CHARACTERISTIC CLASSES OF HYPERSURFACES AND CHARACTERISTIC CYCLES

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Abstract. We give a new formula for the Chern-Schwartz-MacPherson class of a hypersurface with arbitrary singularities, generalizing the main result of [P-P], which was a formula for the Euler characteristic. Two different approaches are presented. The first is based on the theory of characteristic cycle of a D-module (or a holonomic system) and the work of Sabbah [S], Briançon-Maisonobe-Merle [B-M-M], and Lê-Mebkhout [L-M]. In particular, this approach leads to a simple proof of a formula of Aluffi [A] for the above mentioned class. The second approach uses Verdier's [V] specialization property of the Chern-Schwartz-MacPherson classes. Some related new formulas for complexes of nearby cycles and vanishing cycles are also given.

Introduction and statement of the main result

Let X be a nonsingular compact complex analytic variety of pure dimension n and let L be a holomorphic line bundle on X. Take $f \in H^0(X, L)$ a holomorphic section of L such that the variety Z of zeros of f is a (nowhere dense) hypersurface in X. Denoting by TX the tangent bundle of X, we will call

$$c^{FJ}(Z) := c(TX|_Z - L|_Z) \cap [Z]\,, \tag{1}$$

the *Fulton-Johnson class* of Z. This terminology is justified by the fact that both canonical classes defined in [F-J] by $c(TX|_Z) \cap s(\mathcal{N}_Z X)$, and in [F, Ex.4.2.6] by $c(TX|_Z) \cap s(Z, X)$, are equal in the present situation to the right-hand side of (1). Here, $\mathcal{N}_Z X$ is the conormal sheaf to Z in X and $s(Z, X)$ is the Segre class of Z in X (cf. [F]). For more on this, consult [Su]; see also [B-L-S-S] and [Y3]. By $c_*(Z)$ we denote the *Chern-Schwartz-MacPherson class* of Z, see [McP]. We recall its definition later in Section 1.

Note that if Z is nonsingular then

$$c^{FJ}(Z) = c_*(Z) = c(TZ) \cap [Z].$$

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After [Y1,2,3] (see also [B-L-S-S]), we shall call

$$\mathcal{M}(Z) := (-1)^{n-1}\big(c^{FJ}(Z) - c_*(Z)\big) \tag{2}$$

the *Milnor class* of Z. This class is supported on the singular locus of Z; it is convenient, however, to treat it as an element of $H_*(Z)$.

Example 0.1. Suppose that the singular set of Z is finite and equals $x_1, \dots, x_k$. Let μ_{x_i} denote the Milnor number of Z at x_i (see [M]). Then

$$\mathcal{M}(Z) = \sum_{i=1}^{k} \mu_{x_i}[x_i] \in H_0(Z),$$

see, for instance, Suwa [Su] where this result is generalized to complete intersections.

Consider the function $\chi : Z \to \mathbb{Z}$ defined for $x \in Z$ by $\chi(x) := \chi(F_x)$, where F_x denotes the Milnor fibre at x (see [M]) and $\chi(F_x)$ its Euler characteristic. Define also the function $\mu : Z \to \mathbb{Z}$ by $\mu := (-1)^{n-1}(\chi - 1\!\!1_Z)$.

Fix now any stratification $\mathcal{S} = \{S\}$ of Z such that μ is constant on the strata of $\mathcal{S}$. For instance, any Whitney stratification of Z satisfies this property, see [B-M-M] and [Pa]. Actually, it is not difficult to see that the topological type of the Milnor fibres is constant along the strata of Whitney stratification of Z. Let us denote the value of μ on the stratum S by μ_S. Let

$$\alpha(S) := \mu_S - \sum_{S' \neq S, S \subset \overline{S'}} \alpha(S') \tag{3}$$

be the numbers defined inductively on descending dimension of S. (These numbers appear as the coefficients in the development of μ as a combination of the $1\!\!1_{\overline{S}}$'s – see Lemma 4.1.)

The main result of the present paper is

Theorem 0.2. *In the above notation,*

$$\mathcal{M}(Z) = \sum_{S \in \mathcal{S}} \alpha(S) c(L|_Z)^{-1} \cap (i_{\overline{S},Z})_* c_*(\overline{S}), \tag{4}$$

where $i_{\overline{S},Z} : \overline{S} \to Z$ denotes the inclusion.

When X is projective, (4) was conjectured by Yokura in [Y2]. Under this last assumption, the equality

$$\int_Z \mathcal{M}(Z) = \chi(Z) = \sum_{S \in \mathcal{S}} \alpha(S) \int_{\overline{S}} c(L|_{\overline{S}})^{-1} \cap c_*(\overline{S}) \tag{5}$$

was proved in [P-P]; hence the theorem gives, in particular, a generalization of the main result (5) of [P-P] to compact varieties. Perhaps, it is in order to note at this point that when Z is a curve on a complex surface X, (5) is nothing but a classical "adjunction formula" [Ko, (2.2)].

Our proof of the theorem is based on a formula due to Sabbah [S], which allows one to calculate the Chern-Schwartz-MacPherson class of a subvariety in terms of the associated *characteristic cycle*. In the case of hypersurface Z, this characteristic cycle was calculated in [B-M-M] and [L-M] in terms of the blow-up of the Jacobian ideal of a local equation of Z in X. So the proof of Theorem 0.2 is obtained by putting this local description and the global data together, and expressing the characteristic cycle of Z in terms of the global blow-up of the singular subscheme of Z. Here by the *singular subscheme* of Z we mean the one defined locally by the ideal $\left(f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$, where $(z_1, \dots, z_n)$ are local coordinates on X.

The approach used leads to a very simple proof of a formula for the Chern-Schwartz-MacPherson class of hypersurface in terms of some divisors associated with the above blow-up. This formula was originally obtained by Aluffi [A] by different methods. Some new formulas for the Chern-Schwartz-MacPherson classes of the constructible functions χ and μ are also given.

