On Motion Planning for Robotic Manipulation with Permanent Rolling Contacts*

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Abstract

We address the motion planning problem (open-loop trajectory design) for manipulating rigid bodies with permanent rolling contact without slipping. This problem is related in particular to dextrous manipulation with robotic hands, consisting in changing the position and orientation of the manipulated object together with its grasp. We prove the flatness property for planar structures allowing to solve the motion planning problem by simple interpolation, without need to integrate the system differential equations and without quasi-static approximations. Though this property fails to be valid for general three-dimensional hand structures, similar results can be obtained for special structures thanks to the notion of Liouvillian systems. The links between flat or Liouvillian hand-object structures and non-holonomy of the contacts between fingers and object are also discussed. Several examples are studied in details.

Keywords: Dextrous manipulation, Nonholonomic mechanical systems, Motion planning, Flat systems, Liouvillian systems

1 Introduction

We study the motion planning problem (MPP) or, roughly speaking the open-loop trajectory design of robotic hands manipulating objects via one-point rolling contacts without slipping. In such systems all the fingers' degrees of freedom are actuated as opposed to the object's ones. The control objective is to change the orientation, position and grasp of the manipulated object.

The modelling of hand-object structures (HOS) and the kinematics of contact in the context of dextrous manipulation is treated in details in the robotics literature (see e.g. [6, 18, 19, 20], and [15, 16] including controllability aspects). An exhaustive review of static grasping methods is presented in [26].

Motion planning algorithms which can be applied to a large class of dextrous manipulation problems are reported in [5, 28] using quasi-static approximations. These algorithms involve

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optimization in order to find contact forces, for a given constant velocity, based on Peshkin's minimum power principle [24].

Due to the constraints of rolling without slipping, three-dimensional HOSs are nonholonomic. The underlying motion planning problem can be solved without quasi-static approximations by methods reported in [13, 21] using piecewise constant or sinusoidal inputs. An explicit solution in this framework is provided for the case of two parallel planar fingers with a symmetric object including non-pivoting constraints by [16]. An extension of the method of [13] when the fingers are allowed to break contact with the object (i.e for stratified configuration spaces) can be found in [11].

Our approach uses the notions of (differential) flatness [7, 8, 9, 10, 17] and Liouvillian systems. The latter notion was defined in an algebraic context in [4] and already used for motion planning purposes in [27].

Note that, though flatness can be also exploited for feedback design purposes, the feedback design and tracking aspects are not dealt with in this paper. For the sake of completeness, let us mention some contributions concerning feedback control of object manipulation: [6] for the rigid body case, [2, 3] with soft fingers and with uncertainties on the Jacobian matrices, and the learning control scheme proposed in [22] with unknown inertia parameters of the object.

Coming back to motion planning, the flatness property allows an easy parameterization of all feasible trajectories of the system which does not require integration of differential equations, and does not restrict to quasi-static approximations. More precisely, the definition of flatness directly implies the existence of a so-called flat output, of the same dimension as the input, such that there is a one-to-one mapping between sufficiently smooth trajectories of the flat output and feasible trajectories of the system, including the inputs, making the integration of the system dynamics useless. Thus, the motion planning can be done in the flat output space using elementary interpolation methods.

The Liouvillian property is slightly weaker. Roughly speaking, a system is Liouvillian if there exists an output of the same dimension as the input such that all the system variables can be expressed as functions of this output, a finite number of its derivatives, and a finite number of functions of its integrals. The output in this case is called a partially flat output, and the additional functions of the output integrals are referred to as integral variables. Thus, the flatness based approach to motion planning can be extended up to the computation of a finite number of integrals of functions of time, which is still simpler than integrating the system differential equations.

We prove that the dynamic model of planar HOSs are flat as a consequence of the constraints holonomy. Here the inputs are the joint torques of the fingers and a flat output can be chosen as the position and orientation of the object and combinations of the contact forces. In the three-dimensional case the rolling without slipping condition gives rise to nonholonomic constraints. As is, non-holonomy does not exclude flatness (see for instance the examples of the car with n-trailers or of the rolling hoop, or penny, in [17]), but in our case the kinematic equations form a Liouvillian system.

