ when (d,s) is odd, and $2+((d,s)-2)/2$ when (d,s) is even. □

In the next three theorems, we describe moduli spaces for groups F in $\mathcal{F}_p$. We shall also separate the moduli spaces into two parts. One part involves some obvious invariants of F, and the other is determined by subgroups $F\cap\mathcal{A}_p$.

To be proved later, some obvious invariants of $F\in\mathcal{A}_p$, under mappings tangent to the identity, are p,q,δ,s,d and $a\in\mathbf{C}^*$ with $\arg a\in[0,\pi)$, where a is determined by $F\cap\mathcal{A}_p=<f>$ and by $f(z)=z+az^{p+1}+O(p+2)$, $s=|F/F\cap\mathcal{A}_0|$, $d=d_f$, $q=q_f$, and

$\delta = \delta_f$; denote this set of groups F by $\mathcal{F}_{p,q,\delta,d,s,a}$, and by $\mathcal{F}_{p,q,\delta,d,s,a}/\overset{*}{\sim}$, the set of equivalence classes under holomorphic mappings tangent to the identity.

Define

$$\hat{\mathcal{M}}^*_{p,q,\delta,d} = \{\Phi \in \mathcal{M}_p \colon d_\Phi = d, \delta_\Phi = \delta, q_\Phi = q\}.$$

Given $\tilde{\Phi}, \Phi \in \hat{\mathcal{M}}^*_{p,q,\delta,d}$, we denote $\tilde{\Phi} \overset{*}{\sim} \Phi$, if $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_j L_{-c_j}$ for some $c_j \in \mathbf{C}$. Let $\mathcal{M}^*_{p,q,\delta,d}/\overset{*}{\sim}$ be the set of equivalence classes.

Theorem 7.3. *Let p, q, δ, d, s be positive integers, and let $a \in \mathbf{C}^*$ with $\arg a \in [0, \pi)$. Assume that $q|p$, $s|q$ and $(d,s)|\delta$. There is a one-to-one correspondence between $\mathcal{F}_{p,q,\delta,d,s,a}/\overset{*}{\sim}$ and (s,d) copies of $\hat{\mathcal{M}}^*_{p,q,\delta,d}/\overset{*}{\sim}$.*

Proof. For $F \in \mathcal{F} = F_p$, take $f_0 \in F \cap \mathcal{A}_p$. Since F is abelian, then $F \cap \mathcal{A}_0$ is a subgroup of $\mathcal{C}(f_0)$ and $F \cap \mathcal{A}_0 = F \cap \mathcal{C}(f_0)$. The latter is an infinite cyclic group, if $d_{f_0} \neq 0$. Now $F \cap \mathcal{A}_0$ has a unique generator $f(z) = z + f_{p+1} z^{p+1} + O(|z|^{p+2})$ with $\arg f_{p+1} \in [0, \pi)$. (Note that f^{-1} is the other generator and $f_{p+1} = a$ is a holomorphic invariant for the equivalence class of F in $\mathcal{F}_p/\overset{*}{\sim}$.) Let $s = s_F$ and $u = u_F$ be the unique integers such that $F = F_{s,u,\tau}(f)$, where τ satisfies $\tau^q = f^{\delta/d}$ so that $\tau'(0)$ is a primitive q-th root of the smallest non-negative argument.

Choose R_λ with $\arg \lambda \in [0, 2\pi/p)$ such that $f_1 = R_\lambda^{-1} f R_\lambda \in \mathcal{A}^*_p$, and define $\Phi_F = \Phi_{f_1}$.

Let $F, \tilde{F} \in \mathcal{F}_p$ with $F \cap \mathcal{A}_p = <f>$ and $\tilde{F} \cap \mathcal{A}_p = <\tilde{f}>$, and $\arg f_{p+1}, \arg \tilde{f}_{p+1} \in [0, \pi)$. Assume that $gFg^{-1} = \tilde{F}$ and $g'(0) = 1$. It is clear that $s_F = |F/(F \cap \mathcal{A}^*_0)| = s_{\tilde{F}}$, which is denoted by s, and that $gfg^{-1} = \tilde{f}$. Now Lemma 7.1 and $g^{-1}Fg = \tilde{F}$ imply that $\mathcal{F}_{s,u}(f) = g\tilde{F}g^{-1} = g\mathcal{F}_{s,\tilde{u}}(g^{-1}fg)g^{-1} = \mathcal{F}_{s,\tilde{u}}(f)$. Hence $\tilde{u} = u$,

From $gfg^{-1} = \tilde{f}$, we see that $f_{p+1} = \tilde{f}_{p+1}$. There is a unique R_λ with $\arg \lambda \in [0, 2\pi/p)$ such that $f_1 = R_\lambda f R_\lambda^{-1}$ and $\tilde{f}_1 = R_\lambda g^{-1} f g R_\lambda^{-1}$ are in $\mathcal{A}^*_p$. Thus for $\tilde{g}_1 = R_\lambda g R_\lambda^{-1}$ we see that $\Phi_{\tilde{f}} = \Phi_{\tilde{g}_1^{-1} f_1 \tilde{g}_1}$ is equivalent to $\Phi_{f_1} = \Phi_f$.

Let $F, \tilde{F} \in \mathcal{F}_p$ be two abelian subgroups such that Φ_F and $\Phi_{\tilde{F}}$ are equivalent. More specifically, $a_F = a_{\tilde{F}} = a$, $s_F = s_{\tilde{F}} = s$, and $\Phi_{f_1,j} = L_{c_{j+1}} \Phi_{\tilde{f}_1,j} L_{-c_j}$ for some constants c_j so that $\tilde{f}_1 = \tilde{g}^{-1} f_1 \tilde{g}$ for some $\tilde{g}$ tangent to the identity, where $f_1 = R_\nu^{-1} f R_\nu$, $\tilde{f}_1 = R_\nu^{-1} \tilde{f} R_\nu$ are in $\mathcal{A}^*_p$ with $f(z) = z + az^{p+1} + O(|z|^{p+2})$, $\tilde{f}(z) = z + az^{p+1} + O(|z|^{p+2})$ generating $F \cap \mathcal{A}_0$ and $\tilde{F} \cap \mathcal{A}_0$, respectively. We have $\tilde{f} = g^{-1} f g$ for $g = R_\nu \tilde{g} R_\nu^{-1}$. By definition we have $F = \mathcal{F}_{s,u}(f)$ and $\tilde{F} = \mathcal{F}_{s,u}(\tilde{f})$. Now Lemma 7.1 says that $\tilde{F} = g^{-1} F g$, which shows that F and $\tilde{F}$ are equivalent.

For the realization, we assume that $\Phi \in \hat{\mathcal{M}}^*_{p,q,\delta,d}$. Take $f \in \mathcal{A}_p$ with $f(z) = z + az^{p+1} + O(|z|^{p+2})$ such that $\Phi_{R_\lambda^{-1} f R_\lambda, j} = \Phi_j$ for some λ with $\arg \lambda \in [0, 2\pi/p)$. Since $s|q$ and $(d,s)|\delta$, then $F_{s,u}(f)$ is a realization. □

Let $\hat{\mathcal{M}}'_{p,q,\delta,d}$ and $\hat{\mathcal{M}}''_{p,q,\delta,d}$ be the set of $\Phi \in \mathcal{M}_p$ satisfy $d_\Phi = d, q_\Phi = q, \delta_\Phi = \delta$ and satisfy $\Phi_{j+2k} \neq L_{c_{j+1}} R_{-1} \Phi_{j+1} R_{-1} L_{c_j}$ $(j = 1, 2, \ldots, 2p)$ for all c_j, k and $\Phi_{j+2k} = L_{c_{j+1}} R_{-1} \Phi_{j+1} R_{-1} L_{c_j}$ $(j = 1, 2, \ldots, 2p)$ for some c_j, k, respectively. If $\tilde{\Phi}, \Phi$ are both in $\hat{\mathcal{M}}'_{p,q,\delta,d}$, or in $\hat{\mathcal{M}}''_{p,q,\delta,d}$, we denote $\tilde{\Phi} \sim \Phi$, if $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_{j+2k} L_{-c_j}$ for some $k \in \mathbf{Z}$ and for all j or $\tilde{\Phi}_{j+2k} = L_{c_{j+1}} R_{-1} \Phi_{j+1} R_{-1} L_{-c_j}$. Let $\mathcal{F}'_{p,q,\delta,d,s}$ be the set of $F \in \mathcal{F}_p$ such that $|F/F \cap \mathcal{A}_p| = s$,

$q_f = q, \delta_f = \delta, d_f = d$ for $F \cap \mathcal{A}_p = <f>$, and such that f and f^{-1} are not holomorphically equivalent. Define $\mathcal{F}'_{p,q,\delta,d,s}$ analogously, but f, f^{-1} are now holomorphically equivalent. Let $\mathcal{F}'_{p,q,\delta,d,s}/\sim$ and $\mathcal{F}''_{p,q,\delta,d,s}/\sim$ the sets of holomorphic equivalence classes.

Theorem 7.4. *There is a one-to-one correspondence between $\mathcal{F}'_{p,q,\delta,d,s}/\sim$ and (s,d) copies of $\hat{\mathcal{M}}'_{p,q,\delta,d}/\sim$.*

Proof. For $F \in \mathcal{F}_p$, let f be the generator of $F \cap \mathcal{A}_0$ with $\arg f_{p+1} \in [0,\pi)$ for $f(z) = z + f_{p+1}z^{p+1} + O(|z|^{p+2})$. We choose λ such that $f_1 = R_\lambda f R_\lambda^{-1} \in \mathcal{A}_p^*$. Define $\Phi_F = \Phi_{f_1}$. Then the equivalence class of Φ_F in $\hat{\mathcal{M}}'_{p,q,\delta,d}/\sim$ is well-defined. Define $s = |F/F \cap \mathcal{A}_0|$. To define $u_F = u$, we first choose an element $f_* \in \mathcal{A}_p^*$ for each *equivalence* class of $\mathcal{A}_p$. Here two elements $f, \tilde{f} \in \mathcal{A}_p$ are equivalent, if $\tilde{f}$ is equivalent to f, or f^{-1}. For each representative f_*, let τ_{f_*} be the unique element such that $\tau_{f_*}^q = f_*^{\delta/d}$ and $\tau'_{f_*}(0)$ has the smallest non-negative argument. Now let f_* be the representative of two generators of $F \cap \mathcal{A}_p$. Then F is equivalent to $F_{s,u,\tau_{f_*}}(f_*)$. Define $u = u_F$. Since f_* and f_*^{-1} are not equivalent, Proposition 7.2 says that u depends only on the equivalence class of F.

Assume that $\tilde{F} = gFg^{-1}$. Let $\tilde{f}$ be the generator of $\tilde{F}$ with $\arg \tilde{f}_{p+1} \in [0,\pi)$. There are two cases: either $gfg^{-1} = \tilde{f}$ or $gfg^{-1} = \tilde{f}^{-1}$. We need to show that $\Phi_{\tilde{f}_1}$ and Φ_{f_1} are equivalent, which is obvious when $gfg^{-1} = \tilde{f}$. Assume now that $gfg^{-1} = \tilde{f}^{-1}$. We have $\tilde{f}_1 = R_\lambda \tilde{f} R_\lambda^{-1} \in \mathcal{A}_p^*$. Then $\tilde{f}_2 = R_{\sqrt{\mu}} \tilde{f}_1^{-1} R_{\sqrt{\mu}}^{-1}$ is in $\mathcal{A}_p^*$. Write $\Phi_{\tilde{f}_1,j} = A_{j+1}A_j^{-1}$, where A_j, together with Ω_j, form a canonical collection of $\tilde{f}_1$. We have

$$R_{\sqrt{\mu}} \tilde{f}_1^{-1} R_{\sqrt{\mu}}^{-1} = R_{\sqrt{\mu}} A_{2j}^{-1} R_{-1} F R_{-1} A_{2j} R_{\sqrt{\mu}}^{-1}$$

on $R_{\sqrt{\mu}}\Omega_{2j}$. Now $\tilde{A}_{j+1} = R_{-1}A_j R_{\sqrt{\mu}}^{-1}$ and $\tilde{\Omega}_{j+1} = R_{\sqrt{\mu}}\Omega_j$ form a canonical collection for $\tilde{f}_2$, from which we get $\Phi_{\tilde{f}_2,j+1} = R_{-1}\Phi_{\tilde{f}_1,j}R_{-1}$. Returning to $R_{\sqrt{\mu}} gfg^{-1} R_{\sqrt{\mu}}^{-1} = \tilde{f}_2$, we see that $\Phi_{F,j}$ and $R_{-1}\Phi_{\tilde{F},j+1}R_{-1}$ have the same equivalence class in $\mathcal{M}_p/\sim$.