In the last section, we show, using Verdier's specialization property of the Chern-Schwartz-MacPherson classes (see [V], and also [S] and [K2]), how to prove another conjecture of Yokura, which, combined with a result from [Y2,3], gives an alternative proof of Theorem 0.2. (More precisely, this comment concerns a variant of Theorem 0.2, where X is projective and the classes are pushed forwarded to the homology of the ambient space X. See the remark after Theorem 5.3.) We find that this specialization argument somewhat better explains the essence of the theorem.

Another expression for the Milnor class $\mathcal{M}(Z)$ was given by Aluffi in [A].

Finally, we note that one motivation for studying the Milnor classes comes from Riemann-Roch-type problems. Namely, it is pointed out by Yokura in [Y1] that the knowledge of the Milnor class is necessary to understand a generalized Verdier-type Riemann-Roch theorem for the Chern-Schwartz-MacPherson class.

1. Chern-Mather classes and Chern-Schwartz-MacPherson classes

We start by recalling some results of Sabbah [S]. Let for X as in the introduction, T^*X denote the cotangent bundle of X. Let V be an (irreducible) subvariety of X. Denote by $c_M(V)$ (resp. $c_M^\vee(V)$) the *Chern-Mather class* of V (resp. the *dual Chern-Mather class*). Let us recall briefly their definitions. Let $\nu : NB(V) \to V$ be the Nash blow-up of V. By definition on $NB(V)$ there exists the "Nash tangent bundle" T_V which extends $\nu^* TV^0$, where V^0 is the regular part of V. Define the following elements of $H_*(V)$

$$(6)\qquad \begin{aligned} c_M(V) &:= \nu_*\big(c(T_V) \cap [NB(V)]\big) \\ c_M^\vee(V) &:= \nu_*\big(c(T_V^*) \cap [NB(V)]\big), \end{aligned}$$

where T_V^* is the dual bundle of T_V. It is easy to see that

$$c_M^\vee(V) = (-1)^{\dim V} c_M(V)^\vee, \tag{7}$$

where for a homology class $a = a_0 + a_1 + a_2 + \dots$, where $a_i \in H_{2i}(V)$, we denote $a^\vee := a_0 - a_1 + a_2 - \dots$.

By $T_V^*X \subset T^*X$ we denote the *conormal space* to V :

$$T_V^*X := \text{Closure}\left\{(x,\xi) \in T^*X \mid x \in V^0,\ \xi|_{T_xV^0} \equiv 0\right\},$$

and by $C(V) \subset \mathbb{P}T^*X$ its projectivization. Let $\pi : C(V) \to V$ be the restriction of the projection $\mathbb{P}T^*X \to X$ to $C(V)$. Then by [S], we have

$$\begin{aligned} c_M^\vee(V) &= c(T^*X|_V) \cap \pi_*\left(c\left(\mathcal{O}(-1)\right)^{-1} \cap [C(V)]\right) \\ c_M(V) &= (-1)^{n-1-\dim V} c\left(TX|_V\right) \cap \pi_*\left(c\left(\mathcal{O}(1)\right)^{-1} \cap [C(V)]\right) , \end{aligned} \tag{8}$$

where $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{P}T^*X$ restricted to $C(V)$.

Let now φ be a constructible function on X,

$$\varphi = \sum a_j \mathbb{1}_{Y_j} ,$$

where Y_j are (closed) subvarieties of X and $a_j \in \mathbb{Z}$. By the *characteristic cycle* of φ we mean the Lagrangian conical cycle in T^*X defined by

$$\text{Ch}(\varphi) := \text{Ch}\left(\bigoplus_j \left(i_{Y_j,X}\right)_* \mathbb{C}_{Y_j}^{\oplus a_j}\right) , \tag{9}$$

where $\mathbb{C}_{Y_j}$ is the constant sheaf on Y_j and $i_{Y_j,X} : Y_j \to X$ denotes the inclusion. For a general definition of the characteristic cycle of a sheaf, we refer the reader to [B]. The characteristic cycle of a constructible function admits the following interpretation. Let $F(X)$ and $L(X)$ denote the groups of constructible functions on X and conical Lagrangian cycles in T^*X respectively. It is known that the assignment

$$T_V^*X \mapsto (-1)^{\dim V} Eu_V , \tag{10}$$

where Eu_V stands for the Euler obstruction (see [McP] and also [S], [K1]), defines a natural transformation of the functors of Lagrangian conical cycles and constructible functions, that is an isomorphism. In particular, we have an isomorphism between $L(X)$ and $F(X)$. The operation of taking the characteristic cycle is the inverse of this isomorphism; that is, it is given by

$$\text{Ch}(Eu_V) = (-1)^{\dim V} T_V^*X . \tag{11}$$

Since every constructible function is a combination of the Eu_V's (see [McP]), this allows "in theory" to compute $\mathrm{Ch}(\varphi)$ for a constructible function φ. However, even for $\varphi = 1\!\!1_V$, this would involve not only the Euler obstruction of V itself but also of some subvarieties of V.

Now we associate with a constructible function φ on X its *Chern-Schwartz-MacPherson class* (abbreviation: CSM-class). Let $\pi : \operatorname{Supp} \mathbb{P}\,\mathrm{Ch}(\varphi) \to \operatorname{Supp} \varphi$ be the restriction of the projection $\mathbb{P}T^*X \to X$. Set

$$c_*(\varphi) := (-1)^{n-1} c\,(TX|_{\operatorname{Supp}\varphi}) \cap \pi_* \left(c\,(\mathcal{O}(1))^{-1} \cap [\mathbb{P}\,\mathrm{Ch}\,\varphi] \right) \tag{12}$$