According to the dimension, the number of fingers, and the model type (i.e. dynamic or kinematic) we show that HOSs can be classified as summarized in Table 1.

The paper is organized as follows. The next section is devoted to the modelling of HOSs including planar configurations and HOSs with special morphology. The holonomy, flatness, and Liouvillian properties of HOSs are studied in Section 3. Exploiting these properties, solutions for the motion planning problem, in the HOS configurations whose models are presented in section 2, are proposed in Section 4. The paper ends with a short conclusion and, for the reader's convenience, an Appendix summarizes some notions from analytical mechanics used

hand structure	model	no. of fingers	holonomy	flatness	Liouvillian
2D	dynamics	> 1	yes	yes	-
3D	kinematics	1	no	yes	-
3D	kinematics	> 1	no	?	yes
3D + symmetry	dynamics	>2	no	?	yes

Table 1: Classification of HOSs

throughout the text.

2 Modelling

Consider the system setup illustrated in Figure 1. It comprises an object to be manipulated, and, without loss of generality, three fingers, enumerated from 1 to 3. Each finger is considered as a fully actuated manipulator with three joints. We assume that each segment of every finger and the manipulated object are rigid bodies. Contacts between the fingertips (last segments of the fingers) and the object are one point rolling contacts without slipping and are permanently maintained. We also assume that the object surface is smooth and convex and the finger surfaces are smooth and strictly convex. The convexity assumptions imply that the contact point between any of the fingers and the object is globally unique.

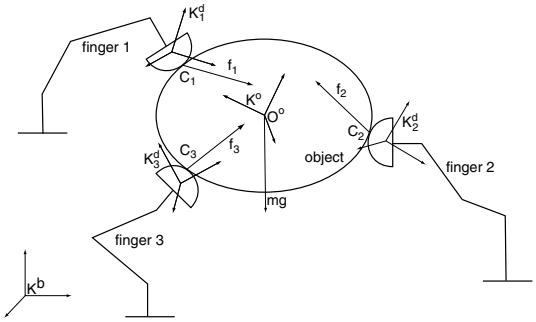

Figure 1: Robot hand with the manipulated object

The HOS is described by a set of differential and algebraic equations and inequality constraints ensuring that the above assumptions remain valid. They are presented in the next two subsections. A more detailed presentation of the modelling may be found in [6, 12, 20].

2.1 Geometry and Kinematics

The geometric contact conditions are derived first together with the geometric description of the fingers. Then we express the constraints on the velocities.

Denote the inertial base frame by K^b. Let K_i^d $(i = 1, \ldots, 3)$ and K^o be frames fixed to the center of gravity (c-o-g) of the ith fingertip and of the object, respectively[1]. The position of the c-o-g of the object (resp. finger) w.r.t. the origin of the frame K^b is denoted by $p^o = (x^o, y^o, z^o)$ (resp. $p_i^d = (x_i^d, y_i^d, z_i^d)$). The orientation of the object (resp. finger) w.r.t the inertial base

[1]The notation K^{d_i} would be more consequent, but we prefer to avoid double subscripts for readability's sake.

frame is given by the matrix $\Omega(\phi^o)$ (resp. $\Omega(\phi_i^d)$). See Appendix for a short presentation of the properties of such matrices. Define the pair

$$(p_i^r, \Omega(\phi_i^r)) = (\Omega(\phi_i^d)^T(p^o - p_i^d), \Omega(\phi_i^d)^T\Omega(\phi^o)), \tag{1}$$

where p_i^r (resp. $\Omega(\phi_i^r)$) represents the relative position (resp. orientation) of the object w.r.t. the finger i.