Conversely, let $F = hF_{s,u,\tau_{f_*}}(f_*)h^{-1}$ and $\tilde{F} = \tilde{h}F_{s,\tilde{u},\tau_{\tilde{f}_*}}(\tilde{f}_*)\tilde{h}$. Let $F \cap \mathcal{A}_p = <f>$ and $\tilde{F} \cap \mathcal{A}_p = <\tilde{f}>$ with $\arg f_{p+1}, \arg \tilde{f}_{p+1} \in [0,\pi)$. We first consider the case that $\Phi_{f_1,j}$ and $\Phi_{\tilde{f}_1,j}$ are equivalent. This implies that $\tilde{f} = gfg^{-1}$. By the choice of f_*, we get $\tilde{f}_* = f_*$ and $ghf_*h^{-1}g^{-1} = \tilde{h}f_*\tilde{h}^{-1}$ or $\tilde{h}f_*\tilde{h}^{-1}$. The latter does not occur since f^* and $\tilde{f}^{*-1}$ are not equivalent. With $\tilde{u} = u$, we get $gFg^{-1} = ghF_{s,u,\tau_{f_*}}(f_*)h^{-1}g^{-1} = \tilde{h}F_{s,u,\tau_{\tilde{f}_*}}(\tilde{f}_*)\tilde{h}^{-1} = \tilde{F}$. Next we assume that $\Phi_{F,j}$ and $R_{-1}\Phi_{\tilde{F},j+1}R_{-1}$ are equivalent in $\mathcal{M}_p/\sim$, that is that $\tilde{f}^{-1} = gfg^{-1}$ as seen above. In this case, we still have $\tilde{f}_* = f_*$ and $ghf_*h^{-1}g^{-1} = \tilde{h}f_*\tilde{h}^{-1}$. So F and $\tilde{F}$ are equivalent.

The realization is already proved in Theorem 7.3. Finally the set of equivalence classes is identified with (s,d) copies of $\hat{\mathcal{M}}'_{p,q,\delta,d}/\sim$, since $F_{s,u,\tau_{f_*}}(f_*) = <\tau_{f_*}^{q/s} f_*^{d'_0/d+u/(s,d)}, f>$, d'_0 depends only on s, d, δ, and $u = 0, 1, \ldots, (s,d)-1$. □

Theorem 7.5. *Let p, q, δ, d, s are positive integers with $s|q|p$ and $(s,d)|\delta$. There is a one-to-one correspondence between $\mathcal{F}''_{p,q,\delta,d,s}/\sim$ and $1 + [(s,d)/2]$ copies of $\hat{\mathcal{M}}''_{p,q,\delta,d}/\sim$.*

Proof. We briefly modify the proof of the previous theorem. For $F \in \mathcal{F}''_{p,q,\delta,d,s}$ and $F \cap \mathcal{A}_p = <f>$ the construction of $\Phi_{f_1} \in \hat{\mathcal{M}}''_{p,q,\delta,d}$ is the same as above. We also let $u_F = u$ be given

by $F = hF_{s,u,\tau_{f_*}}(f_*)h^{-1}$. This time, u is not an invariant of F. Instead, if $F, \tilde{F}$ are equivalent then $u_{\tilde{F}} = u_F$ or $\kappa(u_F)$ for an involution κ of $\{0, 1, \dots, (d,s)-1\}$, described in Proposition 7.2. More precisely, let $F = hF_{s,u,\tau_{f_*}}(f_*)h^{-1}$, $\tilde{F} = \tilde{h}F_{s,\tilde{u},\tau_{\tilde{f}_*}}(\tilde{f}_*)\tilde{h}^{-1}$, and $gFg^{-1} = \tilde{F}$. Then $\tilde{f}_* = f_*$ and $\tilde{h}^{-1}ghf_*h^{-1}g^{-1}\tilde{h} = \tilde{f}_*$ or $\tilde{f}_*^{-1}$. Thus $\tilde{u} = u$ or $\tilde{u} = \kappa(u)$.

Conversely, let $F, \tilde{F} \in \mathcal{F}''_{p,q,\delta,d,s}$ such that the corresponding Φ_{f_1} and $\Phi_{\tilde{f}_1}$ are equivalent and $u_{\tilde{F}} = u_F$ or $\kappa(u_F)$. Hence $gfg^{-1} = \tilde{f}$ or $\tilde{f}^{-1}$, $F = hF_{s,u,\tau_{f_*}}(f_*)h^{-1}$ and $\tilde{F} = \tilde{h}F_{s,\tilde{u},\tau_{\tilde{f}_*}}(\tilde{f}_*)\tilde{h}^{-1}$ imply that $\tilde{f}_* = f_*$. We have either $\tilde{h}^{-1}ghf_*h^{-1}g^{-1}\tilde{h} = f_*$ or f_*^{-1}. From the former, we see that $\tilde{h}^{-1}ghF_{s,u,\tau_{f_*}}(f_*)h^{-1}g^{-1}\tilde{h} = F_{s,u,\tau_{f_*}}(f_*)$, which is also equivalent to $F_{s,\kappa(u),\tau_{f_*}}(f_*)$. So F and $\tilde{F}$ are equivalent. From the latter, we see that $\tilde{h}^{-1}ghF_{s,u,\tau_{f_*}}(f_*)h^{-1}g^{-1}\tilde{h} = F_{s,u,\tau^{-1}_{f_*}}(f_*^{-1}) = F_{s,u,\tau_{f_*}}(f_*)$. Again, F and $\tilde{F}$ are equivalent.

Finally the set of equivalence classes is identified with $1 + [(s,d)/2]$ copies of $\hat{\mathcal{M}}''_{p,q,\delta,d}/\sim$, since κ depends only on δ, s, d. □

Proposition 7.6. *Let $f \in \mathcal{A}_p$ with $d_f \neq 0$. Assume that $\hat{\mathcal{C}}(f) = <\tau, f^{1/d}>$ and $\tau^{q/\delta} = \omega f^{1/d}$. Then for any $\tilde{\tau} = \tau\omega^a$ with $\tilde{\tau}'(0)$ a primitive q-th root of 1, we have $F_{s,j,\tau}(f) = F_{s,\tilde{j},\tilde{\tau}}(f)$ where $\tilde{j} = (1 + aq/\delta)j + a\alpha q/s \mod (d,s)$ and $(d,s) = \alpha d + \beta s$.*

Proof. Assume that $F_{s,j,\tau}(f) = F_{s,\tilde{j},\tilde{\tau}}(f)$. We have $\tilde{\tau} = \tau^{1+aq/\delta}f^{-a/d}$ and $F_{s,\tilde{j},\tilde{\tau}}(f) = <\tau^{q/s(1+aq/\delta)}f^{d'_0/d+\tilde{j}/(d,s)-qa/(sd)}, f>$. Thus $\tilde{j} = (d,s)(aq/(ds) - d'_0/d) + (1+aq/\delta)(d'_0(d,s)/d + j) \mod (d,s) = (1 + aq/\delta)j + (d'_0(d,s)/d + (d,s)\delta/(ds))qa/\delta \mod (d,s) = (1 + aq/\delta)j + a\alpha q/s \mod (d,s)$, where the last identity is computed as in the proof of Proposition 7.2. □

Note that if τ_1, τ_2 satisfy (3.3), then (7.1) defines two sets $\{F_{s,j,\tau_1}(f)\}$ and $\{F_{s,j,\tau_2}(f)\}$, with one being a permutation of another. There is a permutation $j \to j^-$ so that $F_{s,j}(f) = F_{s,j^-}(f^{-1})$, where the latter is defined in terms of $\tau_{f^{-1}}$ whose derivative $\tau'_{f^{-1}}(0)$ is of the smallest non-negative argument. We now compute the permutation $j \to j^-$, without requiring $\tau'_f(0)$ and $\tau'_{f^{-1}}(0)$ have the smallest non-negative arguments. Let $\tau_* = \tau_{f*}$ satisfy

$$\hat{\mathcal{C}}(f) = <\tau_*, f^{1/d}> = <\tau_*> \otimes <\omega_*>, \quad \tau_*^{q/\delta} = \omega_* f^{1/d}. \tag{7.2}$$

so that $\tau'_*(0) = \mu^{p\tilde{q}/q}$ has the smallest absolute value $|\tilde{q}|$. If there are two such elements, denote by τ^+ the one of derivative $\mu^{p|\tilde{q}|/q}$, and the other by τ_-. Put $\omega_+ = \tau_+^{q/\delta}f^{-1/d}$ and $\omega_- = \tau_-^{q/\delta}f^{-1/d}$

Proposition 7.7. *Let $f \in \mathcal{A}_p$ with $d_f \neq 0$. Then*

(i) τ_* is unique, if and only if $q \neq \delta$ or 2δ.

(ii) If $q = \delta$ or 2δ, then $\tau_- = \tau_+^{-1}f^{\frac{2\delta}{qd}}$.

(iii) If $q = \delta$ or 2δ, then $F_{s,j_+,\tau_+}(f) = F_{s,j_-,\tau_-}(f)$ for $j_+ = j_- = 0$ or $j_- = (d,s) - j_+$.

Proof. Let τ_* with $\tau'_*(0) = \mu^{p\tilde{q}/q}$ satisfy (7.2). We have $\omega'_*(0) = \mu^{p\tilde{q}/\delta}$. If τ_* is not unique, there is $a \in \mathbf{Z}$ so that $(\tau_+\omega_+^a)'(0) = \tau'_-(0) = \tau'_+(0)^{-1}$. We get $2\tilde{q}/q + a\tilde{q}/\delta = k$; equivalently, $2\delta\tilde{q}/q + a\tilde{q} = \delta k$. Since $\tilde{q}$ is prime to q, then q divides 2δ. Conversely, if q divides 2δ, then $(\tau_+\omega_+^{-2\delta/q})'(0) = \tau'_+(0)^{-1}$. This shows that $\tau_- = \tau_+\omega_+^{-2\delta/q} = \tau_+^{-1}f^{\frac{2\delta}{qd}}$.

Assume now that q divides 2δ. We have $\tau_- = \tau_+^{-1} f^{\frac{2\delta}{qd}}$. Write $(d,s) = \alpha d + \beta s$. We get $\delta = \frac{\delta\alpha}{(d,s)}d + \frac{\delta\beta}{(d,s)}s$ and $d_0' = -\frac{\delta\beta}{(d,s)} \mod \frac{d}{(d,s)}$. Hence $-\frac{2\delta(d,s)}{sd} - \frac{2d_0'(d,s)}{d} = -\frac{2\delta}{s}\alpha \mod 1$ is an integer, since $s|q$ and $q|2s$. We get

$$F_{s,j_+,\tau_+}(f) = <\tau_-^{-q/s} f^{\frac{2\delta}{sd}+\frac{d_0'}{d}+\frac{j_+}{(d,s)}}, f> = <\tau_-^{q/s} f^{-\frac{2\delta}{sd}-\frac{d_0'}{d}-\frac{j_+}{(d,s)}}, f> = F_{s,-j_+,\tau_-}(f),$$

which implies that $j_- = -j_+ \mod (d,s)$. □

Corollary 7.8. *If $f \in \mathcal{A}_p$ with $d_f \neq 0$, then $\tau_{f^{-1}*} = \tau_{f*}$ for $q \neq \delta$ or 2δ. If $q = \delta$ or 2δ, then $\tau_{f^{-1}+} = \tau_{f-}^{-1}$ and $\tau_{f^{-1}-} = \tau_{f+}^{-1}$.*

Let $\sigma(z) = -\overline{z}$. Let F be an abelian subgroup of $\mathcal{A}_0$. We say that F is *σ-invariant*, if $\sigma f \sigma \in F$ for all $f \in F$. Since $\sigma^2 = \mathrm{Id}$, then $\sigma F \sigma \subset F$ if and only if $\sigma F \sigma = F$. Denote by $\mathcal{F}_{p,\sigma}$ be the set of $F \in \mathcal{F}_p$ such that $\sigma F \sigma = F$. We are interested in classifying F under a single change of holomorphic coordinates that preserves σ.

Proposition 7.9. *Let $f \in \mathcal{A}_p$, and let $\tau = \tau_f, \omega = \omega_f, q = q_f, d = d_f$ be as in Proposition 3.5. Assume that $d_f \neq 0$. If $\sigma F \sigma = F$ with $F \cap \mathcal{A}_0 = <f> \neq \{1\}$, exact one of the following (three) cases occurs:*

a) $\sigma f \sigma = f^{-1}$.
b) $\sigma f^{1/d} \sigma = f^{1/d}$, $\sigma\omega\sigma = \omega^{-1}$ and
i) $\sigma\tau\sigma = \tau^{-1} f^{1/d}$, $q = 2\delta$; or
ii) $\sigma\tau\sigma = \tau^{-1} f^{2/d}$, $q = \delta$.

Moreover, if $\sigma f \sigma = f^{-1}$, all $F_{s,u}(f)$ belongs to $\mathcal{F}_{p,\sigma}$; if $\sigma f \sigma = f$, the total number of $F_{s,u}(f)$ that are σ-invariant is two when $s|\delta$ and (d,s) is even, and is one when $s|\delta$ and (d,s) is odd (and $F = <\tau^{q/s} f^{-\frac{\delta}{sd}}, f>$), or when $s \not| \delta$ and s divides $\delta - sd/2$ (and $F = <\tau^{q/s} f^{\frac{1}{2}-\frac{\delta}{sd}}, f>$), and is zero otherwise.

