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Relativistically Covariant Formulation in Non-Linear Lagrangian Theories and Factor Ordering Problems

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Factor ordering problems in non-linear Lagrangian theories are discussed. It is shown that symmetrization of the Hamiltonian is necessary in order to get the covariant Dyson formula by considering the factor ordering. As a result, there appear terms proportional to $\hbar^2(\delta^3(0))^2$ in the effective Hamiltonian density. In the evaluation of divergent integrals appearing in non-linear theories it is also necessary to take care of the order of integrations. Following the prescriptions thus obtained, Charap's discussions of the soft-pion theorems in chiral dynamics are reexamined in each order of the perturbation expansion. It is concluded that the soft-pion theorems are maintained, order by order, in perturbation theory also under the condition that the factor ordering is completely taken into account.

§ 1. Introduction

Recently many physicists have their interest in the problems how to quantize non-linear Lagrangian theories and how to modify the covariant Dyson formula.[1)~8)] Here we consider the non-linear Lagrangian density that is generally expressed as

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\varphi_\alpha G_{\alpha\beta}(\varphi)\partial^\mu\varphi_\beta-V(\varphi), \tag{1·1}$$

where $G_{\alpha\beta}(\varphi)$ and $V(\varphi)$ are functions of field variables φ_α. If $G_{\alpha\beta}(\varphi)$ does not depend on φ_α, the Lagrangian is called the linear one. Incidentally if Eq. (1·1) is reduced to a linear Lagrangian density under a certain point transformation, we call it reducible Lagrangian density. Otherwise we call it irreducible one. As is well known, in the case of the mechanical system which is described by a linear Lagrangian density, the S-matrix in the interaction representation

$$S=T\exp\left(-\frac{i}{\hbar}\int\mathcal{H}_{\text{int}}(x)\,d^4x\right) \tag{1·2}$$

is also reexpressed by the Dyson formula

$$S=T^*\exp\left(+\frac{i}{\hbar}\int\mathcal{L}_{\text{int}}(x)\,d^4x\right). \tag{1·3}$$

However in non-linear cases this is not a correct prescription. Lee et al.[5)] and others[7)] showed that the Dyson formula must be modified in the following way:

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$$S = T^* \exp\left(+\frac{i}{\hbar}\int(\mathcal{L}_{\rm int} - \delta\mathcal{H})\,d^4x\right), \tag{1·4}$$

where the additional term $\delta\mathcal{H}$ is given by

$$\delta\mathcal{H} = \frac{i\hbar}{2}\delta^4(0)\ \ln\det\,(G_{\alpha\beta})\,. \tag{1·5}$$

In non-linear Lagrangian theories, however, there is another problem, that is, factor ordering problem, because there necessarily appear terms which contain non-commutable factors φ and $\dot{\varphi}$. As is shown by Kamefuchi et al.,[3] we must adopt the method of constructing the Hamiltonian from the Lagrangian in which the order of factors is taken into account, in order to satisfy the reasonable requirement that the S-matrix evaluated from the Hamiltonian remains unchanged under point transformations such as $\varphi_\alpha \to \varphi_\beta f_{\alpha\beta}(\varphi^2)$ where $f_{\alpha\beta}(0) = \delta_{\alpha\beta}$. Lee et al.[5] neglected this problem. How do we modify Lee et al.'s results when the factor ordering becomes important? This is the main theme in the first part of this paper.

Non-linear Lagrangians arise in many applications of field theory to situations of physical interest, for example, to the gravitation field and chiral dynamics. Charap and others[7] argued, on the basis of the naive perturbation calculation in which the factor ordering is neglected, that the masslessness of the pion and the soft pion theorems hold good in each order. Moreover they showed that their conclusion remains unchanged under the point transformation with respect to the pion field. As will be shown, however, their arguments are not completely correct. As one of the characteristic features of non-linear theories, there appear more divergent integrals than in linear theories and in the calculation of them we cannot exchange the order of integrations in momentum and coordinate space. If we make a careful calculation of the integrals, we get the additional divergent quantity proportional to $\hbar^2(\delta^3(0))^2$ and therefore Charap's conclusion appears to be altered. However if we take account of the factor ordering correctly, the counterterm arises in the Hamiltonian to cancel the $\hbar^2(\delta^3(0))^2$ term and then Charap's results are recovered in chiral dynamics.

In § 2 we shall consider a quantum mechanical system and give the method how to treat non-linear Lagrangians. There, considering the factor ordering problems, we shall have a correct effective Hamiltonian when we calculate in terms of the T^* product. In § 3 some remarks will be made on the evaluation of integrals in the covariant formulation. It will be shown that the naive perturbation calculation is not correct and the prescription in the momentum representation will be given to get a correct result. We shall apply the above consideration to quantum field theory in § 4 and especially to chiral dynamics. If we use the modified Dyson formula and evaluate integrals carefully, the resultant amplitudes satisfy the soft pion theorems and the masslessness of the pion in chiral dynamics. Section 5 is devoted to conclusion and discussion.

§ 2. Factor ordering problems in quantum mechanics

The aim of this section is that, considering a quantum mechanical system, we give a method with which we can treat a non-linear Hamiltonian taking account of the order of factors correctly and with which we can construct the modified Dyson formula in non-linear theories.

It has already been shown by Lee et al. that, when we express the S-matrix by means of the T^* product to construct the Feynman rule in non-linear theories, there appears generally the following additional term δH, which is proportional to $\hbar\delta(0)$, in the effective interaction Hamiltonian $H_{\rm int}$:

$$H'_{\rm int} = -L_{\rm int} + \delta H. \qquad (2\cdot1)$$

Then the S-matrix is given by

$$S = T^* \exp\left(-\frac{i}{\hbar}\int H'_{\rm int}(t)\,dt\right). \qquad (2\cdot2)$$

Here we take, as an example, a non-linear Lagrangian which describes one dimensional quantum mechanical system

$$L = \tfrac{1}{2}F(q)\dot{q}G(q)\dot{q}F(q) - \tfrac{1}{2}qq, \qquad (2\cdot3)$$

where q denotes the general coordinate of the system and $F(q)$ and $G(q)$ are functions of q. Then δH is expressed by

$$\delta H = \frac{i\hbar}{2}\delta(0)\ln(F^2G). \qquad (2\cdot4)$$

In their discussion, however, the factor ordering was neglected. So, necessary terms of the order $\hbar^2$ were completely disregarded. Accordingly, their discussion is regarded as correct for graphs containing one closed loop, but not for those containing more than two closed loops. In order to examine whether the neglection of the terms of the order $\hbar^2$ is justified or not, we consider as an example the following particular Lagrangian:

$$L = \tfrac{1}{2}(1+fq)^{-1}\dot{q}(1+fq)^{-2}\dot{q}(1+fq)^{-1} - \frac{1}{2f^2}[(1+fq)^{-1}-1]^2, \qquad (2\cdot5)$$

which can be reduced to the free Lagrangian

$$L = \tfrac{1}{2}\dot{x}\dot{x} - \tfrac{1}{2}x^2 \qquad (2\cdot6)$$

by the coordinate transformation

$$x = \frac{1}{f}[(1+fq)^{-1}-1]. \qquad (2\cdot7)$$

Coordinate transformations should leave the S-matrix invariant, so that each term in the perturbation expansion must also be invariant since now it does not contain any divergences. Therefore the Lagrangian (2·5) should lead us to the result $S=1$. But when the formula (2·1) is used, we have the incorrect result. (See Appendix A.) This example indicates that the neglection of the factor ordering is not justified.

