1 Values in Game Theory

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A century ago the pioneers in game theory John von Neumann (1903–1957) and Oskar Morgenstern (1902–1978) were born. At the age of 23 (December 7, 1926) John von Neumann presented for the Mathematical Society in Göttingen his paper 'Zur Theorie der Gesellschaftspiele', which appeared in 1928 ([1]; for an English translation, see [2]). This paper, inspired by parlour games (cf. [3]), contained already the outline of the book [4], by von Neumann and Morgenstern in 1944. Present were here in the germ already games in extensive, normal and coalitional form. The main theorem was the existence theorem of values for mixed extensions of matrix games. Using this theorem games in coalitional form could be defined. Now 75 years later we can look back in great admiration to this paper, which revolutionarised social sciences so much.

About 350 years ago a similar impact on science had games of chance in the hands of Blaise Pascal, which made probability theory and statistics flourishing.

Common in the contributions of von Neumann and Pascal is the starting question of the value of a position of a player in a situation, where nature and/or other players play a role in the final result. This lecture deals with values in all kind of interactive situations.

Let us look first at zero-sum games with arbitrary action spaces. Here values not necessarily exist (cf. Wald [5]) but we can define upper and lower values. A huge literature on 'minimax' theorems exists, where sufficient conditions on action spaces and the payoff function are given to guarantee that the upper value is equal to the lower value. Also extensions to stochastic games (cf. Shapley, [6]) to vector payoffs (cf. Blackwell, [7]) and to 'zero-sum like' arbitration games (cf. [8]) can be mentioned.

An axiomatic characterization of the value for matrix games is given by Vilkas [9] (cf. Vorobev [10], p. 95). Here the properties monotonicity, dominance, symmetry and objectivity play a role. Inspired by this beautiful paper extensions to a class of infinite zero-sum games [11], to stochastic games [12] and to linear programming problems [13] were deduced.

Attempts to characterize value sets for non-zero sum games were undertaken by Vilkas [14] and Jansen and Tijs [15].

For 'generic' extensive form games with complete information (and also for other classes of non-cooperative games with a unique Nash equilibrium) a value vector is uniquely determined.

In the spirit of von Neumann, for n-person zero-sum games, where the corresponding two-person games have not necessarily a value, one can associate a

game in coalitional form, where the values of the coalitions are closed intervals with the lower and upper value as end points. Such cooperative games deserve some attention in a future study.

Let us turn now to bargaining problems. What is the value of a player to be in such a bargaining situation? This question posed by J. Nash [16] led to his now famous bargaining solution, where the axiomatic method was helpful. Another approach was taken by Raiffa [17], who took as one of his solutions the feasible compromise between the disagreement vector and the utopia vector. An axiomatic characterization of this solution was given by Kalai and Smorodinsky [18], and the solutions is nowadays called the Raiffa-Kalai-Smorodinsky solution.

For TU-games, via an axiomatic approach, Shapley [19] found his famous value. Variants were introduced by Myerson [20], Borm, Owen and Tijs [21] and others. Another approach was undertaken by Tijs [22], who considered as value vector for a TU-game the feasible compromise of an upper and a lower vector (cf. [23]). An extension to NTU-games was given in [24]. For a survey of compromise values, see [25].

We will discuss in the lecture more extensively the following recent contributions.

(i) The value of information in information collecting situations, [26].

(ii) The compromise value for games on matroids, [27].

(iii) Hypercubes and compromise values for cooperative fuzzy games, [28].

(iv) Stable and additive values on perfect cones of cooperative games, [29].

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