Proof. Let f be a generator of $F \cap \mathcal{A}_0$. Since $\sigma f \sigma$ remains a generator, then $\sigma f \sigma = f$ or $\sigma f \sigma^{-1} = f^{-1}$. Since f is not an involution, then two cases cannot occur simultaneously. For the latter we already know by Lemma 3.1 that σ reverses each element of $\hat{\mathcal{C}}(f)$. In particular all $F_{s,u}(f)$ are σ-invariant.

Assume now that σ and f commute. We have $\sigma\hat{\mathcal{C}}(f)\sigma = \hat{\mathcal{C}}(f)$. In the proof of Theorem 1.4, we proved that $\sigma\omega\sigma = \omega^{-1}$. Since $\hat{\mathcal{C}}(f) = <\tau, f^{1/d}>$, we can write $\tau\sigma\tau\sigma = f^{l/d}$. Raising both side to q-th power yields $2\delta = ql$. We have either $q = 2\delta$ with $l = 1$, or $q = \delta$ with $l = 2$. For the former, we get $\sigma\tau\sigma = \tau^{-1} f^{1/d}$. For the latter, we get $\sigma\tau\sigma = \tau^{-1} f^{2/d}$. By Proposition 7.9 we know that $F = <\tau^{\frac{q}{s}} f^{d'/d}, f>$ with $0 \leq d' < d$. Now $\sigma F \sigma = F$, if and only if

$$\sigma\tau^{q/s}\sigma f^{d'/d} = (\tau^{q/s} f^{d'/d})^a f^b. \tag{7.3}$$

Looking at the linear parts of both sides we see that $a = -1$. When $l = 1$ (and $q = 2\delta$), (7.3) holds if and only if d divides $q/s + 2d'$, i.e,

$$d|2(d' + \delta/s). \tag{7.4}$$

When $l = 2$ (and $q = \delta$), then (7.3) holds if and only if d divides $2q/s + 2d' = 2\delta/s + 2d'$, i.e., (7.4) holds. Recall that $d' = d_0' + jd/(d,s)$ for some $0 \leq j < (d,s)$ and for some d_0' and

s_0' satisfy $\delta = s_0' d - d_0' s$. Consequently, the number of groups $F_{p,u}(f)$ is 0 when d does not divides any $2d' + 2\delta/s$, 1 when d divides some $2d' + 2\delta/s$ and (d, s) is odd, and 2 when d divides some $2d' + 2\delta/s$ and (d, s) is even.

Consider the case s divides δ first. For $d' = -\delta/s$ and $s' = 0$, d divides $2d' + 2\delta/s$.

Consider now that s does not divide δ. Since s divides q and the latter is δ or 2δ, then s is even, $q = 2\delta$ and q/s is odd. Assume now that d divides some $2d' + 2\delta/s = 2d' + q/s$ (with $\delta = s'd - d's$), then d is odd. In particular, (d, s) is odd.

Finally, note that d divides some $2(d' + \delta/s) = 2ds'/s$, if and only if s divides some $2s'$ for $\delta = s'd - d's$. Furthermore, we may also assume that $0 \le s' < s$. Thus, we have either $s' = 0$ or $s = 2s'$. The case $s' = 0$ occurs, if and only if s divides δ. The case $s = 2s'$ occurs, if and only if $\frac{\delta}{s} - \frac{d}{2}$ is an integer. □

Let $\sigma_1(z) = -\sqrt{\mu}^{2m-1}\overline{z}$ with m being given by (6.2). Recall that $\mathcal{N}_{p,\sigma_1}$, modulo a suitable equivalence relation, is the moduli space for the classification of holomorphic maps $f \in \mathcal{A}_p^*$ commuting with σ_1 under changes of coordinates that preserve σ_1. Define

$$\begin{aligned}\hat{\mathcal{M}}_{p,\sigma}^* = \big\{&(a, s, u, \Phi)\colon \Phi \in \mathcal{M}_{p,\sigma}, d_\Phi \neq 0, a > 0,\\ &s|q_\Phi, (s, d_\Phi)|\delta_\Phi, 0 \le u < (d, s)\big\}\\ \bigsqcup\big\{&(a, s, u, \Phi)\colon \Phi \in \mathcal{N}_{p,\sigma_1}, d_\Phi \neq 0, a > 0,\\ &s|q_\Phi, (s, d_\Phi)|\delta_\Phi, u = u_0, \text{or } u = u_0 + \frac{(d, s)}{2} \in \mathbf{Z}\big\},\end{aligned}$$

in which u_0 is the smallest integer u_0 in $[0, (d, s))$ such that $F_{s,u_0}(f) \in \mathcal{F}_{p,\sigma}$. Proposition 7.9 says that u_0 exists if and only if s divides some $2s'$ for $\delta = s'd - d's$, and that $F_{s,u_0+(d,s)/2}$ is in $\mathcal{F}_{p,\sigma}$ if and only if $s|\delta$ and (d, s) is even. Given $(\tilde{a}, \tilde{s}, u, \tilde{\Phi}), (a, s, u, \Phi) \in \hat{\mathcal{M}}_p^*$, we denote $(\tilde{a}, \tilde{s}, \tilde{u}, \tilde{\Phi}) \overset{*}{\sim} (a, s, u, \Phi)$, if $(\tilde{a}, \tilde{s}, \tilde{u}) = (a, s, u)$ and if $\Phi, \tilde{\Phi}$ are in $\mathcal{M}_{p,\sigma}$ and $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_j L_{-c_j}$ for all j and some $c_j = -\overline{c}_{2p-j+1} \in \mathbf{C}$, or if $\Phi, \tilde{\Phi}$ are in $\mathcal{N}_{p,\sigma_1}$ and $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_j L_{-c_j}$ for all j and some $c_j = \overline{c}_{2m-j} \in \mathbf{C}$.

Theorem 7.10. *There is a one-to-one correspondences between* $\{F \in \mathcal{F}_{p,\sigma}\colon d_F \neq 0\}/\overset{*}{\sim}$ *and* $\hat{\mathcal{M}}_{p,\sigma}^*/\overset{*}{\sim}$.

Proof. For $F \in \mathcal{F}_{p,\sigma}$ with $d_F \neq 0$, we need to define a, s, u and Φ for F. We take a generator f_0 of $F \cap \mathcal{A}_0$. By Proposition 7.9, there are two cases: either $\sigma f_0 \sigma = f_0^{-1}$, or $\sigma f_0 \sigma = f_0$. Note that exact one case occurs, since f_0 is not an involution.

Let s_F be the order of $F/F \cap \mathcal{A}_0$. We now define Φ_F and a_F. We first consider the case that f_0 is reversible by σ. By (4.1), $F \cap \mathcal{A}_0$ has a unique generator $f(z) = z + a\mu_0 z^{p+1} + O(|z|^{p+2})$ with $a > 0$, and we assign $a_F = a$. Choose a unique R_λ with $\lambda > 0$ such that $f_1 = R_\lambda^{-1} f R_\lambda \in \mathcal{A}_{p,\sigma}^*$, and define $\Phi_F = \Phi_{f_1} \in \mathcal{M}_{p,\sigma}$. When f_0 and σ commutes, we first choose a linear map R_{λ_0}, dependent of μ and p only, such that $R_{\lambda_0} \sigma R_{\lambda_0}^{-1} = \sigma_1$. Now, by (6.1), choose a generator f of $F \cap \mathcal{A}_0$ such that $R_{\lambda_0} f R_{\lambda_0}^{-1}(z) = z + a\mu_0 z^{p+1} + O(|z|^{p+2})$ with $a > 0$. Define $a_F = a$ and choose $\lambda > 0$ such that $f_1 = R_{\lambda\lambda_0} f R_{\lambda\lambda_0}^{-1} \in \mathcal{I}_{p,\sigma_1}^+$. Define $\Phi_F = \Phi_{f_1} \in \mathcal{N}_{p,\sigma_1}$. Let u be the unique integer such that $F = F_{s,u}(f)$.

We need to show that a, Φ, s, u depend only on the equivalence class of $F \cap A_0$ in $\mathcal{F}_{p,\sigma}/\overset{*}{\sim}$. Assume that $\tilde{F} = g^{-1} F g$ with $g'(0) = 1$ and $g\sigma = \sigma g$. Let $f_1 = R_\nu f R_\nu^{-1}$ be as above ($\nu = \lambda$

or $\lambda_0\lambda$). Then $\tilde{f}_1 = R_\nu g f g^{-1} R_\nu^{-1}$ is in $\mathcal{A}^*_{p,\sigma}$ or in $\mathcal{I}^+_{p,\sigma_1}$, and $\tilde{f} = g^{-1}fg$ is a generator of $\tilde{F}$. Since $g\sigma = \sigma g$, we have either f and $\tilde{f}$ are reversible by σ, or f and $\tilde{f}$ commute with σ. This shows that $a_F = a_{\tilde{F}}$. Put $\tilde{g} = R_\nu g R_\nu^{-1}$. We see that $\Phi_{\tilde{F}} = \Phi_{\tilde{g}f_1\tilde{g}^{-1}}$ is equivalent to $\Phi_{f_1} = \Phi_F$ in $\mathcal{M}_{p,\sigma}/\overset{*}{\sim}$ or in $\mathcal{N}_{p,\sigma_1}/\overset{*}{\sim}$, since $R_\nu g^{-1} R_\nu^{-1}$ is tangent to the identity and commutes with σ or with σ_1. By a similar argument in Theorem 7.3, we obtain $s_F = s_{\tilde{F}}$ and $u_F = u_{\tilde{F}}$.

Let $F, \tilde{F} \in \mathcal{F}_{p,\sigma}$ be two abelian subgroups such that $a_F = a_{\tilde{F}} = a, s_F = s_{\tilde{F}} = s$, and $u_F = u_{\tilde{F}} = u$, while Φ_F and $\Phi_{\tilde{F}}$ are equivalent in $\mathcal{M}_{p,\sigma}/\overset{*}{\sim}$ or in $\mathcal{N}_{p,\sigma_1}/\overset{*}{\sim}$. Let $f(z) = z + a\mu_0 z^{p+1} + O(|z|^{p+2})$ and $\tilde{f}(z) = z + a\mu_0 z^{p+1} + O(|z|^{p+2})$ be the generators of $F \cap \mathcal{A}_0$ and $\tilde{F} \cap \mathcal{A}_0$, respectively. There exists ν dependent of a only such that $f_1 = R_\nu f R_\nu^{-1}$ and $\tilde{f}_1 = R_\nu \tilde{f} R_\nu^{-1}$ are both in $\mathcal{A}^*_p$ or in $\mathcal{I}^+_{p,\sigma_1}$. Since Φ_F and $\Phi_{\tilde{F}}$ are equivalent, there exists a biholomorphic map $\tilde{g}$, satisfying $\tilde{g}'(0) = 1$ and $\tilde{g} = \sigma\tilde{g}\sigma$ or $\tilde{g} = \sigma_1\tilde{g}\sigma_1$, such that $\tilde{g}\tilde{f}_1\tilde{g}^{-1} = f_1$. Put $g = R_\nu^{-1}\tilde{g}R_\nu$. Then $\tilde{f} = g^{-1}fg$. By definition, we have $F = \mathcal{F}_{s,u}(f)$ and $\tilde{F} = \mathcal{F}_{s,u}(\tilde{f})$. By Lemma 7.1, we get $\tilde{F} = g^{-1}Fg$, which shows that F and $\tilde{F}$ are equivalent.

For the realization, we assume that $(a, s, u, \Phi) \in \hat{\mathcal{M}}^*_p$. Take f_1 in $\mathcal{A}^*_p$ or in $\mathcal{I}^+_{p,\sigma_1}$ such that $\Phi_{f_1,j} = \Phi_j$. Put $f = R_\nu^{-1} f_1 R_\nu$, where ν is determined by a as above. Now $F_{s,u}(f)$ is a realization. □

Define

$$\hat{\mathcal{M}}_{p,\sigma} = \big\{(s, u, \Phi)\colon (1, s, u, \Phi) \in \hat{\mathcal{M}}^*_{p,\sigma}\big\}.$$

Given $(\tilde{s}, \tilde{u}, \tilde{\Phi}), (s, u, \Phi) \in \hat{\mathcal{M}}_{p,\sigma}$, we denote $(\tilde{s}, \tilde{u}, \tilde{\Phi}) \sim (s, u, \Phi)$, if one the following occurs:

i) $(\tilde{s}, \tilde{u}) = (s, u)$, and $\tilde{\Phi} \sim \Phi$ in $\mathcal{M}_{p,\sigma}/\sim$, i.e.

$$\tilde{\Phi}_j = L_{c_{j+1}} \Phi_{j+2k} L_{-c_j}, \quad j = 1, \dots, 2p \tag{7.5}$$

hold with $c_j = -\overline{c}_{2p-j+1}$ and with $k = 0$ or $k = p/2 \in \mathbf{Z}$.

ii) $(\tilde{s}, \tilde{u}) = (s, u)$, $\Phi \sim \tilde{\Phi}$ in $\mathcal{N}_{p,\sigma_1}/\sim$, i.e. (7.5) hold with $c_j = \overline{c}_{2m-j}$ and $k = 0$ or $k = p/2 \in \mathbf{Z}$.

iii) $(\tilde{s}, \tilde{u}) = (s, u^-)$, p is odd, and $\tilde{\Phi} \overset{*}{\sim} R_{-1}\Phi' R_{-1}$ in $\mathcal{M}_{p,\sigma}/\overset{*}{\sim}$, i.e.

$$\tilde{\Phi}_j = L_{c_{j+1}} R_{-1} \Phi_{j+p} R_{-1} L_{-c_j}, \quad j = 1, \dots, 2p \tag{7.6}$$

hold with $c_j = -\overline{c}_{2p-j+1}$, where $\Phi'_j = \Phi_{j+p}$. (Recall that $F_{s,u,\tau_f}(f) = F_{s,u^-,\tau_{f^{-1}}}(f^{-1})$, with both $\tau_f, \tau_{f^{-1}}$ have smallest non-negative arguments for $\tau'_f(0)$ and $\tau'_{f^{-1}}(0)$.)

iv) $(\tilde{s}, \tilde{u}) = (s, u^-)$, p is odd, and $\tilde{\Phi} \overset{*}{\sim} R_{-1}\Phi' R_{-1}$ in $\mathcal{N}_{p,\sigma_1}/\overset{*}{\sim}$, i.e. (7.6) hold with $c_j = \overline{c}_{2m-j}$.