In the following, we shall discuss how to construct the Feynman rule which gives the correct S-matrix, starting with a given non-linear Hamiltonian in which factor ordering is meaningful. For simplicity, we confine ourselves to an one-dimensional system described by an interaction Hamiltonian $H_{\text{int}}(q)$ in the interaction representation, where q denotes the coordinate. Generalization to a n-dimensional one can be made easily. As usual, the S-matrix for this system is expressed by

$$S=T\exp\left(-\frac{i}{\hbar}\int H_{\text{int}}(t)\,dt\right). \tag{2·8}$$

Even if we take account of ordering for the contraction between non-commutable factors, we can get the following Wick's theorem for T product:

$$\begin{aligned} &T[H_{\text{int}}(t)H_{\text{int}}(t')\cdots] \\ &=:X(t)\,Y(t)\cdots: \\ &\quad+:X^{\cdot}(t)\,Y^{\cdot}(t)\cdots X(t')\,Y(t')\cdots: \\ &\quad+:X^{\cdot}(t)\,Y(t)\cdots X^{\cdot}(t')\,Y(t')\cdots: \\ &\quad+\cdots, \end{aligned} \tag{2·9}$$

where $H_{\text{int}}(t)=X(t)\,Y(t)\cdots$. It is important to maintain the order of factors in the above expression. The T product is defined only among H_{int}'s and is irrelevant to the orders of factors in the respective Hamiltonians. The two kinds of contractions appearing in Eq. (2·9) are expressed by

$$\begin{aligned} X^{\cdot}(t)\,Y^{\cdot}(t') &= \langle T(X(t),\,Y(t'))\rangle_0 \\ &= \langle X(t)\,Y(t')\rangle_0\;\theta(t-t')+\langle Y(t')\,X(t)\rangle_0\theta(t'-t) \end{aligned} \tag{2·10}$$

for the operators in the different Hamiltonians and

$$X^{\cdot}(t)\,Y^{\cdot}(t)=\langle X(t)\,Y(t)\rangle_0 \tag{2·11}$$

for those in the same Hamiltonian respectively. The value of (2·10) at $t\neq t'$ remains invariant under the exchange of factors by definition of the T product. The T product of operators at equal-time point or the value of $\theta(0)$ is not usually defined. If we express, however, the S-matrix in the form of (1·2) by means of the T product, the indefiniteness of $\theta(0)$ has no effects on the resultant value after integration. In other words, we can define the value of $\theta(0)$ freely. However in order to represent the S-matrix in terms of the T^* product we extract the singular $\delta(t-t')$-term from $\langle T(\dot{q}(t)\,\dot{q}(t'))\rangle_0$ and carry out the t' integration, so that the value of $\theta(0)$ has finite influences on the result after the t' integration. Therefore, once we define the value of $\theta(0)$, we must throughout use this definition. For the sake of later convenience we adopt $\theta(0)=1/2$. Then it can be found that the value of (2·10) becomes always independent of the order of factors even though the factors are not commutable with each other. On the other hand, Eq. (2·11) depends explicitly on the order of factors. In fact for non-commutable factors the following inequality holds:

$$\langle X(t)\,Y(t)\rangle_0 \neq \langle T(X(t),\,Y(t'))\rangle_0|_{t=t'}. \tag{2·12}$$

Therefore, we cannot always express contractions by the vacuum expectation values of T products, although we do so in the usual derivation of the naive Feynman rule. Lee et al.[5] tacitly regarded the left-hand side of (2·12) as equivalent to the right-hand side by neglecting the factor ordering.

In order to get rid of the above difference between two kinds of contractions, we need only to symmetrize H_{int}, that is, rewrite H_{int} as the sum of terms in which all factors are symmetrized. Then, it is justified that we regard the contraction between any two factors as the vacuum expectation value of the T product, because the following equality always holds good:

$$\langle\{X(t)\,Y(t)\}\rangle_0=\langle T\{X(t)\,Y(t')\}\rangle_0|_{t=t'}, \tag{2·13}$$

where the brace denotes the symmetrization of factors, that is,

$$\{X(t)\,Y(t)\}=\tfrac{1}{2}(X(t)\,Y(t)+Y(t)\,X(t)). \tag{2·14}$$

In order to rewrite the S-matrix in terms of the T^* product, let us make the contraction of symmetrized terms with the use of only the $\delta(t-t')$-term in $\langle T(\dot{q}(t)\,\dot{q}(t'))\rangle_0$. Then we have again automatically the symmetrized quantity. Therefore, only if starting with the symmetrized expression for H_{int}, we need not take the order of factors into account and can adopt Lee et al.'s method straightforwardly to get a correct expression for the S-matrix. It is easily shown that the symmetrization can always be performed using the canonical commutation relations. As shown in Appendix B, we get the relations

$$\frac{1}{2}[a(q)\,\dot{q}b(q)\,\dot{q}c(q)+c(q)\,\dot{q}b(q)\,\dot{q}a(q)]$$

$$=\{\dot{q}abc\dot{q}\}+\frac{\hbar^2}{4}(ab''c-a''bc-abc''+2a'bc') \tag{2·15}$$

and

$$\tfrac{1}{2}(d(q)\,\dot{q}e(q)+e(q)\,\dot{q}d(q))=\{\dot{q}de\}. \tag{2·16}$$

With these relations, an arbitrary interaction Hamiltonian can be symmetrized easily.

Now, by means of the above-mentioned procedure, we shall express the S-matrix in terms of the T^* product explicitly for the following example and compare our result with that of Lee et al. When we take the non-linear Lagrangian (2·3) and regard the order of factors as meaningful, we get the following Hamiltonian in the Heisenberg representation by means of the method proposed by Kamefuchi et al.:[3)]

$$H=\frac{1}{8}(G^{-1}p+pG^{-1})F^{-1}GF^{-1}(G^{-1}p+pG^{-1})+\frac{1}{2}qq\,, \tag{2·17}$$

where p denotes the canonical momentum of q. This Hamiltonian is so constructed that the S-matrix is invariant under coordinate transformations. From Eq. (2·17), we obtain the following Hamiltonian H_{int} in the interaction representation:

$$H_{\text{int}}=\frac{1}{8}(G^{-1}\dot{q}+\dot{q}G^{-1})F^{-1}GF^{-1}(G^{-1}\dot{q}+\dot{q}G^{-1})-\frac{1}{2}\dot{q}\dot{q}\,, \tag{2·18}$$

where the canonical momentum in the interaction representation is rewritten by $\dot{q}$. With the use of Eq. (2·15), this Hamiltonian can be symmetrized and turns out to be

$$H_{\text{int}}=\{H_{\text{int}}\}+\delta' H\,, \tag{2·19}$$

where

$$\delta' H=\frac{\hbar^2}{8}[G^{-1}(F^{-2})''+G^{-2}F^{-2}G''-G^{-3}F^{-2}(G')^2]. \tag{2·20}$$

Applying Lee et al.'s method to this symmetrized H_{int}, we get a correct expression for the S-matrix:

$$S=T^*\exp\left(-\frac{i}{\hbar}\int H'_{\text{int}}dt\right), \tag{2·21}$$

where

$$H'_{\text{int}}=-\{L_{\text{int}}\}+\delta H+\delta' H \tag{2·22}$$

and δH is already given by Eq. (2·4). This result differs from that of Lee et al. by the new additional term $\delta' H$ which comes from the symmetrization of H_{int} and is proportional to $\hbar^2$. We are easily led to the result that the $\delta' H$ term certainly recovers the invariance of the S-matrix elements in the above-mentioned reducible case (2·5). (See Appendix A.) This example shows that the factor ordering problem is quite important in non-linear Lagrangian theories and our formalism is correct at least in this case.