Denote by $p_{C_i}^o = (x_{C_i}^o, y_{C_i}^o, z_{C_i}^o)^T$ (resp. $p_{C_i}^d = (x_{C_i}^d, y_{C_i}^d, z_{C_i}^d)^T$) the coordinates of the contact point C_i expressed in the frame K^o (resp. K_i^d). From the permanent contact assumption we have

$$p_{C_i}^d - p_i^r - \Omega(\phi_i^r)p_{C_i}^o = 0. \tag{2}$$

Let the smooth function[2] $c_i^d : \mathbb{R}^3 \to \mathbb{R}$ (resp. $c^o : \mathbb{R}^3 \to \mathbb{R}$) characterize the surface of fingertip i (resp. object) such that any surface point P of the ith fingertip (resp. object) with coordinates $p_P^{d_i}$ (resp. p_P^o) expressed in the basis of K_i^d (resp. K^o) satisfies $c_i^d(p_P^{d_i}) = 0$ (resp. $c^o(p_P^o) = 0$). Moreover, for any interior point P of the corresponding rigid bodies we have $c_i^d(p_P^{d_i}) < 0$ (resp. $c^o(p_P^o) < 0$). Since C_i is a point of both surfaces in contact one has

$$c_i^d(p_{C_i}^d) = 0 \tag{3}$$
$$c^o(p_{C_i}^o) = 0. \tag{4}$$

Denote by $Dc = \left(\frac{\partial c}{\partial x}, \frac{\partial c}{\partial y}, \frac{\partial c}{\partial z}\right)$ the derivative of a surface function c. Since the tangent planes of the surfaces at C_i must coincide, the unit normal vectors must be colinear with opposite directions:

$$\frac{Dc_i^d(p_{C_i}^d)}{\parallel Dc_i^d(p_{C_i}^d) \parallel} = -\frac{Dc^o(p_{C_i}^o)\Omega(\phi_i^r)^T}{\parallel Dc^o(p_{C_i}^o) \parallel}. \tag{5}$$

It remains to give the position and orientation of the frame K_i^d fixed to the fingertip as functions of the joint coordinates. For, denote by q_i the vector of joint coordinates of the finger i and by d_i the direct geometry function of the finger i [14]. Since each finger has three joints by assumption, the dimension of q_i is equal to 3 for every $i = 1, 2, 3$. Then we have

$$d_i : U \subset \mathbb{R}^3 \to \mathbb{R}^3 \times SO(3), \qquad d_i(q_i) = (p_i^d, \Omega(\phi_i^d)), \tag{6}$$

where U is the set of admissible joint configurations.

Constraints (2)-(6) are geometric. However, the rolling without slipping assumption imposes additional constraints on the relative velocities at the contact points. Consider the object and the ith finger, $i = 1, 2, 3$ (see Figure 2). Recall that the two-dimensional vector space $\ker Dc_i^d(p_{C_i}^d) = \{w \in \mathbb{R}^3 | Dc_i^d(p_{C_i}^d)w = 0\}$ is identified to the tangent plane of the surface of the finger at the point C_i. Denote by $\{w_{1,C_i}^d, w_{2,C_i}^d\}$ a basis of $\ker Dc_i^d(p_{C_i}^d)$. Let $v_{C_i}^r$ denote the instantaneous relative velocity of the bodies in contact at the point C_i, such that $v_i^r = \dot{p}_i^r$. Using (81) of the Appendix, we have:

$$v_{C_i}^r = \dot{p}_i^r + [\omega_i^r \times]\Omega(\phi_i^r)p_{C_i}^o.$$

The geometric assumption of permanent and unique contact point implies that $v_{C_i}^r$ lies in the tangent plane at the contact point, i.e. $v_{C_i}^r \in \ker Dc_i^d(p_{C_i}^d)$. Moreover, it vanishes according

[2]Unless explicitly mentioned, all the functions we consider throughout the paper are smooth, i.e. infinitely differentiable in all their variables in open dense subsets of the corresponding manifolds.

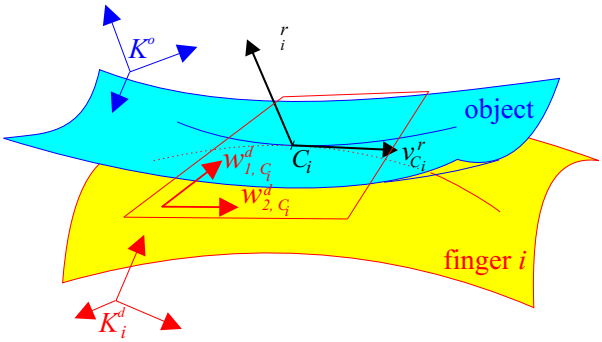

Figure 2: Rolling without slipping of surfaces

to the rolling without slipping constraint. We therefore obtain two independent equations by expressing that the components of $v^r_{C_i}$ in the basis $\{w^d_{1,C_i}, w^d_{2,C_i}\}$, vanish:

$$(w^d_{j,C_i})^T v^r_{C_i} = 0, \quad j = 1, 2. \tag{7}$$

Constraints (3)-(6) and (7) must hold true at every contact point, namely for $i = 1, 2, 3$.