Theorem 7.11. *There is a one-to-one correspondences between* $\{F \in \mathcal{F}_{p,\sigma}\colon d_F \neq 0\}/\sim$ *and* $\hat{\mathcal{M}}_{p,\sigma}/\sim$.

Proof. We need to modify the proof of Theorem 7.10.

For $F \in \mathcal{F}_{p,\sigma}$, we need to define s, u and Φ for F. As before we take a generator f_0 of $F \cap \mathcal{A}_0$. By Proposition 7.9, there are two cases: either $\sigma f_0 \sigma = f_0^{-1}$, or $\sigma f_0 \sigma = f_0$.

The definitions of s_F, Φ_F and u_F are the same as in Theorem 7.10.

We need to show that Φ, s, u depend only on the equivalence class of $F \cap A_0$ in $\mathcal{F}_{p,\sigma}/\sim$. Assume that $\tilde{F} = g^{-1}Fg$ with $g\sigma = \sigma g$. Let $f_1 = R_\nu f R_\nu^{-1}$, $\tilde{f}_1 = R_{\tilde{\nu}} \tilde{f} R_{\tilde{\nu}}^{-1}$ be as in the proof

of Theorem 7.10 ($\nu = \lambda$ and $\lambda_0\lambda$, or $\tilde{\nu} = \tilde{\lambda}$ and $\tilde{\nu} = \lambda_0\tilde{\lambda}$, and $\lambda, \tilde{\lambda} > 0$). Put $\tilde{g} = R_\nu g R_{\tilde{\nu}}^{-1}$. We have

i) $f_1, \tilde{f}_1 = \tilde{g}^{-1} f_1 \tilde{g}$ are in $\mathcal{A}^*_{p,\sigma}$ and $\tilde{g}\sigma = \sigma\tilde{g}$; or
ii) $f_1, \tilde{f}_1 = \tilde{g}^{-1} f_1 \tilde{g}$ are in $\mathcal{I}^+_{p,\sigma_1}$ and $\tilde{g}\sigma_1 = \sigma_1\tilde{g}$; or
iii) $f_1, \tilde{f}$ are in $\mathcal{A}^*_{p,\sigma}$, $\tilde{f}_1^{-1} = \tilde{g}^{-1} f_1 \tilde{g}$, and $\tilde{g}\sigma = \sigma\tilde{g}$; or
iv) $f_1, \tilde{f}_1$ are in $\mathcal{I}^+_{p,\sigma_1}$, $\tilde{f}_1^{-1} = \tilde{g}^{-1} f_1 \tilde{g}$ and $\tilde{g}\sigma_1 = \sigma_1\tilde{g}$.

For (i) or (ii), we see immediately that $\Phi_{\tilde{F}} = \Phi_{\tilde{g}^{-1} f_1 \tilde{g}} \sim \Phi_{f_1} = \Phi_F$ in $\mathcal{M}_{p,\sigma}/\sim$ or in $\mathcal{N}_{p,\sigma_1}/\sim$. Consider case (iii). Since $f_1, \tilde{f}_1 \in \mathcal{A}^*_p$ and $\tilde{g}$ commutes with σ, then $\tilde{g}'(0)$ is real. Thus $\tilde{g}'(0) = -1$ and p is odd. Put $h = \tilde{g}R_{-1}$. Then $h\sigma = \sigma h$ and $\tilde{f}_2 = R_{-1}\tilde{f}_1^{-1}R_{-1} = h^{-1} f_1 h$. Consequently, $\Phi_{\tilde{f}_2}$ and Φ_{f_1} are equivalent in $\mathcal{M}_{p,\sigma}/\overset{*}{\sim}$. We need to compute the former. Write $\tilde{f}_1 = A_{2j+1}^{-1} F A_{2j+1}$. Then

$$R_{-1}\tilde{f}_1^{-1}R_{-1} = R_{-1}\sigma A_{2j+1}^{-1} F A_{2j+1}\sigma R_{-1} = R_{-1}\tilde{A}_{-2j}^{-1} R_{-1} F R_{-1}\tilde{A}_{-2j} R_{-1}.$$

Set $\tilde{A}_j = R_{-1}A_{j+p}R_{-1}$. Then $\Phi_{\tilde{f}_2} = R_{-1}\Phi'_{\tilde{f}_1}R_{-1}$. Therefore, $\Phi'_{\tilde{f}_1}$ and $R_{-1}\Phi_{f_1}R_{-1}$ are equivalent in $\mathcal{M}_{p,\sigma}/\overset{*}{\sim}$. Consider case (iv). Again we know that $\tilde{g}'(0) = -1$ and p is odd. Put $h = \tilde{g}R_{-1}$. Then $h\sigma_1 = \sigma_1 h$ and $\tilde{f}_2 = R_{-1}\tilde{f}_1^{-1}R_{-1} = h^{-1} f_1 h$. Consequently, $\Phi_{\tilde{f}_2}$ and Φ_{f_1} are equivalent in $\mathcal{N}_{p,\sigma_1}/\overset{*}{\sim}$. To compute the former, write $\tilde{f}_1 = A_{2j+1}^{-1} F A_{2j+1}$. Then

$$R_{-1}\tilde{f}_1 R_{-1} = R_{-1}\sigma_1 A_{2j+1}^{-1} F A_{2j+1}\sigma_1 R_{-1} = R_{-1}\tilde{A}_{p-2j}^{-1} R_{-1} F R_{-1}\tilde{A}_{p-2j} R_{-1}.$$

Set $\tilde{A}_j = R_{-1}A_{j+p}R_{-1}$. Then $\Phi_{\tilde{f}_2} = R_{-1}\Phi'_{\tilde{f}_1}R_{-1}$. Therefore, $\Phi'_{\tilde{f}_1}$ and $R_{-1}\Phi_{f_1}R_{-1}$ are equivalent in $\mathcal{N}_{p,\sigma_1}/\sim$.

Still assume that $\tilde{F} = g^{-1}Fg$ with $g\sigma = \sigma g$. Return to cases (i) and (ii). We also have $s_F = |F/(F \cap \mathcal{A}_0^*)| = s_{\tilde{F}}$. To show that $u_F = u_{\tilde{F}}$, we may assume that $\tilde{f} = f$ (so $g = \mathrm{Id}$). From $F = \tilde{F}$ we know that F and $\tilde{F}$ have the same u value. For cases (iii) and (iv), we may assume that generators $\tilde{f} = f^{-1}$, which gives us $u_F^- = u_{\tilde{F}}$.

Let $F, \tilde{F} \in \mathcal{F}_{p,\sigma}$ be two abelian subgroups such that (s_F, u_F, Φ_F) and $(s_{\tilde{F}}, u_{\tilde{F}}, \Phi_{\tilde{F}})$ are equivalent according to (i)-(iv) above. There exist $\nu, \tilde{\nu}$ such that $f_1 = R_\nu f R_\nu^{-1}$ and $\tilde{f}_1 = R_{\tilde{\nu}}\tilde{f}R_{\tilde{\nu}}^{-1}$ are both in $\mathcal{A}^*_p$ or in $\mathcal{I}^+_{p,\sigma_1}$, where $f, \tilde{f}$ generate $F \cap \mathcal{A}_0$ and $\tilde{F} \cap \mathcal{A}_0$, respectively. For (i) and (ii), there exists a biholomorphic map $\tilde{g}$, satisfying $\tilde{g} = \sigma\tilde{g}\sigma$ or $\tilde{g} = \sigma_1\tilde{g}\sigma_1$, such that $\tilde{g}\tilde{f}_1\tilde{g}^{-1} = f_1$. Put $g = R_\nu^{-1}\tilde{g}R_{\tilde{\nu}}$, $F^* = g\tilde{F}g^{-1}$ and $f^* = gfg^{-1}$. By Lemma 7.1 we get For (iii) and (iv), we find $\tilde{g}$, satisfying $\tilde{g} = \sigma\tilde{g}\sigma$ or $\tilde{g} = \sigma_1\tilde{g}\sigma_1$, such that $\tilde{g}\tilde{f}_1\tilde{g}^{-1} = f_1^{-1}$. Now $\tilde{F} = \mathcal{F}_{s_{\tilde{F}}, u_{\tilde{F}}}(\tilde{f}) = \mathcal{F}_{s_F, u_F^-}(g^{-1}f^{-1}g) = g^{-1}Fg$.

The realization is given by Theorem 7.10. □

8. pairs of holomorphic involutions

In this section we shall classify pairs of holomorphic involutions, and abelian groups in $\mathcal{F}_p$ invariant under a holomorphic involution.

Let I_1, I_2 be a pair of holomorphic involutions such that none of I_1, I_2 and I_1I_2 is the identity. We want to classify the pairs under a change of holomorphic coordinates. We shall always normalize I_1 as $I\colon z \to -z$. Put $f = I_1I_2$. Notice that $f^{-1} = IfI$. We shall

see that f is in some $\mathcal{A}_p$ for which p is odd. Thus, when p is odd, let $\mathcal{I}_p$ be the set of $f \in \mathcal{A}_p$ satisfying $IfI = f^{-1}$, and let $\mathcal{I}_p^* = I_p \cap \mathcal{A}_p^*$. We need to classify $\mathcal{I}_p$ under biholomorphisms g commuting with I.

Let us first look at the formal normal form of $f \in \mathcal{I}_p$. Assume that $f^{-1} = IfI$. We have $IfI(z) = z + (-1)^p f_{p+1} z^{p+1} + O(|z|^{p+2})$ and $f^{-1}(z) = z - f_{p+1} z^{p+1} + O(|z|^{p+2})$. Thus p is odd. We also know that there is a unique polynomial $g(z)$ of degree less than $p+1$ such that $gfg^{-1} = z + z^{p+1} + \lambda z^{2p+1} + O(|z|^{2p+2})$. Now $\tilde{g}f\tilde{g}^{-1}(z) = z + z^{p+1} + \lambda' z^{2p+1} + O(|z|^{2p+2})$ for $\tilde{g} = IgI$. The uniqueness of g also implies that $IgI = g$. Assume that $f(z) = z + z^{p+1} + \lambda z^{2p+1} + O(|z|^{2p+2})$. From $f^{-1} = IfI$, a simple computation shows that $\lambda = \frac{p+1}{2}$. Now let g be the unique formal map with $g_{p+1} = 0$ such that gfg^{-1} is the time-one map v_1 of $z^{p+1}\frac{\partial}{\partial z}$. Since both f and $v_1^{-1} = Iv_1I$ are reversible by I, the uniqueness of g implies that $IgI = g$. One also sees that g is the unique map commuting with I and transforming f into v_1.

Let $\mathcal{I}_p/\sim$ be holomorphic equivalence classes of $\mathcal{I}_p$ under biholomorphisms g that commute with I. Let $\mathcal{I}_p^*/\overset{*}{\sim}$ be holomorphic equivalence classes of $\mathcal{I}_p^*$ under g satisfying $gI = Ig$ and $g'(0) = 1$.

As Lemma 4.1, we have

Lemma 8.1. *Let $p > 0$ be odd. For each $f \in \mathcal{I}_p^*$, there exists a canonical collection Ω_j, A_j such that A_j is conformal on $\Omega_j \bigcup I\Omega_{p+j}$, and $A_{j+p} = IA_jI$ on Ω_{p+j}. Moreover, for any $\epsilon > 0$,*

$$\bigcup_{j=1}^{p} \{A_j^{-1}(R(\epsilon)) \cup IA_j^{-1}(R(\epsilon))\}$$

is a punctured neighborhood of the origin, and is contained in any given neighborhood of the origin when ϵ is sufficiently small.

For $f \in \mathcal{I}_p^*$, let Ω_j, A_j be a canonical collection of f satisfying I*i-v*. Based on Lemma 8.1, we choose $A_{p+j} = IA_jI$ for $j = 1, \ldots, p$, when p is odd. Now define $\Phi_{f,j} = A_{j+1}A_j^{-1}$, so $\Phi_{p+j} = I\Phi_{f,j}I$. Note that $\Phi_{f,j}$ are well-defined up to the form $L_{c_{j+1}}\Phi_{f,j}L_{-c_j}$ with $c_{p+j} = -c_j$.