§ 3. Some remarks on the evaluation of integrals

We must pay special attention to the evaluation of integrals in the covariant formulation, because we are confronted with integrands involving many-time derivatives in non-linear Lagrangian theories. As was already mentioned in § 2, there appears the second derivative of the propagator function $\ddot{S}(t)$ which contains the singular $\delta(t)$-function as follows:

$$\ddot{S}(t) = -S(t) - 2i\delta(t), \tag{3·1}$$

where

$$S(t) = \frac{2}{\hbar}\langle T(q(t)q(0))\rangle_0 = e^{-it}\theta(t) + e^{it}\theta(-t). \tag{3·2}$$

Therefore the values of other integrands at $t=0$ contribute to the results of the integration and we must use the same value of $\theta(0)$ as defined in § 2 which leads to

$$S(0) = 1 \quad \text{and} \quad \dot{S}(0) = 0. \tag{3·3}$$

In the model (2·3), the term $\dot{q}^2$ appears in the interaction Lagrangian. Therefore, in perturbation calculations, we must evaluate the integrals whose integrands involve the following types of functions:

$$\mathrm{I} = \ddot{S}(t_1 - t_2)\ddot{S}(t_1 - t_2)S^l(t_1 - t_2), \tag{3·4}$$

$$\mathrm{II} = \ddot{S}(t_1 - t_2)\dot{S}^h(t_1 - t_2)S^m(t_1 - t_2) \tag{3·5}$$

and

$$\mathrm{III} = \ddot{S}(t_1 - t_2)S^n(t_1 - t_2), \tag{3·6}$$

where l, m and n are non-negative integers and h is 1 or 2. With Eq. (3·1) and Eq. (3·3), Eqs. (3·4)$\sim$(3·6) are transformed into the forms, respectively,

$$\begin{aligned}\mathrm{I} &= S^{l+2}(t_1 - t_2) + 4i\delta(t_1 - t_2)S^{l+1}(0) - 4\delta(t_1 - t_2)\delta(0)S^l(0) \\ &= S^{l+2}(t_1 - t_2) + 4i\delta(t_1 - t_2)(1 + i\delta(0)),\end{aligned} \tag{3·7}$$

$$\begin{aligned}\mathrm{II} &= -S^{m+1}(t_1 - t_2)\dot{S}^h(t_1 - t_2) - 2i\delta(t_1 - t_2)\dot{S}^h(0)S^m(0) \\ &= -S^{m+1}(t_1 - t_2)\dot{S}^h(t_1 - t_2)\end{aligned} \tag{3·8}$$

and

$$\begin{aligned}\mathrm{III} &= -S^{n+1}(t_1 - t_2) - 2i\delta(t_1 - t_2)S^n(0) \\ &= -S^{n+1}(t_1 - t_2) - 2i\delta(t_1 - t_2).\end{aligned} \tag{3·9}$$

The role of δ-functions in the above expressions is to replace t_1 with t_2 in the residual part of the integrand after the integration over t_1. Accordingly, integrals for any diagrams reduce to those that contain functions S and $\dot{S}$ only in integrands. These integrals are easily evaluated as usual. To illustrate, let us consider the following integral:

$$\mathrm{I} = \int_{-\infty}^{+\infty} \ddot{S}(t)[\dot{S}(t)]^2 dt. \tag{3·10}$$

Using the relations (3·1) and (3·3), we get

$$\mathrm{I} = \int_{-\infty}^{+\infty} [\dot{S}(t)]^2[-S(t) - 2i\delta(t)]dt$$

$$= -\int_{-\infty}^{+\infty}[\dot{S}(t)]^2 S(t)\,dt$$

$$= \int_{-\infty}^{+\infty}[e^{-3it}\theta(t)+e^{3it}\theta(-t)]\,dt$$

$$= -\frac{2}{3}i, \qquad (3\cdot11)$$

where an appropriate damping factor is assumed as usual. It is worthwhile to mention that the usual differential calculus is not always valid for products of the propagator functions. If we use the relation

$$\frac{d}{dt}[\dot{S}(t)]^3 = 3\ddot{S}(t)\,[\dot{S}(t)]^2, \qquad (3\cdot12)$$

the integral (3·10) becomes zero:

$$I = \frac{1}{3}\int_{-\infty}^{+\infty}\frac{d}{dt}[\dot{S}(t)]^3 dt = \frac{1}{3}[\dot{S}(t)]^3\Big|_{-\infty}^{+\infty} = 0\,. \qquad (3\cdot13)$$

However the relation (3·12) is found incorrect, since $S(t)$ involves the step function $\theta(t)$ and the derivative of $\theta^n(t)$ is defined only as a distribution in the following sense:

$$\int_{-\infty}^{+\infty} f(t)\,\frac{d}{dt}\theta^n(t)\,dt = -\int_{-\infty}^{+\infty} f'(t)\,\theta^n(t)\,dt$$

$$= -\int_{-\infty}^{+\infty} f'(t)\,\theta(t)\,dt$$

$$= f(0), \qquad (3\cdot14)$$

where $f(t)$ is a regular function. So we have

$$\frac{d}{dt}\theta^n(t) = \delta(t)$$

$$\neq n\theta^{n-1}(0)\,\delta(t)\,. \qquad (3\cdot15)$$

Then the relation (3·12) must be modified to be

$$\frac{d}{dt}[\dot{S}(t)]^3 = 3\ddot{S}(t)\,[\dot{S}(t)]^2 + 2i\delta(t)\,. \qquad (3\cdot16)$$

When use is made of this relation, we get

$$I = \int_{-\infty}^{+\infty}\left(\frac{1}{3}\,\frac{d}{dt}[\dot{S}(t)]^3 - \frac{2}{3}i\delta(t)\right)dt$$

$$= -\frac{2}{3}i, \qquad (3\cdot17)$$

which is equal to the result (3·11).