– an element in $H_*(\operatorname{Supp}\varphi)$. We note that, in particular, by (8), (11) and (12) one has

$$c_*(Eu_V) = c_M(V)\,. \tag{13}$$

If $V \subset X$ is a (closed) subvariety, we will write $c_*(V) := c_*(1\!\!1_V)$ as is customary. Note that (12) is in agreement with [McP] because for $1\!\!1_V = \sum_i b_i Eu_{Y_i}$, where $b_i \in \mathbb{Z}$ and $Y_i \subset X$ are (closed) subvarieties, we have

$$c_*(1\!\!1_V) = \sum_i b_i c_*(Eu_{Y_i}) = \sum_i b_i c_M(Y_i) = c_*(V)\,.$$

Thus, denoting by $\pi : \operatorname{Supp}\mathrm{Ch}(1\!\!1_V) \to V$ the restriction of the projection $\mathbb{P}T^*X \to X$, we have

$$c_*(V) = (-1)^{n-1} c\,(TX|_V) \cap \pi_* \left(c\,(\mathcal{O}(1))^{-1} \cap [\mathbb{P}\,\mathrm{Ch}(1\!\!1_V)] \right)\,. \tag{14}$$

2. Characteristic cycle of a hypersurface (local case)

Suppose that $U \subset \mathbb{C}^n$ is an open subset and $Z \subset U$ is a hypersurface of zeros of a holomorphic function $f : U \to \mathbb{C}$. Let $\mathcal{J}_f$ denote the *Jacobian ideal* $\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$ of f, where $(z_1, \dots, z_n)$ are the standard coordinates of $\mathbb{C}^n$. Consider the blow-up $\pi : \mathrm{Bl}_{\mathcal{J}_f} U \to U$ of $\mathcal{J}_f$. Recall that we may interpret it as follows

$$\mathrm{Bl}_{\mathcal{J}_f} U = \mathrm{Closure} \left\{ (x,\eta) \in U \times \check{\mathbb{P}}^{n-1} \mid x \notin \operatorname{Sing} Z, \eta = \left[\frac{\partial f}{\partial z_1}(x) : \dots : \frac{\partial f}{\partial z_n}(x) \right] \right\}\,,$$

where $\operatorname{Sing} Z$ denotes the singular subscheme of Z, and $\check{\mathbb{P}}^{n-1}$ stands for the dual projective $(n-1)$-space.

Remark 2.1. $\mathrm{Bl}_{\mathcal{J}_f} U$ can be also interpreted as the projectivization of the *relative conormal space* $T^*_f \subset T^*U$ (see [B-M-M, § 2], where we put $\Omega = X = U$). Then by the Lagrangian specialization all fibres of the restriction of $\tilde{f} : T^*U \to U \xrightarrow{f} \mathbb{C}$

to T_f^* are conical Lagrangian subvarieties of T^*U. In particular, every irreducible component of $\tilde{f}^{-1}(0) \cap T_f^*$ is conormal to its projection on U. For details, we refer to [B-M-M, § 2] and to references therein.

Let $\mathcal{Z}$ be the total transform $\pi^{-1}(Z)$ of Z in $\mathrm{Bl}_{\mathcal{J}_f} U$ and $\mathcal{Z} = \bigcup_i D_i$ be the decomposition of $\mathcal{Z}$ into irreducible components. Set $C_i := \pi(D_i)$ and denote by $\mathcal{I}_{C_i}$ the ideal defining C_i. Then define

$$\begin{array}{rcl} n_i & := & \text{multiplicity of } \mathcal{I}_{C_i} \text{ along } D_i \\ m_i & := & \text{multiplicity of } f \text{ along } D_i \\ p_i & := & \text{multiplicity of } \mathcal{J}_f \text{ along } D_i \end{array}$$

Let us now record the following result.

Proposition 2.2. : $m_i = n_i + p_i$.

Proof. Observe that by Remark 2.1 we have $D_i = \mathbb{P}T^*_{C_i}U$. Let x be a generic point of C_i and choose a system of coordinates $(z_1, \ldots, z_n)$ at x such that $C_i = \{z_1 = \ldots = z_k = 0\}$ in a neighborhood of x. Then, over a neighborhood of x,

$$D_i = C_i \times \check{\mathbb{P}}^{k-1}, \tag{15}$$

where

$$\check{\mathbb{P}}^{k-1} = \{[\eta_1 : \ldots : \eta_n] \in \check{\mathbb{P}}^{n-1} \,|\, \eta_{k+1} = \ldots = \eta_n = 0\}.$$

Let $\zeta : E \to U$ denote the blow-up of the product of $\mathcal{J}_f$ and $\mathcal{I}_{C_i}$. So

$$E = \text{Closure}\left\{\Big(x, [z_1(x) : \ldots : z_k(x)], \Big[\frac{\partial f}{\partial z_1}(x) : \ldots : \frac{\partial f}{\partial z_n}(x)\Big]\Big) | x \notin \mathrm{Sing}\, Z\right\}$$

in $U \times \mathbb{P}^{k-1} \times \check{\mathbb{P}}^{n-1}$. Then ζ factors through π

$$\begin{array}{ccc} E & \longrightarrow & Bl_{\mathcal{J}_f}U \\ & \zeta \searrow & \downarrow \pi \\ & & U \end{array}$$

and there exists at least one irreducible component, say B_{ij}, of the exceptional divisor of ζ which projects surjectively onto D_i. Let $\gamma(t) = \big(z(t), v(t), \eta(t)\big)$ be an analytic curve in E such that $\big(z(0), v(0), \eta(0)\big)$ is a generic point of B_{ij}, $z_{k+1}(t) \equiv \ldots \equiv z_n(t) \equiv 0$ and $f\big(z(t)\big) \neq 0$ for $t \neq 0$. Then we have for $t \neq 0$,

$$v(t) = [z_1(t) : \ \ldots \ : z_k(t)] \in \mathbb{P}^{k-1},$$
$$\eta(t) = \left[\frac{\partial f}{\partial z_1}\big(z(t)\big) : \ \ldots \ : \frac{\partial f}{\partial z_n}\big(z(t)\big)\right] \in \check{\mathbb{P}}^{n-1}$$

and $\eta(0) = \big[\eta_1(0) : \ldots : \eta_k(0) : 0 : \ldots : 0\big]$ by (15).