For p odd, let $\mathcal{N}_p$ be the set of $\Phi \in \mathcal{M}_p$ satisfying $\Phi_{p+j} = I\Phi_j I, j = 1 \ldots, 2p$. Denote $\tilde{\Phi} \sim \Phi$, if $\tilde{\Phi}_j = L_{c_{j+1}}\Phi_{j+2k}L_{-c_j}$ for some $k \in \mathbf{Z}$ and $c_j = -c_{j+p}$, and denote $\tilde{\Phi} \overset{*}{\sim} \Phi$, if the additional condition $k = 0$ holds.

We have

Theorem 8.2. *Let $p > 0$ be odd. Let $f, \tilde{f} \in \mathcal{I}_p^*$. If $\tilde{f} = g^{-1}fg$ for some holomorphic mapping g commuting with I, there exist $k \in \mathbf{Z}$ and $c_j = -c_{p+j} \in \mathbf{C}$ such that $g'(0) = \mu^k$*

$$\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_{f,j+2k}L_{-c_j}, \quad j = 1, \ldots, 2p. \tag{8.1}$$

Conversely, if (8.1) holds, then $\tilde{f} = g^{-1}fg$ for some g with $gI = Ig$ and $g'(0) = \mu^k$. For each $\Phi \in \mathcal{N}_p$, there exists $f \in \mathcal{I}_p^*$ with $\Phi_{f,j} = \Phi_j$ for all j.

Proof. Let $\tilde{f}, f \in \mathcal{I}_p^*$. Assume that $\tilde{f} = g^{-1}fg$ with $gI = Ig$. Then $g'(0) = \mu^k$.

Let Ω_j, A_j be a canonical collection of f with $A_{p+j} = IA_jI$. Then $A'_j = A_{j+2k}g$ and $\Omega'_j = g^{-1}(\Omega_{j+2k})$ are a canonical collection of $\tilde{f}$. Thus, $\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_{f,j+2k}L_{-c_j}$ with $c_{p+j} = -c_j$.

Next, assume that $\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_{f,j+2k}L_{-c_j}$ with $c_{p+j} = -c_j$. Recall that $\Phi_{\tilde{f},j} = \tilde{A}_{j+1}\tilde{A}_j^{-1}$, $\Phi_{f,j} = A_{j+1}A_j^{-1}$ with $\tilde{A}_{p+j} = I\tilde{A}_jI$ and $A_{p+j} = IA_jI$. Replacing $\tilde{A}_j$ with $L_{-c_j}\tilde{A}_j$, we get $\tilde{A}_{j+1}\tilde{A}_j^{-1} = A_{j+2k+1}A_{j+2k}^{-1}$. As in the proof of Theorem 2.5, we know that $g = A_{j+2k}^{-1}\tilde{A}_j$ ($j = 1, \ldots, 2p$) extends to a holomorphic mapping defined near the origin, $\tilde{f} = g^{-1}fg$ and $g'(0) = \mu^k$. Also $IgI = g$ in a neighborhood of the origin.

Given $\Phi \in \mathcal{N}_p$, we need to find $f \in \mathcal{I}_p^*$ such that $A_{j+1}A_j^{-1} = \Phi_j$, where $A_j = IA_{p+j}I, j = 1, \ldots, 2p$, together with some domains $\Omega_1, \ldots, \Omega_{2p}$, are a canonical collection of f. By Theorem 2.5, there exists $f \in \mathcal{A}_p^*$ such that $\Phi_j = A_{j+1}A_j^{-1} = \Phi_{f,j}$, where A_j, Ω_j form a canonical collection of f ($A_{p+j} = IA_jI$ does not necessarily hold). Since $\Phi_{p+j} = I\Phi_jI$, we have $IA_{j+1}A_j^{-1}I = A_{p+j+1}A_{p+j}^{-1}$ on some half-plane $\Pi_{p,j+p}(\rho_1)$. Now $\hat{I} = A_{p+j}^{-1}IA_j$ is well-defined and extends to a holomorphic involution defined near the origin.

Note that

$$\hat{I}f\hat{I} = \hat{I}A_1^{-1}FA_1\hat{I} = A_{p+1}^{-1}IFIA_{p+1} = f^{-1}.$$

Since I is not an identity, we get $\hat{I}(z) = -z + O(|z|^2)$.

Put $g(z) = \frac{1}{2}(z + I\hat{I}(z)) = z + O(|z|^2)$. Now g is biholomorphic, and $g\hat{I} = Ig$. Thus $I\tilde{f}I = \tilde{f}^{-1}$ for $\tilde{f} = gfg^{-1}$. We now have $\Phi_{\tilde{f},j} = \tilde{A}_{j+1}\tilde{A}_j^{-1}$ and $\Phi_{\tilde{f},p+j} = I\Phi_{\tilde{f},j}I$. Since $\tilde{f} = gfg^{-1}$ and $g'(0) = 1$, Theorem 2.5 says $\Phi_{\tilde{f},j} = L_{c_{j+1}}\Phi_jL_{-c_j}$. Using $I\Phi_jI = \Phi_{p+j}$, we get

$$L_{c_{j+1}}\Phi_jL_{-c_j} = \Phi_{\tilde{f},j} = IL_{c_{p+j+1}}\Phi_{p+j}L_{-c_{p+j}}I = L_{-c_{p+j+1}}\Phi_jL_{c_{p+j}}.$$

Thus $c' = c_j + c_{p+j}$ for all j and $L_{-c'}\Phi_jL_{c'} = \Phi_j$. When $d_\Phi = 0$, we put $A'_j = L_{-c_j+\frac{c'}{2}}\tilde{A}_j$ for $j = 1, \ldots, 2p$. Clearly, $A'_{j+1}A_j^{'-1} = \Phi_j$. Now $c_j - \frac{c'}{2} = -c_{p+j} + \frac{c'}{2}$ implies that $IA'_jI = \tilde{A}_{p+j} + c_j - \frac{c'}{2} = A'_{p+j}$, which gives us the desired realization. When $d = d_\Phi = d_{\Phi_f} \neq 0$, then $c' = \frac{k}{d}$, $k \in \mathbf{Z}$. When k is even, $A'_j = \tilde{A}_j - c_j + \frac{c'}{2}$ gives us the realization as above. Assume now that $k = 2l+1$. Put $A'_j = L_{-c_j+\frac{l}{d}}\tilde{A}_j$ for $j = 1, \ldots, p$ and $A'_j = L_{-c_j+\frac{l+1}{d}}\tilde{A}_j$ for $j = p+1, \ldots, 2p$. Then we have

$$\begin{aligned} IA'_jI = A'_{p+j}; \quad A'_{j+1}A_j^{'-1} = \Phi_j, \quad j \neq p, 2p; \\ A'_{p+1}A_p^{'-1} = L_{\frac{1}{d}}\Phi_p, \quad A'_1A_{2p}^{'-1} = L_{-\frac{1}{d}}\Phi_{2p}. \end{aligned} \tag{8.2}$$

Take $\tilde{I} = \tilde{f}^{\frac{1}{d}}I$. Then $\tilde{I}$ is a holomorphic involution with $\tilde{I}(z) = -z + O(|z|^2)$. Put $h = \frac{1}{2}(\mathrm{Id} + I\tilde{I})$, so $h\tilde{I} = Ih$. Put $f^* = h\tilde{f}h^{-1}$. Then $f^{*-1} = If^*I$. Put $A_j^* = A'_jh^{-1} = IA_{p+j}^*I$ for $j = 1, \ldots, p$. Obviously, $A_{j+1}^*A_j^{*-1} = A'_{j+1}A_j^{'-1} = \Phi_j$ for $j = 1, \ldots, p-1$. Now rewrite $h\tilde{I} = Ih$ as

$$h^{-1}Ih = \tilde{I} = A_1^{'-1}L_{\frac{1}{d}}A'_1I.$$

We have

$$A_{p+1}^*A_p^{*-1} = IA'_1h^{-1}IhA_p^{'-1} = IL_{\frac{1}{d}}A'_1IA_p^{'-1} = \Phi_p.$$

For $j = p+1, \ldots, 2p$, we obtain $\Phi_{f^*,j} = I\Phi_{f^*,p+j}I = I\Phi_{p+j}I = \Phi_j$, which gives us the desired realization. □

Remark. Theorem 8.2 is due to Voronin [20], when $p = 1$. One should also compare the above argument with Voronin's proof. Notice that the f and f^* that appeared in the above proof are not equivalent by any biholomorphism commuting with I. Otherwise, $f^{1/d}$ admits a square root, which is impossible since p and hence q is odd. (The proof of the last assertion of Theorem 5.1 is valid when ν_1, ν_2 and σ are replaced with $f^{1/d}$, Id and I, respectively.)

Next we shall classify subgroups in $\mathcal{F}_p$ that are invariant under I. For this purpose we need to study maps in $\mathcal{A}_p$ that commute with I, i.e. the odd maps. In fact, we shall consider a more general linear periodic map R other than the involution I. For a positive integer k, let $\mathcal{I}_{p,k}$ be the set in $\mathcal{A}_p$ commuting with R_{μ^k}, and let $\mathcal{N}_{p,k}$ be the set of $\Phi \in \mathcal{M}_p$ satisfying $\Phi_{2k+j} = \Phi_j$. Put $I^*_{p,k} = I_{p,k} \cap \mathcal{A}^*_p$. Define $\mathcal{I}_{p,k}/\sim$ to be the set of holomorphic equivalence classes under maps g commuting with R_{μ^k}, and $\mathcal{I}^*_{p,k}/\overset{*}{\sim}$ the set of equivalence classes under g satisfying the additional condition $g'(0) = 1$. Denote $\tilde{\Phi} \sim \Phi$, if $\tilde{\Phi}_j = L_{c_{j+1}}\Phi_{j+2l}L_{-c_j}$ for some $l \in \mathbf{Z}$ and $c_j = c_{j+2k}$, and denote $\tilde{\Phi} \overset{*}{\sim} \Phi$, if $l = 0$ additionally. Let $\mathcal{N}_{p,k}/\sim, \mathcal{N}_{p,k}/\overset{*}{\sim}$ be the corresponding sets of equivalence classes.

We have

Lemma 8.3. *Let k be a positive integer, and let $R = \mu^k$. Let $f, \tilde{f} = z + O(2)$ be formal power series commuting with R, and let g be a formal transformation such that $gfg^{-1} = \tilde{f}$. If f is not the identity, then g and R commute. If $\Phi, \tilde{\Phi} \in \mathcal{M}_p$ satisfy $\Phi_{j+2k} = \Phi_j$ and $\tilde{\Phi}_{j+2k} = \tilde{\Phi}_j$, and if $\tilde{\Phi}_j = L_{c_{j+1}}\Phi_{j+2l}L_{-c_j}$, then $c_j = c_{j+2k}$.*

Proof. Since R and f commute, then $\tilde{f}$ and gRg^{-1} also commute. Since $\tilde{f}(z) = z + O(2)$ is formally equivalent to $v_{\lambda,1}$, the group of formal centralizers of $\tilde{f}$ is abelian. Thus $R^{-1}gRg^{-1}$ is periodic and tangent to the identity, and hence it is the identity.

For the second part, one readily sees that $\Phi_{j+2l} = L_{-c_{j+1}+c_{j+1+2k}}\Phi_{j+2l}L_{c_j-c_{j+2k}}$. Thus $c' = c_{j+2k} - c_j$ are independent of j. Now $2pc' = 0$, i.e., $c_j = c_{j+2k}$. □

Theorem 8.4. *Let k be a positive integer. There is a one-to-one correspondence between $\mathcal{I}^*_{p,k}/\overset{*}{\sim}$ and $\mathcal{N}_{p,k}/\overset{*}{\sim}$. There is a one-to-one correspondence between $\mathcal{I}_{p,k}/\sim$ and $\mathcal{N}_{p,k}/\sim$.*

Proof. Assume that f is in $\mathcal{I}^*_{p,k}$. Let A_j, Ω_j be a canonical collection of f. Then $A'_j = A_{j+2k}R_{\mu^k}, \Omega'_j = R^{-1}_{\mu^k}\Omega_{j+2k}$ form a canonical collection also. Thus we may assume that $A_{j+2k} = A_jR_{\mu^{-k}}$ for all j. Consequently, we get a moduli function $\Phi_{f,j} = A_{j+1}A_j^{-1}$ such that $\Phi_{j+2k} = \Phi_j$.

Conversely, assume that $\Phi \in \mathcal{M}_p$ satisfies $\Phi_{j+2k} = \Phi_j$. Take a canonical collection A_j, Ω_j for some $f \in \mathcal{A}^*_p$ with $A_{j+1}A_j^{-1} = \Phi_j$. Then $\hat{R} = A^{-1}_{j+2k}A_j$ is a holomorphic map satisfying $\hat{R}f\hat{R} = f$. Now we see that $\hat{R}'(0) = \mu^k$. Choose a holomorphic map g, tangent to the identity, such that $g\hat{R}g^{-1} = R_{\mu^k}$. Put $\tilde{f} = gfg^{-1}$. Now let A'_j be a canonical collection of $\tilde{f}$ satisfying $A'_{j+2k} = A'_jR_{\mu^{-k}}$. Then $A'_{j+1}A'^{-1}_j = L_{c_{j+1}}\Phi_jL_{-c_j}$ with $c_j = c_{j+2k}$.

