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An exact Doppler shift formula in the spherical acoustic wave

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Abstract

It is shown that the largely used formula, describing the Doppler shift in an acoustic spherical wave

$$T = \tau(1 + \frac{v}{u}\cos\theta_-)^{-1}$$

has an approximate character, being the result of some approximations made during its derivation. An exact formula is derived and some of its consequences are revealed.

1 Introduction

The nonlongitudinal acoustic Doppler shift ($NADS$) is mentioned by many authors [1,2] using a formula which has an approximate character, as a result of the fact that during its derivation simplificatory assumptions are made[1], like : "it will be assumed that the source-observer distance is sufficiently large that the wave fronts are plane". Of course it is questionable if acoustic wave can be detected at very large distances! Because the formula is such in common use, it is easy to forget (and authors often fail to remind the reader) that the formula is the result of an approximation.

The purpose of our paper is to derive an exact formula for the (NADS) and to compare the exact expression to the frequently-used approximation.

2 An exact Doppler shift formula in the acoustic field

Consider a stationary source L of monochromatic acoustic waves of period τ located at the origin of the $S(XOY)$ reference frame and an observer R travelling at speed v parallel to the OX axis as shown in Fig.1.The medium through which the wave propagates is considered to be at rest relative to the $S(XOY)$ reference frame The observer receives two successive wave crests at times t_1 and t_2 when located at positions $R_1(r_1, \theta_1)$ and $R_2(r_2, \theta_2)$ respectively. The time T between the observer's reception of two crests is

$$T = t_2 - t_1 = \tau + \frac{r_2 - r_1}{u} \tag{1}$$

u representing the propagation velocity of the acoustic wave relative to source. Between the reception of two successive wave crests R travels a distance $v\tau$.

To derive an exact expression for the (NADS) we use the law of cosines

$$r_1 = \sqrt{r_2^2 + v^2T^2 - 2vTr_2\cos\theta_2} \tag{2}$$

applied to Fig.1. Eliminating r_1 between Eqs.(1) and (2) we obtain

$$(1-\beta^2)T^2 - 2(\tau + \frac{r}{u})(1 - \beta\cos\theta)T + \tau(\tau + \frac{2r}{u}) = 0. \tag{3}$$

Where for simplicity we have put $\theta_2 = \theta$ and $r_2 = r$. Eq. (3) is a quadratic equation for T. Choosing the solution with physical meaning ($T = 0$ if $\tau = 0$), we obtain solving it, the Doppler factor

$$D_m = \frac{T}{\tau} = \frac{A - \sqrt{A^2 - B}}{1 - \beta^2} = \frac{B}{(1-\beta^2)(A + \sqrt{A^2 - B})} \tag{4}$$

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where $A = (1 + \frac{r}{\lambda})(1 - \beta\cos\theta)$, $B = (1 - \beta^2)(1 + 2\frac{r}{\lambda})$, $\lambda = uT$, the wavelength of the spherical wave under consideration and $\beta = \frac{v}{u}$. As we see D_m depends on the parameter M

$$M = \frac{r}{\lambda} = \frac{d}{\lambda\sin\theta} = \frac{m}{\sin\theta} = \frac{df}{u\sin\theta} \tag{5}$$

d representing the distance between L and the direction along which R moves and $f = \frac{1}{\tau}$ the frequency in the acoustic wave. We remind that θ represents the angle made by the position vector of R at position R_2 with the positive direction of the Ox axis.

In order to gain in transparency, we will express the results obtained above as a function of the angle θ made by the same position vector with the negative direction of the Ox axis ($\cos\theta = -\cos\theta_-$), which changes A and B to

$$A = (1 + \frac{m}{sin\theta_-})(1 + \beta\cos\theta_-), \tag{6}$$

$$B = (1 - \beta^2)(1 + \frac{2m}{\sin\theta_-}). \tag{7}$$

Under such conditions we can consider that for the angles comprised between ($0^0 < \theta_- < 90^0$) R is "incoming" whereas for the angles comprised between $90^0 < \theta_- < 180^0$ R is "outgoing". Physically $M = \frac{d}{\lambda\sin\theta}$ can be thought as the number of wavelength between the source and the receiver, whereas $m = \frac{d}{\lambda}$ as the number of wavelengths between source and the line long which R moves.