Next, care must be taken also in the course of calculations using the Feynman rules in the momentum representation. In non-linear theories the integrands involve higher order terms with respect to momenta, so that we cannot always exchange the order of integrations. Let us again discuss the integral (3·10) which is reexpressed as

$$I=\int dt\int\frac{k^2qpe^{-i(k+q+p)t}}{(k^2-1+i\epsilon)(q^2-1+i\epsilon)(p^2-1+i\epsilon)}\frac{dkdqdp}{(2\pi)^3} \tag{3·18}$$

$$=\frac{1}{3}\int dt\int\frac{kqp(k+q+p)e^{-i(k+q+p)t}}{(k^2-1+i\epsilon)(q^2-1+i\epsilon)(p^2-1+i\epsilon)}\frac{dkdqdp}{(2\pi)^3} \tag{3·19}$$

$$=\frac{i}{3}\int dt\int\frac{d}{dt}\left(\frac{kqpe^{-i(k+q+p)t}}{(k^2-1+i\epsilon)(q^2-1+i\epsilon)(p^2-1+i\epsilon)}\right)\frac{dkdqdp}{(2\pi)^3}. \tag{3·20}$$

If we exchange the integrations with respect to time and momenta in (3·19) as done in the usual momentum representation, we get

$$I=\frac{1}{3}\int\frac{kqp(k+q+p)\delta(k+q+p)}{(k^2-1+i\epsilon)(q^2-1+i\epsilon)(p^2-1+i\epsilon)}\frac{dkdqdp}{(2\pi)^2}$$

$$=0. \tag{3·21}$$

Or if we exchange the order of differentiation and integration in (3·20), we also have

$$I=\frac{1}{3}\int dt\frac{d}{dt}\int\frac{kqpe^{-i(k+q+p)t}}{(k^2-1+i\epsilon)(q^2-1+i\epsilon)(p^2-1+i\epsilon)}\frac{dkdqdp}{(2\pi)^3}$$

$$=\frac{i}{3}\int dt\frac{d}{dt}(\dot{S})^3=0. \tag{3·22}$$

Of course it is self-evident to have the same result both in (3·21) and (3·22), because the above two incorrect manipulations exactly coincide with each other. Consequently we arive at the conclusion that we cannot exchange freely the order of integrations and differentiations in non-linear theories and that we must conserve the order of integrations to get correct results. It is, nevertheless, convenient to calculate the Feynman diagrams in momentum space. So we give here the corresponding prescription to the correct ones in coordinate space when we go formally into momentum representation after the exchange of integrations.

(i) When there appears a term such as $k^2/(k^2-1)$ in the integrand, it must be understood to be the sum of two pieces:

$$\frac{k^2}{k^2-1}=\frac{1}{k^2-1}+1. \tag{3·23}$$

This corresponds to the extraction of the $\delta(t)$ function as shown in (3·1).

(ii) The integrals of odd functions such as $\int k/(k^2-1+i\varepsilon)dk$ which appear in the residual integrals must be set equal to zero. This corresponds to the use of the relation (3·3).

Taking the integral (3·18) as an example, we explain the above prescriptions. Going into the momentum representation, we get

$$I=\int\frac{k^2qp\delta(k+q+p)}{(k^2-1+i\varepsilon)(q^2-1+i\varepsilon)(p^2-1+i\varepsilon)}\frac{dkdqdp}{(2\pi)^2}$$

$$=\int\left[\frac{qp}{(q^2-1+i\varepsilon)(p^2-1+i\varepsilon)}+\frac{qp}{(q^2-1+i\varepsilon)(p^2-1+i\varepsilon)(k^2-1+i\varepsilon)}\right]$$

$$\times\delta(k+q+p)\frac{dkdqdp}{(2\pi)^2}$$

$$=\int\frac{qp}{(q^2-1+i\varepsilon)(p^2-1+i\varepsilon)(k^2-1+i\varepsilon)}\delta(k+q+p)\frac{dkdqdp}{(2\pi)^2}$$

$$=-\frac{2}{3}i, \qquad (3\cdot24)$$

which is in accord with the correct value (3·11). It is noticeable that following these rules we get correct results for arbitrary higher order terms in the perturbation calculation as already stated in the discussion of $\ddot{S}(t)$.

§ 4. Application to quantum field theory and chiral dynamics

It is important to see if we can extend the above considerations about the factor ordering problems to the case of quantum field theory such as chiral dynamics. Charap and others[7] examined the problems as to whether the soft pion theorems and the masslessness of the pion are maintained, order by order, in perturbation theory when we do not regard a chiral Lagrangian as the effective one but as the basis of a dynamical theory. The answer to this question turned out to be yes if the effective interaction Hamiltonian density $\mathcal{H}'_{\rm int}(=-\mathcal{L}_{\rm int}+\delta\mathcal{H})$ is used in the covariant formulation and $\delta\mathcal{H}$ provides the counterterm for the most divergent term (which violates the theorems) in each order. In their considerations, however, the order of factors was neglected. Moreover they calculated amplitudes with naive Feynman rules in the momentum representation without taking particular care in the evaluation of integrals.

Let us show that non-commutability of the integrals discussed in § 2 is very important also for this case. For example, in the calculation of the self-mass of the pion in the second order perturbation, it is necessary to evaluate the following integral:

$$J=\int d^4y\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\Delta_F(x-y)\cdot\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu}\Delta_F(x-y)\cdot\Delta_F(x-y) \qquad (4\cdot1)$$

$$= \int d^4y \int \frac{(kq)^2 e^{-i(k+q+l)(x-y)}}{(k^2+i\varepsilon)(q^2+i\varepsilon)(l^2+i\varepsilon)} \frac{d^4k d^4q d^4l}{(2\pi)^{12}} . \tag{4·2}$$

Exchange the order of the integrals in Eq. (4·2) and perform the y integration first. Then after some appropriate modifications of the integrand using the relations such as $(k+q+l)^2\delta^4(k+q+l)=0$, we get

$$J= \int \frac{k^2q^2\delta^4(k+q+l)}{(k^2+i\varepsilon)(q^2+i\varepsilon)(l^2+i\varepsilon)} \frac{d^4k d^4q d^4l}{(2\pi)^8}$$
$$=\delta^4(0)\int \frac{d^4l}{(l^2+i\varepsilon)(2\pi)^4} . \tag{4·3}$$

On the other hand we evaluate Eq. (4·1) by the use of the relation

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\Delta_F(x-y) = i\int \frac{d^3\boldsymbol{k}}{2\omega_k(2\pi)^3} k^\mu k^\nu [\theta(x_0-y_0)e^{-ik(x-y)}$$
$$+\theta(y_0-x_0)e^{ik(x-y)}] - \delta^4(x-y)\delta_{\mu 0}\delta_{\nu 0} . \tag{4·4}$$

Then we obtain, as shown in Appendix C,

$$J=\delta^4(0)\Delta_F(0) - \frac{1}{8}(\delta^3(0))^2$$
$$=\delta^4(0)\int \frac{d^4l}{(2\pi)^4(l^2+i\varepsilon)} - \frac{1}{8}(\delta^3(0))^2. \tag{4·5}$$

It has thus been shown in the above correct evaluation that there appears the $(\delta^3(0))^2$ term which was neglected by Charap and others. As a result, without considering the factor ordering, we must argue that the self-mass of the pion does not vanish in the soft-pion limit because the $\hbar^2(\delta^3(0))^2$-term cannot be eliminated. It is concluded that if changing the order of the two kinds of integrations and going into the momentum representation, we must evaluate the integral in momentum space after discriminating the most divergent term in the integrand in order to get the correct results. This prescription is essentially equivalent to the one stated in the rules in § 3.