The remaining assertions in the theorem follow from Theorem 2.5 and Lemma 8.3. □

We say that F is *I-invariant*, if $IfI \in F$ for all $f \in F$. Since $I^2 = \mathrm{Id}$, then $IFI \subset F$ if and only if $IFI = F$. Denote by $\mathcal{F}_{p,I}$ be the set of $F \in \mathcal{F}_p$ such that $IFI = F$. We are interested in classifying F under a single change of holomorphic coordinates that preserves I.

Proposition 8.5. *Let $f \in \mathcal{A}_p$, and let $\tau = \tau_f, \omega = \omega_f, q = q_f, d = d_f$ be as in Proposition 3.5. Assume that $d_f \neq 0$.*

a) Assume that $f = IfI$. Then p is even and each $\mathcal{F}_{s,u}(f)$ belongs to $\mathcal{F}_{p,I}$.

b) Assume that $IfI = f^{-1}$. Then p is odd, $I\tau I = \tau f^{-2/d}$ and $I\omega I = \omega$. There is a group in $\mathcal{F}_{s,u}(f)$ invariant by I, if and only if $q = \delta$, s divides q, in which case $<\tau^{q/s} f^{-\frac{q}{sd}}, f>$ is the unique group in $\mathcal{F}_{s,u}(f)$ invariant under I.

Proof. Let f be a generator of $F \cap \mathcal{A}_0$. Since IfI remains a generator, then $IfI = f$ or $IfI = f^{-1}$. For the former we already know by Lemma 3.1 that I commutes with each element of $\hat{\mathcal{C}}(f)$. In particular all $\mathcal{F}_{s,u}(f)$ are I-invariant.

Assume now that I reverses f. We have $I\hat{\mathcal{C}}(f)I = \hat{\mathcal{C}}(f)$. For if $gfg^{-1} = f$, then $IgI \in \hat{\mathcal{C}}(f)$ and if g is additionally periodic then $IgI = g$. Since $\hat{\mathcal{C}}(f) = <\tau, f^{1/d}>$, we can write $\tau^{-1}I\tau I = f^{l/d}$. Raising both sides to q-th power yields $-2\delta = ql$. Since q divides p and p is odd, we get $q = \delta$ with $l = -2$.

Assume now that $F = <\tau^{q/s} f^{d'/d}, f>$ is invariant by I. Then we have

$$\tau^{q/s} f^{-2\frac{\delta}{ds} - \frac{d'}{d}} = I\tau^{q/s} f^{d'/d} I = \tau^{q/s} f^{d'/d} f^l$$

for some $l \in \mathbf{Z}$. Thus d divides $2(\frac{\delta}{s} + d')$, i.e., s divides $2s'$. Since s, dividing $q = \delta$, is odd, then s divides s'. Since $s' = s'_0 + j\frac{s}{(d,s)}$ with $0 \leq j < (d,s)$ and $s' = l\frac{s}{(d,s)}$, then s' is unique. Conversely, if all the conditions are satisfied, then $<\tau^{q/s} f^{-q/(sd)}, f>$ is I-invariant. □

For p even, define

$$\hat{\mathcal{N}}_p^* = \big\{(a, s, u, \Phi)\colon \Phi \in \mathcal{N}_{p,p/2}, d_\Phi \neq 0, \arg a \in [0, \pi), \\ s | q_\Phi, 0 \leq u < (d, s)\big\},$$

and define $(\tilde{a}, \tilde{s}, \tilde{u}, \tilde{\Phi}) \overset{*}{\sim} (a, s, u, \Phi)$, if $(\tilde{a}, \tilde{s}, \tilde{u}) = (a, s, u)$ and if $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_j L_{-c_j}$ for some $c_j = c_{j+p} \in \mathbf{C}$.

For p odd, define

$$\hat{\mathcal{N}}_p^* = \big\{(a, s, \Phi)\colon \arg a \in [0, \pi), \Phi \in \mathcal{N}_p, d_\Phi \neq 0, q_\Phi = \delta_\Phi, s | q_\Phi\big\},$$

and define $(\tilde{a}, \tilde{s}, \tilde{\Phi}) \overset{*}{\sim} (a, s, \Phi)$, if $(\tilde{a}, \tilde{s}) = (a, s)$ and $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_j L_{-c_j}$ for some $c_j = -c_{j+p} \in \mathbf{C}$.

Let $\mathcal{F}_{p,I}/\!\sim$ be the set of holomorphic equivalence classes of $F \in \mathcal{F}_{p,I}$ under holomorphic maps g satisfying $gI = Ig$, and let $\mathcal{F}_{p,I}/\overset{*}{\sim}$ be the set of holomorphic equivalence classes of $F \in \mathcal{F}_{p,I}$ under g satisfying the additional condition $g'(0) = 1$.

Theorem 8.6. *There is a one-to-one correspondences between $\{F \in \mathcal{F}_{p,I}\colon d_F \neq 0\}/\overset{*}{\sim}$ and $\hat{\mathcal{N}}_p^*/\overset{*}{\sim}$.*

Proof. We shall first find a one-to-one correspondence between $\mathcal{F}_{p,I}/\overset{*}{\sim}$ and $\hat{\mathcal{N}}_p^*/\overset{*}{\sim}$.

For $F \in \mathcal{F}_{p,I}$, we need to define a, s and Φ for F. We take a generator f_0 of $F \cap \mathcal{A}_0$. Proposition 8.5 says that $If_0I = f_0$ with p even, or $If_0I = f_0^{-1}$ with p odd.

Let s_F be the order of $F/F \cap \mathcal{A}_0$. We now define a_F and Φ_F. There is a unique generator $f(z) = z + a\mu_0 z^{p+1} + O(|z|^{p+2})$ with $0 \le \arg a < \pi$, and we assign $a_F = a$. Choose a unique R_λ with $\lambda > 0$ such that $f_1 = R_\lambda^{-1} f R_\lambda \in \mathcal{I}_p^*$, and define $\Phi_F = \Phi_{f_1} \in \mathcal{N}_p$. When p is even, let u be the unique integer such that $F = F_{s,u}(f)$. The rest of proof is similar to that of Theorem 7.10. □

For p even, define

$$\hat{\mathcal{N}}_p = \big\{(s, u, \Phi)\colon (1, s, u, \Phi) \in \hat{\mathcal{N}}_p^*\big\}.$$

Given $(\tilde{s}, u, \tilde{\Phi}), (s, u, \Phi) \in \hat{\mathcal{N}}_p$, we denote $(\tilde{s}, \tilde{u}, \tilde{\Phi}) \sim (s, u, \Phi)$, if $(\tilde{s}, \tilde{u}) = (s, u)$, and $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_{j+2k} L_{-c_j}$ with $c_j = c_{j+p}$, or if $(\tilde{s}, \tilde{u}) = (s, u^-)$, and $\tilde{\Phi}_j = L_{c_{j+1}} R_{-1} \Phi_{j+2k+1} R_{-1} L_{-c_j}$ with $c_j = c_{j+p}$. (Recall that $\mathcal{F}_{s,u}(f) = \mathcal{F}_{s,u^-}(f^{-1})$ and $\Phi_f = \Phi$.) When p is odd, define

$$\hat{\mathcal{N}}_p = \big\{(s, \Phi)\colon (1, s, \Phi) \in \hat{\mathcal{N}}_p^*\big\}.$$

Given $(\tilde{s}, \tilde{\Phi}), (s, \Phi) \in \hat{\mathcal{N}}_p$, we denote $(\tilde{s}, \tilde{\Phi}) \sim (s, \Phi)$, if $\tilde{s} = s$, and $\tilde{\Phi}_j = L_{c_{j+1}} \Phi_{j+2k} L_{-c_j}$ for all j or $\tilde{\Phi}_j = L_{c_{j+1}} R_{-1} \Phi_{j+2k+1} R_{-1} L_{-c_j}$ for all j with $c_j = -c_{j+p}$.

Analogous to Theorem 7.11, we have

Theorem 8.7. *There is a one-to-one correspondences between* $\{F \in \mathcal{F}_{p,I}\colon d_F \neq 0\}/\sim$ *and* $\hat{\mathcal{N}}_p/\sim$.

9. Exceptional non-abelian subgroups of Cerveau and Moussu

Recall that $\mathcal{A}_0$ is the group of germs of local biholomorphic mappings of $\mathbf{C}$ fixing the origin. Denote by $\mathcal{A}_0^* \subset \mathcal{A}_0$ the subgroup consisting of mappings that are tangent to the identity.

According to Cerveau and Moussu [4], a non-abelian subgroup $G \subset \mathcal{A}_0$ is *exceptional*, if $G \cap A_0^*$ is generated by some $h \neq \mathrm{Id}$. A theorem of Cerveau and Moussu [4] says that if $G, \tilde{G} \subset \mathcal{A}_0$ are two non-exceptional subgroups and g is a formal holomorphic map satisfying $gGg^{-1} \equiv \{gfg^{-1}\colon f \in G\} = \tilde{G}$, then g is actually convergent. They also studied relations among the classification of exceptional non-abelian subgroups, mappings in $\mathcal{A}_p$ that are reversible by a periodic maps, and the Ecalle-Voronin theory. The main purpose of this section is to make the results of Cerveau and Moussu [4] on exceptional non-abelian groups more precise; in particular, we shall determine when the classification of the exceptional non-abelian groups G is equivalent to that of a generator of $G \cap \mathcal{A}_0^*$.

By Cerveau and Moussu [4], we know that each exceptional non-abelian subgroup $G \subset \mathcal{A}_0$ is generated by two elements, ω_0 and h_0, where $h_0 = \omega_0 h_0^{-1} \omega_0^{-1}$ is in $\mathcal{A}_p$, ω_0 is periodic, and $\omega_0'(0) = \sqrt{\mu}^m$ with $m > 0$ odd. One can find g so that $\omega_1(z) = g\omega_0 g^{-1}(z) = \sqrt{\mu}^m z$. Then $gGg^{-1} = <\omega_1, h_1>$ for $h_1 = gh_0g^{-1}$. We have $m = m_1 m_2$ with $m_1 = (m, p)$. Since m is odd, then $\omega_1'(0)$ and $\sqrt{\mu}^{m_1}$ are both primitive $\frac{2p}{m_1}$-th roots of unity. Put $\omega(z) = \sqrt{\mu}^{m_1} z$. We have $<\omega_1> = <\omega>$, and $\omega h \omega^{-1} = h^{-1}$ too. By a linear transformation, one may assume

that h is in $\mathcal{A}_p^*$. From now on we assume that the non-abelian exceptional subgroups have the form

$$(9.1) \qquad G = <\omega, h>, \quad \omega h^{-1}\omega^{-1} = h \in \mathcal{A}_p^*, \qquad \omega(z) = \sqrt{\mu}^m z, \quad m|p, m > 0.$$

We first remark that ω is an invariant. If $g<\omega, h>g^{-1} = <\tilde{\omega}, \tilde{h}>$, then $ghg^{-1} = \tilde{h}$, or $\tilde{h}^{-1}$. Note that $2p/m$ is the order of the quotient group $<\omega, h>/<h> \equiv <\omega>$. Thus $\tilde{\omega} = \omega$ (but $g\omega g^{-1} \neq \tilde{\omega}$ in general).