Since $\big(z(0), \eta(0)\big)$ is a generic point of D_i, the following equality would imply the proposition

$$
\begin{aligned}
&\operatorname{ord}_0(f\circ\zeta)\big(\gamma(t)\big) = \operatorname{ord}_0 f\big(z(t)\big) \\
&= \operatorname{ord}_0\big(z_1(t),\ldots,z_k(t)\big) + \operatorname{ord}_0\left(\frac{\partial f}{\partial z_1}\big(z(t)\big),\ldots,\frac{\partial f}{\partial z_n}\big(z(t)\big)\right). \qquad (16)
\end{aligned}
$$

We show (16). First we note that we may suppose that $(z_1\circ\zeta,\ldots,z_k\circ\zeta)$ is generated by $z_{i_0}\circ\zeta$ at $\gamma(0)$ and $\zeta^{-1}\mathcal{J}_f$ is generated by $\frac{\partial f}{\partial z_{j_0}}\circ\zeta$ at $\gamma(0)$, where $j_0\in\{1,\ldots,k\}$ by (15). We have

$$
\begin{aligned}
\frac{d}{dt} f\big(z(t)\big) &= \sum_{i=1}^{k}\frac{\partial f}{\partial z_i}\big(z(t)\big)\dot z_i(t) \\
&= \frac{\partial f}{\partial z_{j_0}}\big(z(t)\big)\cdot \dot z_{i_0}(t)\left(\sum_{i=1}^{k}\frac{\frac{\partial f}{\partial z_i}\big(z(t)\big)}{\frac{\partial f}{\partial z_{j_0}}\big(z(t)\big)}\cdot\frac{\dot z_i(t)}{\dot z_{i_0}(t)}\right), \qquad (17)
\end{aligned}
$$

where $\dot z_i$ stands for $\frac{dz_i}{dt}$. Note that the quotients make sense since $\partial f/\partial z_{j_0}\circ\zeta$ generates $\zeta^{-1}\mathcal{J}_f$, and $\dot z_i(t)/\dot z_{i_0}(t)$ are analytic (because $z_{i_0}\circ\zeta$ generates $\zeta^{-1}(z_1,\ldots,z_k)$).

We may suppose that $\eta_{j_0}=1$ and $v_{i_0}=1$, which corresponds to choosing affine coordinates on $\mathbb{P}^{k-1}\times\check{\mathbb{P}}^{n-1}$. Since

$$\lim_{t\to 0}\,[\dot z_1(t):\ldots:\dot z_k(t)] = \lim_{t\to 0}\,[z_1(t):\ldots:z_k(t)],$$

we get

$$\lim_{t\to 0}\left(\sum_{i=1}^{k}\frac{\frac{\partial f}{\partial z_i}\big(z(t)\big)}{\frac{\partial f}{\partial z_{j_0}}\big(z(t)\big)}\cdot\frac{\dot z_i(t)}{\dot z_{i_0}(t)}\right) = \lim_{t\to 0}\left(\sum_{i=1}^{k}\frac{\eta_i(t)}{\eta_{j_0}(t)}\cdot\frac{v_i(t)}{v_{i_0}(t)}\right) = \sum_{i=1}^{k}\eta_i(0)v_i(0).$$

This last sum is nonzero by the transversality of relative polar varieties, see, for instance, [H-M, 8.7, Lemme de transversalité]. Consequently, (17) implies

$$\operatorname{ord}_0 f\big(z(t)\big) - 1 = \operatorname{ord}_0\frac{\partial f}{\partial z_{j_0}}\big(z(t)\big) + \big(\operatorname{ord}_0 z_{i_0}(t) - 1\big)$$

which gives (16), as required. □

In the following theorem, the equality (i) and the second equality in (ii) were established in [B-M-M] (see also [L-M]).

Theorem 2.3.

$$\text{(i)}\quad \mathrm{Ch}(1\!\!1_Z) = (-1)^{n-1}\sum_i n_i T^*_{C_i}U\,;$$

$$\text{(ii)}\quad \mathrm{Ch}(\chi) = \mathrm{Ch}(\mathrm{R}\,\Psi_f \mathbb{C}_U) = (-1)^{n-1}\sum_i m_i T^*_{C_i}U\,;$$

$$\text{(iii)}\quad \mathrm{Ch}(\mu) = (-1)^{n-1}\,\mathrm{Ch}\big(\mathrm{R}\,\Phi_f \mathbb{C}_U\big) = \sum_i p_i T^*_{C_i}U\,.$$

(For a definition of the complexes of nearby cycles $\mathrm{R}\,\Psi_f$ and vanishing cycles $\mathrm{R}\,\Phi_f$, we refer the reader to [D-K]. The first equalities in (ii) and (iii) are well-known and follow from the local index theorem, see for instance [B-D-K] and [S, (1.3) and (4.4)].)

Assertion (iii) follows from the equation

$$\mathrm{Ch}(\mu) = (-1)^{n-1}\big(\mathrm{Ch}(\chi) - \mathrm{Ch}(1\!\!1_Z)\big)\,,$$

combined with Proposition 2.2.

Let $\mathcal{Y}$ denotes the exceptional divisor in $\mathrm{Bl}_{\mathcal{J}_f} U$. Since $D_i = \mathbb{P}T^*_{C_i}U$, we can rewrite the assertions of the theorem as the following equalities.

Corollary 2.4.

$$\text{(i)}\quad [\mathbb{P}\,\mathrm{Ch}(1\!\!1_Z)] = (-1)^{n-1}\left([\mathcal{Z}] - [\mathcal{Y}]\right)\,;$$

$$\text{(ii)}\quad [\mathbb{P}\,\mathrm{Ch}(\chi)] = (-1)^{n-1}\,[\mathcal{Z}]\,;$$

$$\text{(iii)}\quad [\mathbb{P}\,\mathrm{Ch}(\mu)] = [\mathcal{Y}]\,.$$

Observe that these equalities already take place on the level of cycles.