Two HOS examples illustrate the modelling procedure for geometry and kinematics. Motion planning algorithms are presented for both examples in Section 4.

Example 1. Figure 3 illustrates the simple setup of an object (sphere with radius R) and a finger (plane). Since we have only one finger, the subscript corresponding to the finger number is omitted. The plane can be translated along the x and y coordinate axes of the reference frame K^b, the joint coordinates corresponding to these translations are denoted by q_1 and q_2, respectively. The choice of the three elementary rotations for the representation of the orientation matrices follows the RPY (roll, pitch, yaw) convention (see Equation (79) of the Appendix). The surface equations (3)-(4) read

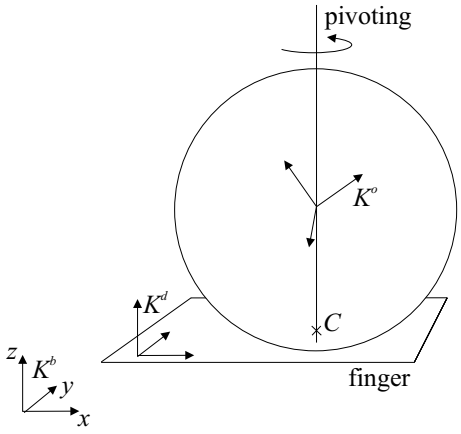

Figure 3: Simple three-dimensional setup with one finger

$$c^o(p^o_C) = (x^o_C)^2 + (y^o_C)^2 + (z^o_C)^2 - R^2 = 0 \qquad c^d(p^d_C) = z^d_C = 0,$$

and their derivatives are

$$Dc^o(p^o_C) = 2(p^o_C)^T \qquad Dc^d(p^d_C) = (0, 0, 1). \tag{8}$$

The direct geometry (6) of the finger is given by

$$\left([q_1, q_2, 0]^T, I\right) = \left(p^d, \Omega(\phi^d)\right).$$

Since $\Omega(\phi^d) = I$, we have that $\Omega(\phi^r) = \Omega(\phi^o)$. Reporting (8) in Equation (5) we get

$$\Omega(\phi^o) p_C^o = -R(0,0,1)^T, \tag{9}$$

thus Equation (2) reads

$$\begin{bmatrix} x^o \\ y^o \\ z^o \end{bmatrix} = \begin{bmatrix} x^d + x_C^d \\ y^d + y_C^d \\ z^d + R \end{bmatrix}.$$

The angular velocity of the sphere, using (80), is given by

$$\omega^o = \begin{bmatrix} \omega_x^o \\ \omega_y^o \\ \omega_z^o \end{bmatrix} = \begin{bmatrix} -\dot{\theta}\sin\varphi + \dot{\psi}\cos\varphi\cos\theta \\ \dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\cos\theta \\ \dot{\varphi} - \dot{\psi}\sin\theta \end{bmatrix}. \tag{10}$$

Using (9), the relative velocity at the contact point reads

$$v_C^r = \dot{p}^r + [\omega^o \times] \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix} = \begin{bmatrix} \dot{x}_C^d - R(\dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\cos\theta) \\ \dot{y}_C^d - R(\dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\cos\theta) \\ 0 \end{bmatrix}$$

and $\ker Dc^d$ is spanned by $w_{1,C}^d = (1,0,0)^T$ and $w_{2,C}^d = (0,1,0)^T$. Thus, the kinematic constraints (7) read

$$\begin{aligned} 0 &= \dot{x}_C^d - R(\dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\cos\theta) \\ 0 &= \dot{y}_C^d - R(\dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\cos\theta). \end{aligned} \tag{11}$$