The variation of D_m with the angle θ_- for several values of the parameter $M = \frac{d}{\lambda\sin\theta}$ is presented in Fig.2.

The possible approximations which can be made are :

- the period τ is very small ($\tau \to 0$), small enough in order to consider that R receives two successive wave crests being located at the same point in space receiving them under the same angle θ_- as we see. In that case M and m takes very large values ($m \to \infty$) and we say that the "very small period" or "very high frequency" approximations are made,

- the distance d defined above is "very large" ($d \to \infty$). We say that the "plane wave approximation" is made, because at very large distances from the source a spherical wave can be considered as being a plane wave. We have in that case $m \to \infty$ as well.

In both cases the (NADS) is described by the Doppler factor

$$D = \lim_{m\to\infty} D_m = \lim_{m\to\infty} \frac{B}{(1-\beta^2)(A + \sqrt{A^2 - B})} = \lim_{m\to\infty} \frac{(1-\beta^2)\frac{2}{\sin\theta_-}}{2(1-\beta^2)\frac{1+\beta\cos\theta_-}{\sin\theta_-}} = \frac{1}{1+\beta\cos\theta_-} \tag{8}$$

an equation largely used,[1,2] without to test if it holds in the studied case.

The variation of D with θ_- is also represent in Fig.2. Eq.(5) shows that the parameter m depends on the angle θ which R receives the wave crest. It has "very big" values in the case of the longitudinal Doppler effect ($\theta = 0^0, \theta = 180^0$) but a minimal value ($\frac{d}{\lambda}$) in the case of the transversal Doppler effect. That fact explains why the biggest discrepancies between the results furnished by Eqs.(4) and (8) occur in the vicinity of $\theta_- = 90^0$.

If for $\theta_- = 90^0$ Eq.(8) gives for D the value $D_{\theta_-=90^0} = 1$ indicating the absence of a transversal Doppler effect, Eq.(4) gives for D_m a value

$$D_{m_{\theta_-=90^0}} = \frac{(1-\beta^2)(1+2m)}{(1-\beta^2)(1+m+\sqrt{(1+m)^2 - (1-\beta^2)(1+2m)})} = \frac{1+2m}{1+m+\sqrt{m^2+\beta^2(1+2m)}} \tag{9}$$

advocating for the existence of the transversal Doppler shift in the acoustic wave.

The variation of $D_{m_{\theta_-=90^0}}$ with m for different values of $\beta = \frac{v}{u}$ is represented in Fig.3. In experiments presented in the literature[1,2] $d = 5$ m, the frequency $f = 1/\tau$ in an acoustic wave changes between 16Hz $< f <$ $2 \cdot 10^4$ Hz and $u = 331.8$ m/s. Under such conditions $m_{\theta_-=90^0}$ changes between $\frac{5\cdot 16}{331.8} \leq m_{\theta_-=90^0} \leq \frac{5\cdot 2\cdot 10^4}{331.8}$ i.e. $0.24 \leq m_{\theta_-=90^0} \leq 301.14$. Defining the error committed working with the approximate formula (Eq.(8)) instead of working with the exact one (Eq.(4)) as

$$\varepsilon\% = \left[1 - \frac{D}{D_m}\right] \cdot 100 \tag{10}$$

we obtain that in the range of $\theta_- = 90^0$ and for $m_{\theta_-=90^0} = 0.24$ the error is $\varepsilon = -28\%$ whereas for $m_{\theta_-=90^0} = 301.14$ the error is $\varepsilon = -0.04\%$ for $\beta = v/u = 0.5$. Taking into account the high accuracy at which time intervals are measured nowadays, the errors calculated above are high enough in order to be taken into account.