Recently Dowker et al.[2] investigated the method of quantization of the chiral invariant Lagrangian. Based on the attitude that the quantized theory should generally be covariant under point transformations, they asserted that the resultant ordering of the correct quantum Hamiltonian density introduces a term proportional to $\hbar^2(\delta^3(0))^2$ into the Hamiltonian density used in the functional integral representation of the generating functional. However they could not cancel out the $\hbar^2(\delta^3(0))^2$ term in the case of the chiral Lagrangian. It is possibly because they did not completely take into account the factor ordering in their discussion. In the following discussion it will be shown that similar considerations in § 2 are also applicable to quantum field theory and that with the chiral Lagrangian we can cancel out the $(\delta^3(0))^2$-term in the soft pion amplitudes.

Let us start with a non-linear Lagrangian density

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\varphi_\alpha G_{\alpha\beta}(\varphi)\partial^\mu\varphi_\beta\,. \tag{4·6}$$

We adopt again the method of Kamefuchi et al.[3)] to determine the Hamiltonian density which assures the invariance of the S-matrix under point transformations. We can immediately write down the effective Hamiltonian density in the relativistically covariant formulation as done in § 2:

$$\mathcal{H}'_{\rm int}(x)=-\{\mathcal{L}_{\rm int}\}+\delta\mathcal{H}+\delta'\mathcal{H}\,, \tag{4·7}$$

where

$$\delta\mathcal{H}=\frac{i\hbar}{2}\delta^4(0)\ln\ \det(G_{\alpha\beta}) \tag{4·8}$$

and

$$\delta'\mathcal{H}=\frac{\hbar^2}{8}(\delta^3(0))^2\left[G_{\alpha\beta}\frac{\partial(G^{-1})_{\alpha\gamma}}{\partial\varphi_\gamma}\frac{\partial(G^{-1})_{\beta\delta}}{\partial\varphi_\delta}-\frac{\partial^2(G^{-1})_{\gamma\delta}}{\partial\varphi_\gamma\partial\varphi_\delta}\right]. \tag{4·9}$$

The $\delta'\mathcal{H}$ term results from the symmetrization of the Hamiltonian density. The S-matrix is then expressed by

$$S=T^*\ \exp\left(-\frac{i}{\hbar}\int\mathcal{H}'_{\rm int}(x)\,d^4x\right). \tag{4·10}$$

We can see that the above Hamiltonian density (4·7) gives correct S-matrix elements for the reducible Lagrangian. As an example of the irreducible Lagrangian, we investigate the non-linear chiral invariant Lagrangian in which the pion alone exists. Let us define the matrix

$$M=\frac{1+if(\phi^2)\phi}{1-if(\phi^2)\phi}\,,\qquad \phi=\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{3}\tau_\alpha\phi_\alpha\,, \tag{4·11}$$

where ϕ_α are pion fields, τ_α denote Pauli matrices. In terms of this matrix M, the chiral invariant Lagrangian density is given by

$$\mathcal{L}=\frac{1}{8f_\pi^2}{\rm Tr}\,\partial_\mu M\partial^\mu M^\dagger. \tag{4·12}$$

In general $f(\phi^2)$ is an arbitrary function of ϕ^2 and, for simplicity, we take $f(\phi^2)=f_\pi$, where f_π is the pion decay constant. According to the method in Ref. 3), we get

$$P_\alpha=(1+f_\pi^2\phi^2)^{-1}\dot{\phi}_\alpha(1+f_\pi^2\phi^2)^{-1}, \tag{4·13}$$

which are modified using the canonical commutation relation and also

$$[\phi_\alpha(x),\,\dot{\phi}_\beta(y)]_{x_0=y_0}=i\hbar\delta_{\alpha\beta}\delta^3(x-y)\,(1+f_\pi^2\phi^2)^2. \tag{4·14}$$

The Hamiltonian density in the Heisenberg picture is

$$\begin{aligned}\mathcal{H}&=\frac{1}{8f_\pi{}^2}\mathrm{Tr}\,(\partial_0 M\partial_0 M^\dagger+\nabla M\nabla M^\dagger)\\&=\frac{1}{2}(1+f_\pi{}^2\phi^2)^{-1}\dot{\phi}_\alpha\dot{\phi}_\alpha(1+f_\pi{}^2\phi^2)^{-1}+\frac{1}{2}(1+f_\pi{}^2\phi^2)^{-2}(\nabla\phi_\alpha)^2\\&\quad-3\hbar^2f_\pi{}^4\phi^2(\delta^3(0))^2\\&=\frac{1}{2}P_\alpha(1+f_\pi{}^2\phi^2)^2P_\alpha+\frac{1}{2}(1+f_\pi{}^2\phi^2)^{-2}(\nabla\phi_\alpha)^2-3f_\pi{}^4\hbar^2\phi^2(\delta^3(0))^2.\end{aligned}\tag{4·15}$$

Now it is convenient to use the interaction representation, regarding

$$\mathcal{H}_{\mathrm{int}}=\mathcal{H}-\tfrac{1}{2}\dot{\phi}_\alpha\dot{\phi}_\alpha-\tfrac{1}{2}(\nabla\phi_\alpha)^2\tag{4·16}$$

as the interaction Hamiltonian density. Then we obtain

$$\begin{aligned}\mathcal{H}_{\mathrm{int}}=&\frac{1}{2}\{\dot{\phi}_\alpha[(1+f_\pi{}^2\phi^2)^2-1]\dot{\phi}_\alpha\}+\frac{1}{2}[(1+f_\pi{}^2\phi^2)^{-2}-1](\nabla\phi_\alpha)^2\\&+\frac{f_\pi{}^2\hbar^2}{2}(3-f_\pi{}^2\phi^2)(\delta^3(0))^2.\end{aligned}\tag{4·17}$$

The effective interaction Hamiltonian density becomes

$$\begin{aligned}\mathcal{H}'_{\mathrm{int}}=&\frac{-1}{2}\partial_\mu\phi_\alpha[(1+f_\pi{}^2\phi^2)^{-2}-1]\partial^\mu\phi_\alpha+\frac{3i\hbar}{2}\delta^4(0)\ln(1+f_\pi{}^2\phi^2)^{-2}\\&+\frac{f_\pi{}^2\hbar^2}{2}(3-f_\pi{}^2\phi^2)(\delta^3(0))^2.\end{aligned}\tag{4·18}$$

Next we evaluate the self-mass of the pion to order $f_\pi{}^4$ in the soft-pion limit and see if the last term in Eq. (4·18) contributes to the self-mass. To order $f_\pi{}^2$ the self-mass vanishes as already shown by Charap and others. The interaction terms for the self-mass to order $f_\pi{}^4$ are given by

$$\begin{aligned}&f_\pi{}^2[\phi^2(\partial_\mu\phi)^2-3i\hbar\delta^4(0)\phi^2]+f_\pi{}^4\Big[-\frac{3}{2}(\phi^2)^2(\partial_\mu\phi)^2+\frac{3i\hbar}{2}\delta^4(0)(\phi^2)^2\\&\qquad-\frac{\hbar^2}{2}(\delta^3(0))^2\phi^2\Big].\end{aligned}\tag{4·19}$$