Observe that these constraints don't exclude rotations of the object around the axis passing through the contact point and normal to the contact surface (see Figure 3). Such motions are referred to as spinning or pivoting motions in the literature. The elimination of these motions is not a consequence of our modelling assumptions. However, translational motions of the plane cannot generate these rotations, therefore it needs to be eliminated if there is no such actuator in the system. The constraint eliminating these pivoting motions reads

$$\omega_z^o = \dot{\varphi} - \dot{\psi}\sin\theta = 0, \tag{12}$$

and referred to as the nonpivoting constraint in the sequel. ∎

Lemma 1. *The Pfaffian system obtained from* (11)*, namely*

$$\begin{aligned} \varpi_1 &= dx_C^d - R(\cos\varphi d\theta + \sin\varphi\cos\theta d\psi) \\ \varpi_2 &= dy_C^d - R(\sin\varphi d\theta - \cos\varphi\cos\theta d\psi) \end{aligned}$$

is fully non holonomic (see Appendix, Definition 4). Moreover, the Pfaffian system obtained by adding the constraint of non-pivoting (12)*:*

$$\varpi_3 = d\varphi - \sin\theta d\psi,$$

to ϖ_1 and ϖ_2 is also fully non holonomic.

Proof. For both cases, the same geometric constraints are used to eliminate variables, hence the kinematic constraints must be satisfied on a seven-dimensional manifold determined by x_C^d, y_C^d, x^d, y^d, and ϕ^o. The one-forms ϖ_1, ϖ_2, and ϖ_3 are linearly independent because of the components in dx_C^d, dy_C^d, and $d\varphi$. Let us calculate first the derived flags (see Appendix and [17]) associated to the Pfaffian systems $I_1 = \{\varpi_1, \varpi_2\}$ and $I_2 = \{\varpi_1, \varpi_2, \varpi_3\}$. After some lengthy but elementary calculations the derived flag of I_1 (resp. I_2) has dimensions $\{2, 0\}$ (resp. $\{3, 2, 0\}$). Denote by Δ_1 (resp. Δ_2) the distribution annihilating I_1 (resp. I_2) and recall that $\dim \Delta_1 = 7 - 2 = 5$ (resp. $\dim \Delta_2 = 7 - 3 = 4$). By the properties of the derived flag, the involutive closure $\bar{\Delta}_1$ (resp. $\bar{\Delta}_2$) of Δ_1 (resp. Δ_2) satisfies $\dim \bar{\Delta}_1 = 7 - 2 + 2 = 7$ (resp. $\dim \bar{\Delta}_2 = 7 - 3 + 1 + 2 = 7$), which is the dimension of the configuration manifold after eliminations and the lemma follows by Definition 4 of the Appendix. □

Example 2. The fingers are the three tangent planes to a sphere of radius R as depicted in Figure 4. They are assumed parallel to the three respective cartesian planes $x = 0, y = 0$ and $z = 0$ of the base frame and their displacements are restricted to translations along the corresponding directions. The frames fixed to the planes have the same orientations as the base

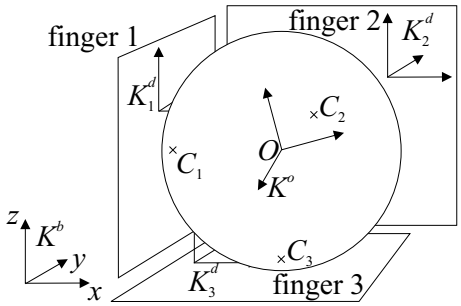

Figure 4: Simple three-dimensional setup

frame. The orientation of the sphere is given by $\phi^o = (\varphi, \theta, \psi)$ as in the previous example. The surface equations (3)-(4) read

$$x_{C_1}^d = 0 \qquad -y_{C_2}^d = 0 \qquad z_{C_3}^d = 0 \qquad (x_{C_i}^o)^2 + (y_{C_i}^o)^2 + (z_{C_i}^o)^2 - R^2 = 0.$$

Their derivatives at the contact points are

$$Dc^o(p_{C_i}^o) = 2(p_{C_i}^o)^T \quad Dc_1^d(p_{C_1}^d) = (1, 0, 0) \quad Dc_2^d(p_{C_2}^d) = -(0, 1, 0) \quad Dc_3^d(p_{C_3}^d) = (0, 0, 1). \quad (13)$$