Remark 2.5. Since f belongs to the integral closure of $\mathcal{J}_f$ (see [LJ-T]) the normalizations of the blow-ups of $\mathcal{J}_f$ and $\left(f, \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$ are equal. Hence Corollary 2.4 holds true if we replace the blow-up of the former ideal by the blow-up of the latter one.

3. Characteristic cycle of a hypersurface (global case)

Let X, L, f and Z be as in the introduction. Let $B = \mathrm{Bl}_Y X \to X$ be the blow-up of X along the singular subscheme Y of Z. Let $\mathcal{Z}$ and $\mathcal{Y}$ denote the total transform of Z and the exceptional divisor in B, respectively. The following description of the CSM-class of Z was established by Aluffi [A] by different methods.

Theorem 3.1. ([A]) Let $\pi : \mathcal{Z} \to Z$ be the restriction of the blow-up to $\mathcal{Z}$. Then

$$c_*(Z) = c(TX|_Z) \cap \pi_*\left(\frac{[\mathcal{Z}] - [\mathcal{Y}]}{1 + \mathcal{Z} - \mathcal{Y}}\right)\,,$$

where on the RHS, $\mathcal{Z}$ and $\mathcal{Y}$ mean the first Chern classes of the line bundles associated with $\mathcal{Z}$ and $\mathcal{Y}$ i.e. those of $\pi^*(L|_Z)$ and $\mathcal{O}_B(-1)$, the latter being the canonical line bundle on B.

Proof. To get a convenient description of B, we use (after [A]) the bundle $\mathcal{P}^1_X L$ of principal parts of L over X (see e.g. [At]). Consider the section $X \to \mathcal{P}^1_X L$ determined by $f \in H^0(X, L)$. Recall that $\mathcal{P}^1_X L$ fits in an exact sequence

$$0 \to T^*X \otimes L \to \mathcal{P}^1_X L \to L \to 0$$

and the section in question is written locally as $(df, f) = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}, f\right)$, where $(z_1, \dots, z_n)$ are local coordinates on X. It follows that the closure of the image of the meromorphic map $X \dashrightarrow \mathbb{P}\mathcal{P}^1_X L$ induced by (df, f) is the blow-up $B \to X$. Thus we may treat B as a subvariety of $\mathbb{P}\mathcal{P}^1_X L$. Clearly, the total transform $\mathcal{Z}$ of Z equals $B \cap \mathbb{P}(T^*X \otimes L)$. The canonical line bundle $\mathcal{O}_B(-1) = \mathcal{O}(\mathcal{Y})$ on B is the restriction of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{P}\mathcal{P}^1_X L$. Observe that the bundle $\mathcal{O}(-1)$ restricted to $\mathcal{Z}$ is contained in $(T^*X \otimes L)|_{\mathcal{Z}}$ (because $f \equiv 0$ over Z). Hence $\mathcal{O}_B(-1)|_{\mathcal{Z}}$ is the restriction of the tautological line bundle $\mathcal{O}_{\widetilde{\mathbb{P}}}(-1)$ on $\widetilde{\mathbb{P}} = \mathbb{P}(T^*X \otimes L)$. Using the natural identification $\mathbb{P}(T^*X \otimes L) \cong \mathbb{P}(T^*X)$ the line bundle $\mathcal{O}_{\widetilde{\mathbb{P}}}(-1)$ corresponds to the line bundle $\mathcal{O}_{\mathbb{P}}(-1) \otimes L$ on $\mathbb{P} = \mathbb{P}(T^*X)$. Thus $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}$ corresponds to $\mathcal{O}_{\widetilde{\mathbb{P}}}(1) \otimes L$ on $\widetilde{\mathbb{P}}$. Hence using the characteristic cycle formula (14), we get

$$\begin{aligned} c_*(Z) &= (-1)^{n-1} c(TX|_Z) \cap \pi_*\Big(c\big(\mathcal{O}_B(1) \otimes \pi^* L|_Z\big)^{-1} \cap \big[\mathbb{P}\,\mathrm{Ch}(1\!\!1_Z)\big]\Big) \\ &= c(TX|_Z) \cap \pi_*\left(\frac{[\mathcal{Z}] - [\mathcal{Y}]}{1 + \mathcal{Z} - \mathcal{Y}}\right) \end{aligned}$$

because by (the global analogue of) Corollary 2.4, we have the equality $[\mathbb{P}\,\mathrm{Ch}(1\!\!1_Z)] = (-1)^{n-1}\big([\mathcal{Z}] - [\mathcal{Y}]\big)$. □

By Corollary 2.4, we have $[\mathbb{P}\,\mathrm{Ch}(\chi)] = (-1)^{n-1}[\mathcal{Z}]$ and $[\mathbb{P}\,\mathrm{Ch}(\mu)] = [\mathcal{Y}]$. Therefore, using similar arguments, we get the following result.

Theorem 3.2. (i) $c_*(\chi) = c(TX|_Z) \cap \pi_*\left(\frac{[\mathcal{Z}]}{1 + \mathcal{Z} - \mathcal{Y}}\right)$;

(ii) $c_*(\mu) = (-1)^{n-1} c(TX|_Z) \cap \pi_*\left(\frac{[\mathcal{Y}]}{1 + \mathcal{Z} - \mathcal{Y}}\right)$.

(The constructible function μ is supported on Y but for later use we consider its CSM-class in $H_*(Z)$.)

Remark 3.3. One can add to the above formulas also

$$c_M(Z) = c_*(Eu_Z) = c(TX|_Z) \cap \pi_*\left(\frac{[\mathcal{Z}']}{1 + \mathcal{Z} - \mathcal{Y}}\right),$$

where $\mathcal{Z}'$ is the proper transform of Z. This equality for the Chern-Mather class was established originally by Aluffi [A] by different methods. Using the technique of characteristic cycles, it is a consequence of the equality $\left[\mathbb{P}\left(\mathrm{Ch}(Eu_Z)\right)\right] = (-1)^{n-1}[\mathcal{Z}']$ (see (11)).