Lemma 9.1. *Let ω be holomorphic and $h \in \mathcal{A}_p$ with $d = d_h \neq 0$. If $h^{-1} = \omega h \omega^{-1}$, then we have (a) ω is periodic, and $\omega'(0) = \sqrt{\mu}^{m_0}$ with m_0 is odd. (b) $\omega\tau\omega^{-1} = \tau h^{-2/d}$, $p/\delta | m_0$, and $q = \delta$. (c) The square root $\omega_h^{1/2} = \sqrt{\mu}^{p/\delta} z + O(2)$ reverses h and satisfies $(\omega_h^{1/2})^k = \omega$ for $k = m_0/(p/\delta)$.*

Proof. Since ω reverses h then $\omega^{2p} = h^{k/d}$ and $h^{-k/d} = \omega h^{k/d}\omega^{-1} = \omega\omega^{2p}\omega^{-1} = h^{k/d}$. Hence $k = 0$ and ω is periodic. Using $\omega h \omega^{-1} = h^{-1}$, we see from the coefficients of z^{p+1} that $\omega'(0)^p = -1$. So $\omega'(0) = \sqrt{\mu}^{m_0}$ for some odd $m_0 > 0$. Since ω is periodic and ω^2 commutes with h, $\omega^2 = \omega_h^a$ and $\omega^{2\delta} = 1$. Hence p/δ divides m_0. Since m_0 is odd, we conclude $q = \delta$. Since ω reverses h, then $\omega\tau\omega^{-1}$ commutes with h. So $\omega\tau\omega^{-1} = \tau h^{l/d}$ Raising both sides to q-th power, we get $h^{-\delta/d} = h^{\delta/d + ql/d}$. Thus $l = -2\delta/q = -2$. Let ω_h be the generator of periodic elements of $\hat{\mathcal{C}}(h)$ satisfying $\omega_h'(0) = \mu^{p/\delta}$. Write $m_0 = kp/\delta$ and $\omega_1 = \omega\omega_h^{-(k-1)/2}$. Then ω_1 reverses h. Since $\omega\omega_h\omega^{-1}$ commutes with h, $\omega\omega_h\omega^{-1} = \omega_h$. Now $(\omega_1^2)'(0) = \mu^{p/\delta} = \omega_h'(0)$, ω_1^2 is periodic and commutes with h. Thus $\omega_1^2 = \omega_h$. □

Proposition 9.2. *Let $\omega(z) = \sqrt{\mu}^m z$ with $m > 0$ and $m|p$. Assume that $h, \tilde{h} \in \mathcal{A}_p$ with $d_h \neq 0$ are reversible by ω. (a) There is a holomorphic g satisfying $g<\omega, h>g^{-1} = <\omega, \tilde{h}>$, if and only if there is a holomorphic g_1 satisfying $g_1 h g_1^{-1} = \tilde{h}$, and $g_1\omega g_1^{-1} = \omega$ or $\omega\tilde{h}$. (b) There is a holomorphic g so that $g\omega h g^{-1} = \omega$ and $ghg^{-1} = h$, if and only if d_h is even.*

Proof. For (a), the existence of g_1 implies $g_1<\omega, h>g_1^{-1} = <\omega, \tilde{h}>$ trivially. Assume now that $g<\omega, h>g^{-1} = <\omega, \tilde{h}>$. We have $g\omega g^{-1} = \omega^a h^b$. Looking at the linear parts, we see that $g\omega g^{-1} = \omega h^b$. We also have $ghg^{-1} = \tilde{h}$, or $\tilde{h}^{-1}$. When the latter occurs, replacing g by ωg, we may assume that $ghg^{-1} = \tilde{h}$. Put $g_1 = \tilde{h}^{b/2} g$ when b is even, and put $g_1 = \tilde{h}^{(b-1)/2} g$ when b is odd. We obtain $g_1 h g_1^{-1} = \tilde{h}$ and $g_1\omega g_1^{-1} = \omega$, or $\omega\tilde{h}$.

For (b), when d_h is even, $h^{1/2}\omega h h^{-1/2} = \omega$ and $h^{1/2} h h^{-1/2} = h$. Assume now that there is g satisfying $ghg^{-1} = h$ and $g\omega h g^{-1} = \omega$. The former implies that $g = \tau^a h^{k/d}$. By Lemma 9.1, we have $\omega\tau\omega^{-1} = \tau h^{-2/d}$. Now $\omega^{-1} g\omega h g^{-1} = 1$ implies that $h^{-2a/d - 2k/d + 1} = 1$, which holds for some choice of a, k, if and only if d is even. □

Remark. Obviously, $<\omega h, h> = <\omega, h>$, $(\omega h)^2 = \omega^2$ (so ωh is periodic) and ωh reverses h. Proposition 9.2 concludes that classifying exceptional non-abelian groups $<\omega, h>$ with $h \in \mathcal{A}_p$ agrees with classifying $h = \omega h^{-1}\omega^{-1} \in \mathcal{A}_p$ under local biholomorphic mappings preserving ω, if and only if d_h is even. This clarifies a remark of Cerveau and Moussu [4].

Proposition 9.3. *Let $\omega(z) = \sqrt{\mu}^m z$ with $m|p$. Assume that $h, \tilde{h} \in \mathcal{A}_p$ are reversible by ω. Let $\omega_h^{1/2}$ and $\omega_{\tilde{h}}^{1/2}$ have the linear parts $\sqrt{\mu}^{p/\delta_h} z$ and $\sqrt{\mu}^{p/\delta_{\tilde{h}}} z$, respectively. If $g\omega g^{-1} = \omega$ and*

$ghg^{-1} = \tilde{h}$, then $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}$; if $g\omega g^{-1} = \omega\tilde{h}$ and $ghg^{-1} = \tilde{h}$, then $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}\tilde{h}$. The converse is also true, if $ghg^{-1} = \tilde{h}$, $(\omega_h^{1/2})^k = (\omega_{\tilde{h}}^{1/2})^k = \omega$ for $m = kp/\delta$ and $\delta = \delta_h$.

Proof. Assume that g satisfies $g\omega g^{-1} = \omega$ and $ghg^{-1} = \tilde{h}$. Then $(g\omega_h^{1/2}g^{-1})^k = (\omega_{\tilde{h}}^{1/2})^k$, where $m = kp/\delta$ and $\delta = \delta_h = \delta_{\tilde{h}}$. Note that both $g\omega_h^{1/2}g^{-1}$ and $\omega_{\tilde{h}}^{1/2}$ reverse $\tilde{h}$, by Lemma 9.1. Thus $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}\tilde{h}^{l/d}$. Raising both sides to k-th power and noting that k is odd, we get $g\omega g^{-1} = \omega\tilde{h}^{l/d}$. Thus $l = 0$, which shows that $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}$. Conversely, start with $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}$ and $ghg^{-1} = \tilde{h}$. Raising the former to k-th power, we get $g\omega g^{-1} = \omega$ for $(\omega_h^{1/2})^k = (\omega_{\tilde{h}}^{1/2})^k = \omega$.

Assume now that $g\omega g^{-1} = \omega\tilde{h}$ and $ghg^{-1} = \tilde{h}$. Again $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}\tilde{h}^{l/d}$. Raising both sides to k-th power, we get $l/d = 1$, i.e., $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}\tilde{h}$. Conversely, assume that $g\omega_h^{1/2}g^{-1} = \omega_{\tilde{h}}^{1/2}\tilde{h}$ and $ghg^{-1} = \tilde{h}$. Rasing the former to the k-th power, we get $g\omega g^{-1} = \omega\tilde{h}$. □

Recall that each exceptional non-abelian group is equivalent to a group $<\omega, h>$. By Proposition 9.3, two groups $<\omega, h>$ and $<\omega, \tilde{h}>$ are equivalent, if and only if $<\omega_h^{1/2}, h>$ and $<\omega_{\tilde{h}}^{1/2}, \tilde{h}>$ are equivalent. By another change of coordinates, one may assume that $\omega_h^{1/2}(z) = \sqrt{\mu}^{p/\delta}z$. This shows that the set of equivalence classes of exceptional non-abelian groups G with $G \cap \mathcal{A}_0^* \subset \mathcal{A}_p$ is identified with a positive odd integer $m|p$ ($2p/m$ is the order of $G/G \cap (\mathcal{A}_p)$) together with the set of equivalence classes of

$$<\omega_h^{1/2}, h>, \quad h = \omega_h^{1/2}h^{-1}\omega_h^{-1/2} \in \mathcal{A}_p^*, \quad \omega_h^{1/2}(z) = \sqrt{\mu}^{p/\delta}z.$$

Fix positive integers δ, p satisfying $\delta|p$, and fix a positive integer d. Let $\mathcal{M}_{p,\delta,d}^r$ be the set of $\Phi \in \mathcal{M}_p$, $d_\Phi = d$, satisfying

$$\Phi_j = \xi + \sum_{k\geq 1} c_{j,k}e^{\epsilon_j 2\pi i k\xi} = I\Phi_{j+p/\delta}I, \quad j = 1, \dots, p/\delta,$$
$$\Phi_{j+2p/\delta} = \Phi_j, \quad j = 1, \dots, 2p, \quad \Phi_{j'+2p/\delta'} \neq \Phi_{j'},$$

for any $\delta' > \delta$ and for some j' dependent of δ'. Note that $\mathcal{M}_{p,\delta,d}^r$ is non-empty; for instance, $\Phi_1 = \xi + e^{\epsilon_1 2\pi i\xi/d} = I^l\Phi_{1+lp/\delta}I^l$ and $\Phi_j = \xi$ for $j \neq 1 + lp/\delta$ form an element in $\mathcal{M}_{p,\delta,d}^r$. Given $\Phi, \tilde{\Phi} \in \mathcal{M}_{p,\delta,d}^r$, we say that $\Phi, \tilde{\Phi}$ are equivalent, and denote $\tilde{\Phi} \sim \Phi$, if there is k so that

$$\tilde{\Phi}_{j+2k} = \Phi_j, \quad j = 1, \dots, 2p$$

or if there is k so that

$$\tilde{\Phi}_{j+2k} = L_{-1/2}\Phi_j L_{1/2}, \quad j = 1, \dots 2p, \quad d_\Phi \text{ odd}.$$

Let $\mathcal{E}_{p,\delta,d,m}$ be the set of non-abelian groups of the form

$$G = <\omega, h>, \quad h \in \mathcal{A}_p^*, \quad \omega(z) = \sqrt{\mu}^m(z), \quad \omega h\omega^{-1} = h^{-1}, \quad d_h = d, \delta_h = \delta, m|p.$$

Denote by $\mathcal{E}_{p,\delta,d,m}/\sim$ the equivalence classes of the groups under local biholomorphic transformations.

Theorem 9.4. *Fix positive integers m, p, δ satisfying $m|p$, $p/\delta|m$ and let d be a positive integer. If m is odd, then $\mathcal{E}_{p,\delta,d,m}$ is non-empty and there is a one-to-one correspondence between $\mathcal{E}_{p,\delta,d,m}/\sim$ and $\mathcal{M}^r_{p,\delta,d}/\sim$.*

Proof. Fix m. Let $G \in \mathcal{E} = \mathcal{E}_{p,\delta,d,m}$. We already know that G is equivalent to some $<\kappa^{m\delta/p}, h>$ with $\kappa(z) = \omega_h^{1/2}(z) = \sqrt{\mu}^{p/\delta} z, h \in \mathcal{A}_p^*$, and $h^{-1} = \kappa h \kappa$. Let A_j^1, Ω_j^1 be a canonical collection of h. Since $\kappa^l h \kappa^{-l} = h^{(-1)^l}$ and $\kappa'(0) = \sqrt{\mu}^{p/\delta}$ then for each k, $A^{k+1}_{j+kp/\delta} = I^k A_j^1 \kappa^{-k}$ and $\Omega^{k+1}_{j+kp/\delta} = \kappa^k \Omega_j^1$ form a canonical collection for h. By the uniqueness of canonical collections, there exists a canonical collection $\tilde{A}_j$, $\Omega_j \subset \cap_{k=1}^{2\delta} \Omega_j^k$ so that $\tilde{A}_j = L_{a_j^k} A_j^k$ on Ω_j for $k = 1, \ldots, 2\delta$. Put

$$A_j = A_j^1, A_{j+lp/\delta} = I^l A_j^1 \kappa^{-l}, \quad j = 1, 2, \ldots, p/\delta, l = 1, 2, \ldots, 2\delta - 1.$$

Clearly, $A_{j+p/\delta} = I A_j \kappa^{-1}$ for $j = 1, \ldots, (2\delta-1)p/\delta$. For $(2\delta-1)p/\delta < j \le 2p$, we have $A_{j+p/\delta} = A_{j+p/\delta-2p} = I^{-1} A^1_{j+2p/\delta-2p} \kappa = I^{-1} A^1_{j-(2\delta-2)p/\delta} \kappa = I A_j \kappa^{-1}$. Obviously $A_j^{-1} L_1 A_j = h$. Write $\Phi_j = A_{j+1} A_j^{-1} = \xi + \sum_{k\ge 0} c_{j,k} e^{\epsilon_j 2\pi i k \xi}$. Then we have $\Phi_{j+p/\delta} = I \Phi_j I$ for all j; in particular $\Phi_{j+2p/\delta} = \Phi_j$ for all j. Next we need to achieve $c_{j0} = 0$ for all j, by replacing A_j with $A_j^* = L_{b_j} A_j$ for $j = 1, 2, \ldots 2p$. Of course, we still need $A^*_{j+p/\delta} = I A_j^* \kappa^{-1}$, which is equivalent to $b_{j+p/\delta} = -b_j$ for all j. Let c^*_{j0} be the constant terms in the expansion of $\Phi_j^* = A^*_{j+1} A^{*-1}_j$. Then $c^*_{j0} = b_{j+1} - b_j + c_{j0}$. To achieve $c^*_{j0} = 0$ for $j = 1, \ldots, p/\delta$, taking the sum yields $-2b_1 = c_{10} + \cdots + c_{m0}$. Taking $b_{j+1} = b_j - c_{j0}$ for $j = 1, \ldots, p/\delta - 1$ gives us $c^*_{10} = \cdots = c^*_{p/\delta 0} = 0$. Put $b_{j+lp/\delta} = (-1)^l b_j$ for $l = 1, \ldots, 2\delta - 1$. Then we have $\Phi^*_{j+lp/\delta} = I^l \Phi^*_j I^l$; in particular $c^*_{j0} = 0$ for all j. Having achieved $c_{j0} = 0$ for $\Phi_j = A_{j+1} A_j^{-1}$, we need only to show that $\Phi_{j+2p/\delta'} = \Phi_j$ does not hold for some j if $\delta' > \delta$. Otherwise, put $A^{-1}_{j+2p/\delta'+1} A_j = A^{-1}_{j+2p/\delta'+1} A_{j+1} = g$. Then g commutes with h, $g^{2\delta'} = 1$ and $g'(0) = \mu^{p/\delta'}$, which contradicts $\delta_h = \delta$.