During the motion of observer R, he is first located in a region where $D_m < 1$ and after that in a region where $D_m > 1$. For $M \to \infty$ (approximate formula) the transition takes place for $\theta_- = 90^0$ but with decreasing values of m it takes place for angles $\theta_- > 90^0$. The values of the angles θ_- for which $D_m = 1$ can be obtained from Fig.2, intersecting the curves presented there with the straight line $D = 1$. From the same figure we can read the values of D and D_m for a given angle θ_- in order to determine the value of the error for that angle.

The approximate formula (Eq.(8)) suggests that R is able to record continuously the time intervals between the reception of two successive wave crests, R receiving the two wave crests being located at the same point in space and receiving them under the some angle θ. The exact formula (Eq.(4)) suggests that R receives the two successive wave crests being located at two different points along which he moves, receiving then under different angles θ_-. Besides that, between the reception of two wave crests, R has no enough information on order to calculate T.

We have not considered the relativistic effects, because their influence is well below the level of physically measurability.

Workers interested in psychology of human perception performance[2], replace the clocks as measuring devices with the human ear. In order to characterize the performance they introduce the concept of pitch as "that attribute of auditory sensation in terms of which sounds may be ordered on a scale extending from high to low". The important result they obtain is the fact that the reception point where $D = 1$ is located at points characterized by an angle $\theta_- > 90^0$, explained by the influence of the intensity of the sound on the pitch, a fact attributed by us, in our approach, to the finite values of the parameter $M = \dfrac{m}{\sin\theta}$, even if it is not so easy to find an equivalence between our T and the pitch attributed by the human ear to the same frequency.

References

[1] R.A.Bachman, "Relativistic acoustic Doppler effect", Am.J.Phys. 50, (1982)

[2] John G. Neuhoff and Michael K McBrath, "Overcoming naive mental models in explaining the Doppler shift : An illusion creates confusion", Am.J.Phys.65,618-621(1997)

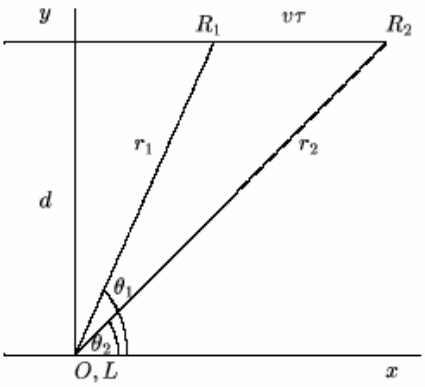

Fig.1

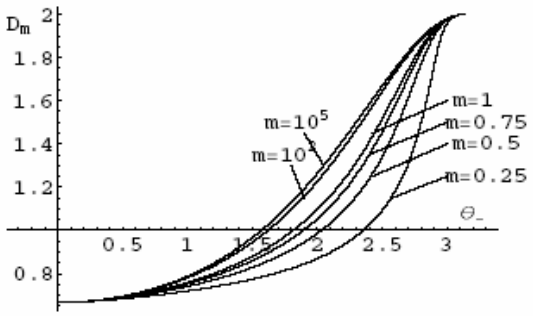

Fig.2

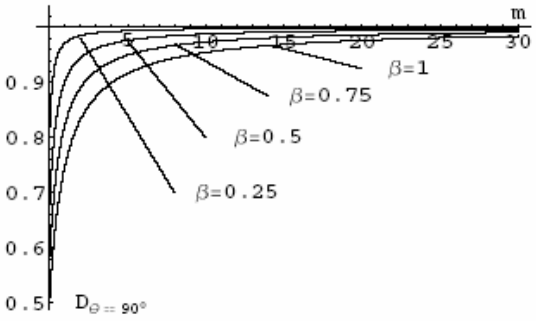

Fig.3

Fig.1. Scenario for deriving an exact Doppler shift formula in the acoustic field

Fig.2 The variation of the Doppler factor D_m with angle θ_- for different values of the factor M and for the same value $\frac{v}{u} = 0.5$. The intersection of $D_m(\theta_-)$ with the line $D_m = 1$ gives the values of θ_- for which no Doppler shift takes place ($D_m = 1$)

Fig.3 The variation of $D_{m_{\theta_-=90^0}}$ with m for $\theta_- = 90^0$ (transversal Doppler effect), for different values of $\beta = \frac{v}{u}$

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