The Feynman diagrams in Fig. 1 give contributions of order $f_\pi{}^4$ and each diagram provides the following amplitudes in the soft-pion limit:

$$\begin{aligned}A_1&=-90i\hbar^2f_\pi{}^4\delta^4(0)\Delta_F(0),\\A_2&=30i\hbar^2f_\pi{}^4\delta^4(0)\Delta_F(0),\\A_3&=72i\hbar^2f_\pi{}^4\delta^4(0)\Delta_F(0),\\A_4&=-36i\hbar^2f_\pi{}^4\delta^4(0)\Delta_F(0),\\A_5&=i\hbar^2f_\pi{}^4(\delta^3(0))^2\end{aligned}\tag{4·20}$$

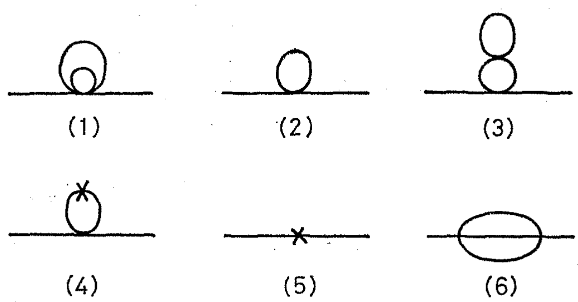

Fig. 1. The Feynman diagrams which contribute to the pion self-mass of order $f_\pi{}^4$.

and

$$A_6 = 24i\hbar^2 f_\pi^4 \delta^4(0) \Delta_F(0) - i\hbar^2 f_\pi^4 (\delta^3(0))^2,$$

where A_i denotes the amplitude coming from the i-th diagram in Fig. 1 and

$$\Delta_F(0) = \int \frac{1}{l^2 + i\varepsilon} \frac{d^4 l}{(2\pi)^4}. \tag{4·21}$$

The self-mass of the pion of order f_π^4 is then given by the sum of A_i, that is,

$$\sum_{i=1}^{6} A_i = 0. \tag{4·22}$$

In conclusion the self-mass of the pion remains zero at least to order f_π^4 in the soft-pion limit even if we consider the factor ordering.

How about the higher order terms than those of order f_π^4? In the interaction Hamiltonian density (4·18) there is the $(\delta^3(0))^2$-term to order f_π^4 only. Then in order to ascertain that in higher order terms the $(\delta^3(0))^2$-contributions are in fact cancelled out without the counterterm in the Hamiltonian density, we turn to the two-closed-loop contributions of order f_π^6 in the soft-pion limit to the 2-pion scattering amplitudesas an example. As it was already shown in Ref. 7) that divergent contributions of the type $\delta^4(0)\Delta_F(0)$ from every diagram are cancelled out each other, we here write down the $(\delta^3(0))^2$-contributions which come from only two diagrams in Fig. 2:

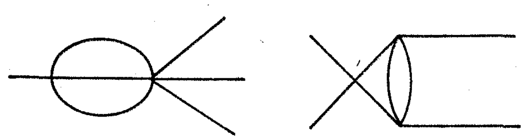

Fig. 2. The Feynman diagrams which provide the $(\delta^3(0))^2$-term in π-π scattering of order f_π^6.

$$B_1 = -24i\hbar^3 f_\pi^6 \int d^4y \int \frac{d^4k d^4q d^4l}{(2\pi)^{12}} \frac{[3(kq)^2 + 2(kq)(kl)]\, e^{-i(k+q+l)(x-y)}}{(k^2+i\varepsilon)(q^2+i\varepsilon)(l^2+i\varepsilon)}$$

$$\to 3i\hbar^3 f_\pi^6 (\delta^3(0))^2 \tag{4·23}$$

and

$$B_2 = 24i\hbar^3 f_\pi^6 \int d^4y d^4z \int \frac{d^4k d^4q d^4l d^4p}{(2\pi)^{16}} \left[\frac{(l\cdot p)[3(k\cdot q)^2 + 2(k\cdot q)(k\cdot l)]}{(k^2+i\varepsilon)(q^2+i\varepsilon)(l^2+i\varepsilon)(p^2+i\varepsilon)} \times e^{-i(k+q)(x-y) - il(x-z) - ip(z-y)} \right]$$

$$\to -3i\hbar^3 f_\pi^6 (\delta^3(0))^2. \tag{4·24}$$

By adding (4·23) and (4·24) we have no contributions of the $(\delta^3(0))^2$-term. In a similar fashion it is easily shown that the $(\delta^3(0))^2$-contributions to the pion self-mass of order f_π^6 are also cancelled out. As a consequence of the above discussions, we arrive at the conclusion that if the factor ordering is correctly considered, the masslessness of the pion and the soft pion theorems are maintained at least to order f_π^6 of the perturbation calculations. Why does the Hamiltonian density (4·15) give the amplitudes which satisfy the soft pion theorems? It is because the Hamiltonian density (4·15) has similar structure concerning the order of factors as the Lagrangian density (4·12) and then the Hamiltonian density is invariant under a chiral transformation in a q-number sense.

§ 5. Conclusion and discussion

In this paper we have studied the method how to treat the factor ordering problems in non-linear theories. In particular it is the most important point to have found the effective Hamiltonian in the relativistically covariant formulation. In addition to the result of Lee et al.,[5] the new term proportional to $\hbar^2(\delta^3(0))^2$, coming from the symmetrization of factors, must further be added to get a correct result. When we express the S-matrix in terms of the T^* product, we have shown it necessary to evaluate the integrals whose integrands involve $\ddot{S}(t)$, after discriminating the singular $\delta(t)$-function and using a certain value of $\theta(0)$ throughout. Prescriptions in the momentum representation which correspond to the calculation in the coordinate space have been obtained. If the original Hamiltonian is given following the method of Kamefuchi et al., our prescription does provide the correct S-matrix elements in the reducible case and also in chiral dynamics. It is emphasized that only by considering the factor ordering correctly the soft pion theorems and the masslessness of the pion hold good for each term, at least, to order f_π^6 in the perturbation calculations.

Now how should we determine the quantum Hamiltonian? In linear theories the action principle gives the Euler equation directly from the Lagrangian, and the Hamiltonian is determined so as to provide the Euler equation. However the action principle does not seem applicable to non-linear theories. Untill now there are two different approaches*) to get the Hamiltonian satisfying the invariance principle:

(I) Kamefuchi et al.'s method. It is this method that we have adopted in the preceding sections. In their approach, Lagrangians related to each other by point transformations give the same Hamiltonians. The order of factors in the Lagrangian as well as in the Hamiltonian is important, and we must not exchange factors in the Lagrangian using the canonical commutation relation because the Lagrangian thus obtained would give the Hamiltonian different from one given by the original Lagrangian.

*) Recently Sugano et al.[4] obtained the same Hamiltonian as in (5·1) in the case when the Riemannian space is of constant curvature using the canonical commutation relations in the Lagrangian. However, application of their method to more general cases seems to us impossible.

(II) De Witt[1] and Dowker et al.[2] proposed that the following Hamiltonian density from Eq. (4·6) is an appropriate one:

$$\mathcal{H}=\tfrac{1}{2}G^{-1/4}P_\alpha G^{1/2}(G^{-1})_{\alpha\beta}P_\beta G^{-1/4}+V(\varphi), \tag{5·1}$$

where $G=\det(G_{\alpha\beta})$ and $V(\varphi)$ is an indefinite potential term. This Hamiltonian density satisfies the general covariance under point transformations. In this approach the order of factors is regarded as meaningless in the Lagrangian.