Assume now that $G, \tilde{G} \in \mathcal{E}$ are equivalent. Then G and $\tilde{G}$ are equivalent to $<\kappa^{m\delta/p}, h>$ and $<\kappa^{m\delta/p}, \tilde{h}>$, respectively. Proposition 9.3 says that there is g so that $ghg^{-1} = \tilde{h}$ and $g\kappa g^{-1} = \kappa$ or $\kappa\tilde{h}$.

Consider first that $g\kappa g^{-1} = \kappa$. Put $\Phi_h = \Phi$ and $\tilde{\Phi} = \Phi_{\tilde{h}}$. We want to show that $\tilde{\Phi}_{j+2k} = \Phi_j$. By Theorem 2.5, we have $\tilde{\Phi}_{j+2k} = L_{a_{j+1}} \Phi_j L_{-a_j}$. Obviously, $a_1 = \cdots = a_{2p} = a$. We also have $g'(0) = \mu^k$ and $g = \tilde{A}^{-1}_{j+2k} L_a A_j$. Recall that $\kappa = A^{-1}_{j+p/\delta} I A_j$ and $\kappa = \tilde{A}^{-1}_{j+p/\delta} I \tilde{A}_j$. Now $1 = \kappa^{-1} g \kappa g^{-1} = \tilde{A}_j^{-1} I L_a I L_{-a} \tilde{A}_j$, i.e. $a = 0$. Assume now that $ghg^{-1} = \tilde{h}$ and $g\kappa g^{-1} = \kappa\tilde{h}$. We want to show that $\tilde{\Phi}_{j+2k} = L_{-1/2} \Phi_j L_{1/2}$. We still have $\tilde{\Phi}_{j+2k} = L_a \Phi_j L_{-a}$, $g = \tilde{A}^{-1}_{j+2k} L_a A_j$. Now $\tilde{A}_j^{-1} L_1 A_j = \tilde{h} = \kappa^{-1} g \kappa g^{-1} = \tilde{A}_j^{-1} I L_a I L_{-a} \tilde{A}_j$, i.e. $a = -1/2$. When $d_h = d_{\tilde{h}}$ is even, we conclude $\Phi_{j+2k} = \tilde{\Phi}_j$.

Assume now that $\tilde{\Phi} = \Phi_{\tilde{h}}$ and $\Phi = \Phi_h$ satisfy $\tilde{\Phi}_{j+2k} = \Phi_j$. We have $A_j^{-1} L_1 A_j = h$, $\tilde{A}_j^{-1} L_1 \tilde{A}_j = h$, $\kappa = A^{-1}_{j+p/\delta} I A_j$, and $\kappa = \tilde{A}^{-1}_{j+p/\delta} I \tilde{A}_j$. Put $g = \tilde{A}^{-1}_{j+2k} A_j$. Then $ghg^{-1} = \tilde{A}^{-1}_{j+2k} L_1 \tilde{A}_{j+2k} = \tilde{h}$. Also $g\kappa g^{-1} = \tilde{A}^{-1}_{j+2k} I \tilde{A}_{j+2k-p/\delta} = \kappa$. Assume now that $\tilde{\Phi}_{j+2k} =$

$L_{-1/2}\Phi_j L_{1/2}$. Put $g = \tilde{A}^{-1}_{j+2k} L_{-1/2} A_j$. Then $ghg^{-1} = \tilde{A}^{-1}_{j+2k} L_{-1/2} L_1 L_{1/2} A_{j+2k} = \tilde{h}$. Also $g\kappa g^{-1} = \tilde{A}^{-1}_{j+2k} L_{-1/2} I L_{1/2} \tilde{A}_{j+2k-p/\delta} = \kappa\tilde{h}$.

For the realization, we need to construct, for each $\Phi \in \mathcal{M}$, an $h \in \mathcal{A}^*_p$ satisfying the following: $\Phi_{h,j} = \Phi_j$, $\kappa h \kappa^{-1} = h^{-1}$ for $\kappa(z) = \sqrt{\mu}^{p/\delta} z + O(2)$. Then $<\kappa^{m\delta/p}, h> \in \mathcal{E}^r_{p,\delta,d}$ is a realization for Φ. We have $h_0 \in \mathcal{A}^*_p$ and a canonical collection $\tilde{A}_j$, $\tilde{\Omega}_j$ of h_0 satisfying $\tilde{A}^{-1}_{j+1}\tilde{A}_j = \Phi_j$. We have $\Phi_{j+p/\delta} = I\Phi_j I$. Thus $\tilde{A}^{-1}_{j+p/\delta+1} I \tilde{A}_{j+1} = \tilde{A}^{-1}_{j+p/\delta} I \tilde{A}_j = \kappa_0$ is periodic and satisfies $\kappa_0'(0) = \sqrt{\mu}^{p/\delta}$ and $\kappa_0^{2p/\delta} = 1$. We also have $\kappa_0 h_0 \kappa_0^{-1} = h_0^{-1}$. So κ_0^2 commutes with h_0. We want to show that κ_0^2 generates all periodic centralizers of h_0. Assume that a periodic $\omega_0(z) = \mu^{p/\delta'} z + O(2)$ commutes with h_0 and that $\delta|\delta'$. Then $\omega_0 h_0 \omega_0^{-1} = h_0$ implies that $\omega_0 = A^{-1}_{j+2p/\delta'} L_{a_j} A_j$ and $\Phi_{j+2p/\delta'} = L_{a_{j+1}} \Phi_j L_{-a_j}$. First we see $a_1 = \cdots = a_{2p} = a$. Now $\omega_0^{\delta'} = 1$ implies that $\delta' a = 0$. So $a = 0$ and $\Phi_{j+2p/\delta'} = \Phi_j$, which violates the definition of $\mathcal{M}$, if $\delta' > \delta$.

Next we choose $g_1'(0) = 1$ so that $g_1 \kappa_0 g^{-1} = \kappa$, and so that $h_1 = g_1 h_0 g_1^{-1}$ is still in $\mathcal{A}^*_p$. In particular, κ^2 generates all periodic centralizers of h_1. Let A'_j, Ω'_j be a set of canonical collection of h_1 satisfying $A'_{j+p/\delta} = IA'_j\kappa_1^{-1}$, constructed above for the group $<\kappa, h_1>$. We only have

$$\Phi_{h_1,j} = L_{a_{j+1}} \Phi_j L_{-a_j}.$$

Since all constant terms in the expansions of $\Phi_{h_1,j}$ and Φ_j are zeros, we get $a_1 = \cdots = a_{2p} = a$. Applying I to both sides, we see that L_{2a} commutes with all Φ_j. Thus $2a = l/d$. When l is even, we get immediately $\Phi_{h_1,j} = \Phi_j$. When $a = \frac{k}{d} + \frac{1}{2d}$, we obtain

$$\Phi_{h_1,j} = L_{\frac{1}{2d}} \Phi_j L_{-\frac{1}{2d}}.$$

Let g_2 be holomorphic with $g_2'(0) = 1$ so that $g_2 h_1^{1/d} \kappa g_2^{-1} = \kappa$, and so that $h = g_2 h_1 g_2^{-1} \in \mathcal{A}^*_p$. Then h is reversible by κ and κ^2 generates all periodic centralizers of h. Define $A_j = L_{-\frac{1}{2d}} A'_j g_2^{-1}$ for $j = 1, \ldots, 2p$. Then $A_j^{-1} L_1 A_j = g_2 A_j'^{-1} L_1 A'_j g_2^{-1} = g_2 h_1 g_2^{-1} = h$, $IA_j\kappa^{-1} = L_{\frac{1}{2d}} A'_{j+p/\delta} \kappa g_2^{-1} \kappa^{-1} = L_{\frac{1}{2d}} A'_{j+p/\delta} h_1^{-1/d} g_2^{-1} = A_{j+p/\delta}$, and $A_{j+1} A_j^{-1} = L_{-\frac{1}{2d}} A'_{j+1} A_j'^{-1} L_{\frac{1}{2d}} = \Phi_j$. □

Fix positive integers m, p, δ satisfying $m|p$, $p/\delta|m$ and let d be a positive integer. Denote by $\mathcal{F}^r_{p,\delta,d,m}$ the set of holomorphic functions $f \in \mathcal{A}_p$ satisfying $h^{-1} = \omega h \omega^{-1}$ for $\omega(z) = \sqrt{\mu}^m z$. Denote $\tilde{h} \overset{*}{\sim} h$, if there is holomorphic g so that $g\omega g^{-1} = \omega$ and $ghg^{-1} = \tilde{h}$. Denote $\tilde{\Phi} \overset{*}{\sim} \Phi$, if $\tilde{\Phi}_{j+2k} = \Phi_j$ for all j and for some k.

Proposition 9.3 and the above proof imply the following

Theorem 9.5. *Fix positive integers p, δ, d, m with m odd, $m|p$ and $p/\delta|m$. There is a one-to-one correspondence between $\mathcal{F}^r_{p,\delta,d,m}/\overset{*}{\sim}$ and $\mathcal{M}^r_{p,\delta,d}/\overset{*}{\sim}$.*

References

[1] P. Ahern and J.-P. Rosay, *Entire functions, in the classification of differentiable germs tangent to the identity, in one or two variables*, Trans. Amer. Math. Soc. **347**(1995), no. 2, 543–572.

[2] I.N. Baker, *Permutable power series and regular iteration*, J. Austral. Math. Soc. **2**(1961/1962) 265-294.

[3] M. Berthier, D. Cerveau and R. Meziani, *Transformations isotropes des germes de feuilletages holomorphes*, J. Math. Pures Appl. (9) **78**(1999), no. 7, 701–722.
[4] D. Cerveau and R. Moussu, *Groupes d'automorphismes de* $(C, 0)$ *et equations diff erentielles* $ydy+\cdots =$ 0, Bull. Soc. Math. France **116**(1988), no. 4, 459–488.
[5] J. Écalle, *Théorie itérative: introduction à la théorie des invariants holomorphes*, J. Math. Pures Appl. (9) **54**(1975), 183–258.
[6] J. Écalle, *Les fonctions résurgentes*, I, II. Publications Mathématiques d'Orsay 81, 5, 6, 1-247, 248-531, Université de Paris-Sud, Département de Mathématique, Orsay, 1981.
[7] P.M. Elizarov, Yu.S. Il′yashenko, A.A. Shcherbakov, and S.M. Voronin, *Finitely generated groups of germs of one-dimensional conformal mappings, and invariants for complex singular points of analytic foliations of the complex plane*, in *Nonlinear Stokes phenomena*, 57-105, Adv. Soviet Math., 14, Amer. Math. Soc., Providence, RI, 1993.
[8] X. Gong, *Conformal maps and non-reversibility of elliptic area-preserving maps*, Math. Z. **237**(2001), no. 2, 275-300.
[9] Yu.S. Il′yashenko, *Nonlinear Stokes phenomena*, in *Nonlinear Stokes phenomena*, 1-55, Adv. Soviet Math., 14, Amer. Math. Soc., Providence, RI, 1993.
[10] L.S.O. Liverpool, *Fractional iteration near a fix point of multiplier* 1, J. London Math. Soc. (2) 9 (1974/75), 599-609.
[11] E. Kasner, *Conformal geometry,* in Proceedings of the Fifth International Mathematical Congress, 2, 81–87, 1913.
[12] B. Malgrange, *Travaux d'Écalle et de Martinet-Ramis sur les systèmes dynamiques,* Bourbaki Seminar, Vol. 1981/1982, pp. 59-73, Astérisque, 92-93, Soc. Math. France, Paris, 1982.
[13] I. Nakai, *The classification of curvilinear angles in the complex plane and the groups of holomorphic diffeomorphisms*, Ann. Fac. Sci. Toulouse Math. (6) **7**(1998), no. 2, 313-334.
[14] G.A. Pfeiffer, *On the conformal geometry of analytic arcs*, Amer. J. Math. **37**(1915), no. 4, 395–430.
[15] G.A. Pfeiffer, *On the conformal mapping of curvilinear angles. The functional equation* $\phi[f(x)] = a_1\phi(x)$. Trans. Amer. Math. Soc. **18**(1917), no. 2, 185–198.
[16] H. Poincaré, *Les fonctions analytiques de deux variables et la representation conforme*, Rend. Circ. Mat. Palermo **23**(1907), 185–220.
[17] A.A. Shcherbakov *Topological classification of germs of conformal mappings with an identical linear part*, (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1982, no. 3, 52-57.
[18] J.-M. Trépreau, *Discrimination analytique des difféomorphismes résonants de* $(\mathbf{C}, 0)$ *et réflexion de Schwarz*, preprint.
[19] S.M. Voronin, *Analytic classification of germs of conformal mappings* $(\mathbf{C}, 0) \to (\mathbf{C}, 0)$, Functional Anal. Appl. **15**(1981), no. 1, 1-13.
[20] S.M. Voronin, *Analytic classification of pairs of involutions and its applications*, Functional Anal. Appl. **16**(1982), no. 2, 94–100.

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