4. Proof of Theorem 0.2

We start this section with the following fact about the constructible functions μ and α defined in the introduction.

Lemma 4.1. :
$$\mu = \sum_{S\in\mathcal{S}} \alpha(S)1\!\!1_{\overline{S}}.$$

Proof. Fix an arbitrary stratum S_0 and a point $x \in S_0$. We have

$$\left(\sum_S \alpha(S)1\!\!1_{\overline{S}}\right)(x) = \sum_{S\neq S_0,\overline{S}\supset S_0} \alpha(S) + \alpha(S_0)$$
$$= \sum_{S\neq S_0,\overline{S}\supset S_0} \alpha(S) + \left(\mu_{S_0} - \sum_{S\neq S_0,\overline{S}\supset S_0} \alpha(S)\right) = \mu(x). \qquad \square$$

Now we pass to the proof of Theorem 0.2. Let $\pi : \mathcal{Z} \to Z$ be the restriction of the blow-up $B = \mathrm{Bl}_Y X \to X$. We have, rewriting (1) as in [A], by using the projection formula,
$$c^{FJ}(Z) = c\left(TX|_Z\right) \cap \pi_*\left(\frac{[\mathcal{Z}]}{1+\mathcal{Z}}\right).$$
(Alternatively, one can use the expression from [F, Ex.4.2.6] and the birational invariance of Segre classes [F, Chap.4]:
$$c(TX|_Z) \cap s(Z,X) = c(TX|_Z) \cap \pi_* s(\mathcal{Z},B)$$
$$= c(TX|_Z) \cap \pi_*\left(\frac{[\mathcal{Z}]}{1+\mathcal{Z}}\right).)$$

Invoking (2) and using Theorem 3.1, we get

$$\begin{aligned} \mathcal{M}(Z) &= (-1)^{n-1}\left(c^{FJ}(Z) - c_*(Z)\right) \\ &= (-1)^{n-1} c\left(TX|_Z\right) \cap \pi_*\left(\frac{[\mathcal{Z}]}{1+\mathcal{Z}} - \frac{[\mathcal{Z}]-[\mathcal{Y}]}{1+\mathcal{Z}-\mathcal{Y}}\right) \\ &= (-1)^{n-1} c\left(TX|_Z\right) \cap \pi_*\left(\frac{[\mathcal{Y}]}{(1+\mathcal{Z})(1+\mathcal{Z}-\mathcal{Y})}\right) \end{aligned} \tag{18}$$

because $\mathcal{Y} \cap [\mathcal{Z}] = \mathcal{Z} \cap [\mathcal{Y}]$ (see [F, Theorem 2.4). If we pass to the characteristic cycle approach, the equality (18) is rewritten, by Corollary 2.4, in the form

$$\mathcal{M}(Z) = (-1)^{n-1} c\left(TX|_Z\right) \cap \pi_*\left(\frac{[\mathbb{P}\,\mathrm{Ch}(\mu)]}{(1+\mathcal{Z})(1+\mathcal{Z}-\mathcal{Y})}\right). \tag{19}$$

Since $\mu = \sum_{S\in\mathcal{S}} \alpha(S) 1\!\!1_{\overline{S}}$ by Lemma 4.1, we have

$$\mathrm{Ch}(\mu) = \sum_{S\in\mathcal{S}} \alpha(S)\,\mathrm{Ch}(1\!\!1_{\overline{S}})$$

and hence

$$\begin{aligned} &\frac{[\mathbb{P}\,\mathrm{Ch}(\mu)]}{(1+\mathcal{Z})(1+\mathcal{Z}-\mathcal{Y})} = \\ &= \sum_{S\in\mathcal{S}} \alpha(S) c(L|_Z)^{-1} \cap \pi_*\Big(c\big(\pi^* L|_Z \otimes \mathcal{O}_B(1)\big)^{-1} \cap [\mathbb{P}\,\mathrm{Ch}(1\!\!1_{\overline{S}})]\Big). \end{aligned} \tag{20}$$

By (14) and the proof of Theorem 3.1, we get

$$\big(i_{\overline{S},Z}\big)_* c_*(\overline{S}) = (-1)^{n-1} c\big(TX|_Z\big) \cap \pi_*\Big(c\big(\pi^* L|_Z \otimes \mathcal{O}_B(1)\big)^{-1} \cap [\mathbb{P}\,\mathrm{Ch}(1\!\!1_{\overline{S}})]\Big) \tag{21}$$

for each stratum $S \in \mathcal{S}$. Finally, using (20) and (21), we rewrite (19) in the form

$$\mathcal{M}(Z) = \sum_{S\in\mathcal{S}} \alpha(S)\, c\big(L|_Z\big)^{-1} \cap \big(i_{\overline{S},Z}\big)_* c_*(\overline{S})$$

which is the required expression. □

5. Another approach via specialization

In this section, the setup is as in the Introduction. Additionally, let us assume that there exists a section $g \in H^0(X, L)$ such that $Z' = g^{-1}(0)$ is smooth and transverse to the strata of a (fixed) Whitney stratification $\mathcal{S} = \{S\}$ of Z. For $t \in \mathbb{C}$, denote $f_t = f - tg$. In this section, by $\mathcal{Z}$ we will denote the following correspondence in $X \times \mathbb{C}$:

$$\mathcal{Z} := \big\{(x,t) \in X \times \mathbb{C} \mid f_t(x) = 0\big\}.$$

Denoting by $p : \mathcal{Z} \to \mathbb{C}$ the restriction to $\mathcal{Z}$ of the projection onto the second factor of $X \times \mathbb{C}$, we have $p^{-1}(t) = \{x \in X \mid f_t(x) = 0\} =: Z_t$ for $t \in \mathbb{C}$.