It is important to observe the difference that the method I starts with the Lagrangian, whereas the latter does with the Hamiltonian. But by taking an appropriate form of $V(\varphi)$ in (5·1), both Hamiltonians coincide with each other. Especially in chiral dynamics we get the same Hamiltonian without the $V(\varphi)$-term in (5·1) and then the soft pion theorems are maintained also on the basis of approach II.

Finally we make a comment on the invariance principle for each term of the perturbation expansion which is, in field theory, terriblly divergent. Then even if the invariance of the total S-matrix under point transformations is maintained, it is not self-evident that each term of the perturbation expansion satisfies the invariance principle. In Appendix D it is shown that in the case of the chiral $U(1)\times U(1)$ symmetric Lagrangian the zero self-mass to order f_π^4 remains unchanged under point transformations. Thus it is highly probable that to all orders in the expansion the invariance principle can be satisfied with the correct Hamiltonian.

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Appendix A

We consider the case of one-dimensional harmonic oscillator described by the following Lagrangian:

$$L=\frac{1}{2}(1+fq)^{-1}\dot{q}(1+fq)^{-2}\dot{q}(1+fq)^{-1}-\frac{1}{2f^2}[(1+fq)^{-1}-1]^2, \tag{A·1}$$

where the frequency $\omega=1$. The Lagrangian (A·1) can be reduced to a linear one by the transformation

$$x=\frac{1}{f}[(1+fq)^{-1}-1]. \tag{A·2}$$

Then the S-matrix should be

$$S=1\,.$$

At first we calculate the vacuum polarization to order f^2 shown in Fig. A1 using Lee et al.'s effective Hamiltonian

$$H'_{\rm int}=-L_{\rm int}+\frac{i\hbar}{2}\delta(0)\ln(1+fq)^{-4}. \qquad ({\rm A}\cdot 3)$$

The relevant interaction terms are

$$f(2\dot{q}q\dot{q}-q^3-2i\delta(0)q)$$
$$+f^2(-5\dot{q}q^2\dot{q}+\tfrac{3}{2}q^4+i\hbar\delta(0)q^2). \qquad ({\rm A}\cdot 4)$$

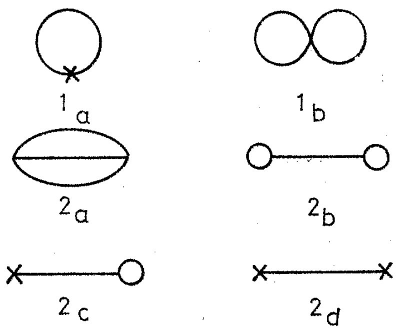

Fig. A1. The Feynman diagrams which contribute to the vacuum polarization of order f^2.

From the diagrams 1_a and 1_b we get the amplitude

$$A_1=\frac{(-i)\hbar}{1!}\Big(-\frac{1}{8}f^2-2i\delta(0)f^2\Big). \qquad ({\rm A}\cdot 5)$$

Similarly from other diagrams we have

$$A_2=\frac{(-i)^2\hbar}{2!}\Big(-\frac{3}{4}if^2-4f^2\delta(0)\Big). \qquad ({\rm A}\cdot 6)$$

Then the total amplitude

$$A=A_1+A_2=i\hbar\frac{f^2}{2}\neq 0\,. \qquad ({\rm A}\cdot 7)$$

As this amplitude A should be zero, we can conclude that Lee et al.'s arguments is insufficient. However following the prescription stated in § 2 we have the effective Hamiltonian

$$H'_{\rm int}=-\{L_{\rm int}\}+\frac{i\hbar}{2}\delta(0)\ln(1+fq)^{-4}+\frac{f^2\hbar^2}{2}(1+fq)^2, \qquad ({\rm A}\cdot 8)$$

the last term of which results from the symmetrization of the Hamiltonian and cancels out the value (A·7). Consequently we have zero vacuum polarization to order f^2.

Appendix B

Let us consider a case of one-dimensional system. Symmetrization of factors is expressed as follows:

1) $$\{\dot{q}q^N\dot{q}\}=\Big(\sum_{l+m+n=N}q^l\dot{q}q^m\dot{q}q^n\Big)\Big/\Big(\sum_{l+m+n=N}1\Big)$$
$$=\tfrac{1}{2}\Big[\sum_{l+m+n=N}(q^l\dot{q}q^m\dot{q}q^n+q^n\dot{q}q^m\dot{q}q^l)\Big]\Big/\Big(\sum_{l+m+n=N}1\Big), \qquad ({\rm B}\cdot 1)$$

where l, m and n run from zero to N. When use is made of the canonical commutation relation $[q, \dot{q}]=i\hbar$, we get

$$q^l\dot{q}q^m\dot{q}q^n+q^n\dot{q}q^m\dot{q}q^l$$
$$=2\dot{q}q^N\dot{q}-\hbar^2\{(l+n)(N-1)-2l\cdot n\}q^{N-2}. \qquad (\mathrm{B}\cdot 2)$$

Generally the following relations can be derived:

$$\sum_{l+m+n=N} f(l,m,n)=\sum_{k+n=N}\sum_{l+m=k} f(l,m,n)$$
$$=\sum_{k=0}^{N}\sum_{l=0}^{k} f(l,k-l,N-k), \qquad (\mathrm{B}\cdot 3)$$

where $f(l, m, n)$ is a function of l, m and n. Then we obtain

$$\sum_{l+m+n=N} 1=\tfrac{1}{2}(N+1)(N+2),$$
$$\sum_{l+m+n=N}(l+n)=\tfrac{1}{3}N(N+1)(N+2)$$

and

$$\sum_{l+m+n=N} ln=\tfrac{1}{24}(N-1)(N+1)(N+2). \qquad (\mathrm{B}\cdot 4)$$

As a result,

$$\{\dot{q}q^N\dot{q}\}=\dot{q}q^N\dot{q}-\frac{\hbar^2}{4}N(N-1)q^{N-2}. \qquad (\mathrm{B}\cdot 5)$$

If $a(q)$ is a given function of q alone, we find

$$\dot{q}a(q)\dot{q}=\{\dot{q}a(q)\dot{q}\}+\frac{\hbar^2}{4}a''(q). \qquad (\mathrm{B}\cdot 6)$$

For the general case, we get

$$\frac{1}{2}(a(q)\dot{q}b(q)\dot{q}c(q)+c(q)\dot{q}b(q)\dot{q}a(q))$$
$$=\{\dot{q}abc\dot{q}\}+\frac{\hbar^2}{4}(ab''c-a''bc-abc''+2a'bc'). \qquad (\mathrm{B}\cdot 7)$$

2) Similarly,

$$\{\dot{q}q^N+q^N\dot{q}\}=\sum_{l+m=N}(q^l\dot{q}q^m+q^m\dot{q}q^l)/(\sum_{l+m=N}1)$$
$$=\sum_{l+m=N}(\dot{q}q^N+q^N\dot{q})/(\sum_{l+m=N}1)$$
$$=\dot{q}q^N+q^N\dot{q}\,. \qquad (\mathrm{B}\cdot 8)$$