Let $F(\mathcal{Z})$ (resp. $F(Z)$) denote the group of constructible functions on $\mathcal{Z}$ (resp. on Z). Denote by

$$\sigma_F : F(\mathcal{Z}) \to F(Z_0 = Z)$$

the *specialization map of constructible functions* (see [V], [S] and [K2], where a different notation is used). Recall briefly its definition. If $Y \subset \mathcal{Z}$ is a (closed) subvariety, one sets for the generator $1\!\!1_Y$,

$$\big(\sigma_F 1\!\!1_Y\big)(x) := \lim_{t\to 0} \chi\big(B(x,\varepsilon) \cap Y_t\big)$$

for any sufficiently small $\varepsilon > 0$, where $B(x, \varepsilon)$ is the closed ball of radius ε about x and $Y_t = Y \cap Z_t$. In our situation, we are aiming to compute $\sigma_F 1\!\!1_{\mathcal{Z}}$. More explicitly, for $x \in Z$ we want to calculate

$$(\sigma_F 1\!\!1_{\mathcal{Z}})(x) = \lim_{t\to 0} \chi\big(B(x,\varepsilon) \cap Z_t\big).$$

This is the content of the following

Proposition 5.1. *One has*

$$(\sigma_F 1\!\!1_Z)(x) = \begin{cases} \chi(x) = 1 + (-1)^{n-1}\mu(x) & \text{for } x \notin Z \cap Z' \\ 1 & \text{for } x \in Z \cap Z'. \end{cases}$$

Proof. If $x \notin Z \cap Z'$ i.e. $g(x) \neq 0$, then

$$Z_t = \{z \mid f(z) - tg(z) = 0\} = \{z \mid f(z)/g(z) = t\}$$

after restriction to a small ball is the Milnor fibre of f/g at x, and f/g also defines Z in a neighborhood of x. The assertion follows.

Let now $x \in Z \cap Z'$. We will use similar arguments to those used in Step 1 of the proof of Proposition 7 in [P-P]. Proceeding locally we can assume that x is the origin in $\mathbb{C}^n$, that in our local coordinates $g(z) \equiv z_n$ and that $\{z_n = 0\}$ is transverse to a fixed Whitney stratification $\mathcal{S} = \{S\}$ of $Z = \{f = 0\}$. Our goal is to show that for sufficiently small $\varepsilon > 0$ and $0 < \delta << \varepsilon$, if $t \in \mathbb{C}$ satisfies $0 < |t| < \delta$, then

$$Z_t \cap B_\varepsilon = \Big\{(z_1, \dots, z_n) \in \mathbb{C}^n \;\Big|\; |z| < \varepsilon,\ f - tz_n = 0\Big\}$$

is contractible, where $B_\varepsilon = B(0, \varepsilon)$. Set $V = \{f = z_n = 0\}$. If ε is sufficiently small then $V \cap B_\varepsilon$ is contractible. So it suffices to retract $Z_t \cap B_\varepsilon$ onto $V \cap B_\varepsilon$. In what follows we shall proceed on $Z_t \setminus V$ for t sufficiently small. First note that since the stratification is Whitney and hence satisfies the a_f condition, we have by the assumption on transversality

$$\left| \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_{n-1}} \right) \right| \geq c \left| \frac{\partial f}{\partial z_n} \right|$$

for some universal $c > 0$. Therefore the linear forms $df(p)$ and $dz_n(p)$ are linearly independent for $p \notin \{f = 0\}$. So are clearly the forms $d(f - tz_n)$ and dz_n. Consequently the orthogonal projection of grad $|z_n|$ onto $Z_t = \{f - tz_n = 0\} \setminus V$ is nonzero, and we may normalize it so that the normalized vector field $\vec{v}$ satisfies

$$(i) \qquad \frac{\partial |z_n|}{\partial \vec{v}} = 1\ ;$$

$$(ii) \qquad \frac{\partial (f - tz_n)}{\partial \vec{v}} = 0.$$

We want, as well, the trajectories of this vector field do not leave B_ε. For this we modify $\vec{v}$ near $S_\varepsilon = \Big\{z \;\Big|\; |z| = \varepsilon\Big\}$. Let $p \in V \cap S_\varepsilon$ and let S be the stratum which contains p. Let $p(s) \to p$ as $s \to 0$ be an analytic curve such that $f\big(p(s)\big) \neq 0$ for $s \neq 0$. Then the limit η of $df\big(p(s)\big)$ in $\check{\mathbb{P}}^{n-1}$ as $s \to 0$, exists. The forms η and dz_n are linearly independent by the assumption on transversality, and both vanish on the tangent space to $S \cap \{z_n = 0\}$. Therefore, by the Whitney condition (b) for the closure of $S \cap \{z_n = 0\}$, we get the linear independence of η, dz_n and $\sum_{i=1}^n z_i dz_i$ at p. Consequently, the orthogonal projection of grad $|z_n|$ onto $S_\varepsilon \cap (Z_t \setminus V)$ is nonzero in a neighborhood of p. Since $S_\varepsilon \cap V$ is compact, there exist a neighborhood U of $S_\varepsilon \cap V$ and a vector field $\vec{w}$ on $U \setminus \big(\{z_n = 0\} \cup \{f = 0\}\big)$ such that for t small enough,

$$(i) \qquad \frac{\partial |z_n|}{\partial \vec{w}} = 1 \ ;$$

$$(ii) \qquad \frac{\partial (f - t z_n)}{\partial \vec{w}} = 0 \ ;$$

$$(iii) \qquad \frac{\partial \rho}{\partial \vec{w}} = 0 \ , \qquad \text{where } \rho(z) = \| z \|^2 \ .$$

Using partition of unity we "glue" $\vec{w}$ and $\vec{v}$ in order to get a vector field $\vec{u}$ defined on $Z_t \setminus V$ such that

$$(i) \qquad \frac{\partial |z_n|}{\partial \vec{u}} = 1 \ ;$$

$$(ii) \qquad \frac{\partial (f - t z_n)}{\partial \vec{u}} = 0 \ ;$$

$$(iii) \qquad \frac{\partial \rho}{\partial \vec{u}} = 0 \qquad \text{on } S_\varepsilon.$$

The flow of $\vec{u}$ allows us to retract $Z_t \cap B_\varepsilon$ onto $Z_{t,c} = Z_t \cap B_\varepsilon \cap \{|z_n| \leq c\}$ for c as small as we want. On the other hand, for c small enough, $Z_{t,c}$ retracts onto $V \cap B_\varepsilon = Z_t \cap B_\varepsilon \cap \{z_n = 0\}$, as required. □

Now we want to pass to the *specialization map of homology classes*

$$\sigma_H : H_*(Z_t) \to H_*(Z_0 = Z)$$

(see [V], [S] and [K2], where a different notation is used). Recall briefly its definition. Let $D \subset \mathbb{C}$ be a disk of a sufficiently small radius such that the inclusion $Z = Z_0 \subset p^{-1}(D)$ is a homotopy equivalence. Thus for small nonzero $t \in D$ one defines the above σ_H as the composition

$$H_*(Z_t) \xrightarrow{i_*} H_*(p^{-1}D) \cong H_*(Z_0 = Z) \, ,$$

where $i : Z_t \to p^{-1}D$ is the inclusion. Recall now that Verdier's specialization property of CSM-classes asserts the following. For $\varphi \in F(\mathcal{Z})$ and t sufficiently small, one has

$$\sigma_H c_*\big(\varphi|_{Z_t}\big) = c_*(\sigma_F \varphi) \, . \tag{22}$$

(see [V] and also [S] and [K2]).

Let us evaluate the both sides of (22) for $\varphi = 1\!\!1_{\mathcal{Z}}$. The LHS reads simply $\sigma_H c_*(Z_t)$. As for the RHS, we have by Proposition 5.1

$$\begin{aligned} \sigma_F 1\!\!1_{\mathcal{Z}} &= 1\!\!1_Z + (-1)^{n-1}\big(\mu \cdot 1\!\!1_{Z \setminus Z \cap Z'}\big) \\ &= 1\!\!1_Z + (-1)^{n-1}\big(\mu \cdot 1\!\!1_Z - \mu \cdot 1\!\!1_{Z \cap Z'}\big) \, . \end{aligned} \tag{23}$$

Invoking the equality $\mu = \sum_S \alpha(S)1\!\!1_{\overline{S}}$ (see Lemma 4.1), Equation (23) is rewritten as

$$\sigma_F 1\!\!1_{\mathcal{Z}} = 1\!\!1_Z + (-1)^{n-1}\left(\sum_S \alpha(S)1\!\!1_{\overline{S}} - \sum_S \alpha(S)1\!\!1_{\overline{S}\cap Z'}\right), \tag{24}$$

and applying c_* to (24) we get that the RHS of (22) is evaluated as

$$\begin{aligned} &c_*\big(\sigma_F 1\!\!1_{\mathcal{Z}}\big) = \\ &= c_*(Z) + (-1)^{n-1}\left\{\sum_S \alpha(S)\big[(i_{\overline{S},Z})_* c_*(\overline{S}) - (i_{\overline{S}\cap Z',Z})_* c_*(\overline{S}\cap Z')\big]\right\}, \end{aligned}$$

where $i_{\overline{S}\cap Z',Z}$ denotes the inclusion $\overline{S}\cap Z' \to Z$.

Suming up, by virtue of the specialization property (22), we have proved

Proposition 5.2. *For the specialization map $\sigma_H : H_*(Z_t) \to H_*(Z)$, where $t \neq 0$ is small enough, one has*

$$\begin{aligned} &\sigma_H c_*(Z_t) = \\ &= c_*(Z) + (-1)^{n-1}\left\{\sum_{S\in\mathcal{S}} \alpha(S)\big[(i_{\overline{S},Z})_* c_*(\overline{S}) - (i_{\overline{S}\cap Z',Z})_* c_*(\overline{S}\cap Z')\big]\right\}. \end{aligned}$$

We now state the folowing result which appeared as a conjecture in [Y2].

Theorem 5.3. *In the above notation, one has*

$$\mathcal{M}(Z) = \sum_{S\in\mathcal{S}} \alpha(S)\big[(i_{\overline{S},Z})_* c_*(\overline{S}) - (i_{\overline{S}\cap Z',Z})_* c_*(\overline{S}\cap Z')\big].$$

Proof. Observe that for t like in Proposition 5.2, we have $c_*(Z_t) = c^{FJ}(Z_t)$ because Z_t is smooth. Moreover, since the Fulton-Johnson class is expressed in terms of the Chern classes of vector bundles, one has $\sigma_H\big(c^{FJ}(Z_t)\big) = c^{FJ}(Z)$. We thus have

$$\begin{aligned} \mathcal{M}(Z) &= (-1)^{n-1}\big(c^{FJ}(Z) - c_*(Z)\big) \\ &= (-1)^{n-1}\big(\sigma_H c^{FJ}(Z_t) - c_*(Z)\big) \\ &= (-1)^{n-1}\big(\sigma_H c_*(Z_t) - c_*(Z)\big) \\ &= \sum_S \alpha(S)\big[(i_{\overline{S},Z})_* c_*(\overline{S}) - (i_{\overline{S}\cap Z',Z})_* c_*(\overline{S}\cap Z')\big] \end{aligned}$$

by Proposition 5.2. □

Finally, arguing as in [Y2,3 § 2] one shows that Theorem 5.3 implies, for X projective,

$$(i_{Z,X})_*\mathcal{M}(Z) = \sum_{S\in\mathcal{S}} \alpha(S)c(L)^{-1} \cap (i_{\overline{S},X})_* c_*(\overline{S}),$$

where $i_{Z,X} : Z \to X$ and $i_{\overline{S},X} : \overline{S} \to X$ denote the inclusions.

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Note. Milnor classes of singular varieties were also studied by Brasselet, Lehmann, Seade and Suwa in [B-L-S-S], as we have been informed by J.-P. Brasselet and S. Yokura. For a survey about Milnor classes, see [Y3].

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