Then for $d(q)$,

$$\tfrac{1}{2}(\dot{q}d(q)+d(q)\dot{q})=\{\dot{q}d(q)\} \qquad (\mathrm{B}\cdot 9)$$

and for the general case

$$\tfrac{1}{2}(d\dot{q}e+e\dot{q}d)=\{de\dot{q}\}. \tag{B·10}$$

Appendix C

Let us here prove Eq. (4·5). The Feynman propagator for the Klein-Gordon equation is

$$\Delta_F(x-y)=-\int\frac{d^3\boldsymbol{k}}{(2\pi)^3 2\omega_k}[\theta(x_0-y_0)e^{-ik(x-y)}+\theta(y_0-x_0)e^{ik(x-y)}]. \tag{C·1}$$

Then using Eqs. (4·4) and (C·1), we have

$$\begin{aligned}J=&\delta^4(0)\Delta_F(0)-\frac{1}{2}\int\left(\frac{\omega_k}{\omega_q}\right)\frac{d^3\boldsymbol{k}d^3\boldsymbol{q}}{(2\pi)^6}\\&+i\int d^4y\int\frac{d^3\boldsymbol{p}d^3\boldsymbol{k}d^3\boldsymbol{q}}{(2\pi)^9 8\omega_k\omega_q\omega_p}[\omega_k^2\omega_q^2-2\omega_k\omega_q(\boldsymbol{k}\cdot\boldsymbol{q})+(\boldsymbol{k}\cdot\boldsymbol{q})^2]\\&\times[\theta(x_0-y_0)e^{-i(k+q+p)(x-y)}+\theta(y_0-x_0)e^{i(k+q+p)(x-y)}],\end{aligned} \tag{C·2}$$

where we take $\theta(0)=1/2$ as discussed in § 2. As we have separated the singular $\delta(t)$-term in Eq. (4·4), we may exchange the order of the integrations in the last term of (C·2). So, using the symmetric property of integral variables, we get

$$\begin{aligned}J=&\delta^4(0)\Delta_F(0)-\frac{1}{2}\int\left(\frac{\omega_k}{\omega_q}\right)\frac{d^3\boldsymbol{k}d^3\boldsymbol{q}}{(2\pi)^6}\\&+\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{q}d^3\boldsymbol{p}\delta^3(\boldsymbol{k}+\boldsymbol{q}+\boldsymbol{p})}{(2\pi)^6 16\omega_k\omega_q\omega_p(\omega_k+\omega_q+\omega_p)}[(3\omega_k^3+2\omega_k^2\omega_q)(\omega_k+\omega_q+\omega_p)-6\omega_k^2\omega_q\omega_p]\\=&\delta^4(0)\Delta_F(0)-\frac{1}{2}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{q}}{(2\pi)^6}\left(\frac{\omega_k}{\omega_q}\right)\\&+\frac{1}{2}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{q}}{(2\pi)^6}\left(\frac{\omega_k}{\omega_q}\right)-\frac{1}{8}\int\frac{d^3\boldsymbol{k}d^3\boldsymbol{q}d^3\boldsymbol{p}}{(2\pi)^6}\delta^3(\boldsymbol{k}+\boldsymbol{q}+\boldsymbol{p})\\=&\delta^4(0)\Delta_F(0)-\frac{1}{8}(\delta^3(0))^2\end{aligned} \tag{C·3}$$

as required.

Appendix D

Let us define the quantity M as

$$M=\frac{1+if(\phi^2)\phi}{1-if(\phi^2)\phi}, \tag{D·1}$$

where ϕ denotes a real pseudoscalar field and $f(\phi^2)$ is an arbitrary function of ϕ^2. Then the chiral invariant Lagrangian is

$$\mathcal{L}=\frac{1}{8f_\pi^2}\partial_\mu M\partial^\mu M^\dagger$$

$$=\frac{1}{8f_\pi^2}(1-if\phi)^{-1}\{(f+f'\phi),\dot{\phi}\}(1+f^2\phi^2)^{-1}\{(f+f'\phi),\dot{\phi}\}(1+if\phi)^{-1}$$

$$+\frac{1}{2f_\pi^2}(1+f^2\phi^2)^{-2}(f+f'\phi)^2(\nabla\phi)^2. \qquad \text{(D·2)}$$

Adopting Kamefuchi et al.'s method and following the prescription in § 2, we have the effective Hamiltonian density in the covariant formulation

$$\mathcal{H}'_{\text{int}}=-\frac{1}{2f_\pi^2}\partial_\mu\phi[(f+f'\phi)^2(1+f^2\phi^2)^{-2}-f_\pi^2]\partial^\mu\phi$$

$$+\frac{i\hbar}{2}\delta^4(0)\ln(f+f'\phi)^2(1+f^2\phi^2)^{-2}f_\pi^{-2}$$

$$+\frac{f_\pi^2\hbar^2}{8}(\delta^3(0))^2[-2f\phi+(1+f^2\phi^2)(f+f'\phi)^{-2}(2f'+f''\phi)]^2. \qquad \text{(D·3)}$$

Now we expand $f(\phi^2)$ as follows:

$$f(\phi^2)=f_\pi[1+\alpha_1(f_\pi\phi)^2+\alpha_2(f_\pi\phi)^4+\cdots]. \qquad \text{(D·4)}$$

To the self-mass to order f_π^4 of the pseudoscalar field the following interactions contribute:

$$f_\pi^2[(1-3\alpha_1)\phi^2(\partial_\mu\phi)^2+i\hbar\delta^4(0)(3\alpha_1-1)\phi^2]$$

$$+\frac{f_\pi^4}{2}[(10\alpha_2+9\alpha_1^2-16\alpha_1+3)\phi^4(\partial_\mu\phi)^2+i\hbar(10\alpha_2-9\alpha_1^2-4\alpha_1+1)$$

$$\times\delta^4(0)\phi^4+\hbar^2(\delta^3(0))^2(3\alpha_1-1)^2\phi^2]. \qquad \text{(D·5)}$$

The Feynman diagrams are exactly the same as shown in Fig. 1. The contributions in the soft-pion limit are evaluated respectively by

$$C_1=-6i\hbar^2(10\alpha_2+9\alpha_1^2-16\alpha_1+3)\delta^4(0)\Delta_F(0),$$

$$C_2=6i\hbar^2(10\alpha_2-9\alpha_1^2-4\alpha_1+1)\delta^4(0)\Delta_F(0),$$

$$C_3=8i\hbar^2(9\alpha_1^2-6\alpha_1+1)\delta^4(0)\Delta_F(0),$$

$$C_4=-4i\hbar^2(9\alpha_1^2-6\alpha_1+1)\delta^4(0)\Delta_F(0),$$

$$C_5=-i\hbar^2(\delta^3(0))^2(9\alpha_1^2-6\alpha_1+1)$$

and

$$C_6=8i\hbar^2(9\alpha_1^2-6\alpha_1+1)[\delta^4(0)\Delta_F(0)+\tfrac{1}{8}(\delta^3(0))^2], \qquad \text{(D·6)}$$

where C_i denotes the contribution from the i-th diagram in Fig. 1. The total contribution vanishes irrespective of α_1 and α_2, that is,

$$\sum_{i=1}^{6} C_i = 0 .$$

From the above discussion it is probable that the other amplitudes are also invariant under point transformations (for an arbitrary $f(\phi^2)$ in this case).

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