Denoting by $q_i = (q_{i1}, q_{i2}, q_{i3})^T$ the vector of joint coordinates of the ith finger, its direct geometry function reads

$$(q_i, I) = \left(p_i^d, \Omega(\phi_i^d)\right), \quad (14)$$

hence we have $\Omega(\phi_i^r)p_{C_i}^o = \Omega(\phi^o)p_{C_i}^o$. Reporting in Equation (5) the expression of $Dc^o(p_{C_1}^o)$ and $Dc_1^d(p_{C_1}^d)$ from (13) we get

$$\Omega(\phi^o)p_{C_1}^o = Re_1,$$

with $e_1 = (-1, 0, 0)^T$. For the remaining two fingers one obtains similar relations: $\Omega(\phi^o)p_{C_2}^o = Re_2$ and $\Omega(\phi^o)p_{C_3}^o = Re_3$ with $e_2 = (0, 1, 0)^T$ and $e_3 = (0, 0, -1)^T$. Using these expressions, the geometric constraints corresponding to (2) can be simplified to obtain

$$p_{C_i}^d = (p^o - p_i^d) + Re_i, \qquad i = 1, 2, 3,$$

or, isolating the coordinates of the frame fixed to the object,

$$\begin{array}{lll} x^o = x_1^d + R & x^o = x_2^d + x_{C_2}^d & x^o = x_3^d + x_{C_3}^d \\ y^o = y_1^d + y_{C_1}^d & y^o = y_2^d - R & y^o = y_3^d + y_{C_3}^d \\ z^o = z_1^d + z_{C_1}^d & z^o = z_2^d + z_{C_2}^d & z^o = z_3^d + R. \end{array} \tag{15}$$

The relative velocity at the contact point C_i reads

$$[\omega^o\times]Re_i + \dot{p}^o - \dot{p}_i^d = 0 \qquad i = 1, 2, 3. \tag{16}$$

The vectors spanning the tangent planes of the fingers are $w_{2,C_2}^d = w_{1,C_3}^d = (1,0,0)^T$, $w_{2,C_1}^d = w_{2,C_3}^d = (0,1,0)^T$, and $w_{1,C_1}^d = w_{1,C_2}^d = (0,0,1)^T$. Reporting this in the expression (7) of the kinematic constraint and using Equation (80) of $[\omega^o\times]$ we get

$$\begin{array}{lll} R\omega_y^o + \dot{z}^o - \dot{z}_1^d = 0 & R\omega_x^o + \dot{z}^o - \dot{z}_2^d = 0 & -R\omega_y^o + \dot{x}^o - \dot{x}_3^d = 0 \\ -R\omega_z^o + \dot{y}^o - \dot{y}_1^d = 0 & -R\omega_z^o + \dot{x}^o - \dot{x}_2^d = 0 & R\omega_x^o + \dot{y}^o - \dot{y}_3^d = 0. \end{array}$$

These kinematic constraints are not fully nonholonomic. For, observe that eliminating $R\omega_x^o$ one gets

$$\dot{z}^o - \dot{z}_2^d = \dot{y}^o - \dot{y}_3^d. \tag{17}$$

Differentiating the geometric constraints $z^o = z_2^d + z_{C_2}^d$ and $y^o = y_3^d + y_{C_3}^d$ obtained in (15), and reporting them in (17) we get $\dot{z}_{C_2}^d = \dot{y}_{C_3}^d$ which can be integrated as $z_{C_2}^d + z_{C_2}^d(0) = y_{C_3}^d + y_{C_3}^d(0)$. One may get similar relations by eliminating ω_y^o and ω_z^o. This gives three integrated constraints:

$$\begin{array}{l} y_{C_1}^d + y_{C_1}^d(0) = x_{C_2}^d + x_{C_2}^d(0) \\ z_{C_1}^d + z_{C_1}^d(0) = -x_{C_3}^d - x_{C_3}^d(0) \\ z_{C_2}^d + z_{C_2}^d(0) = y_{C_3}^d + y_{C_3}^d(0), \end{array} \tag{18}$$

where $y_{C_1}^d(0)$, $x_{C_2}^d(0)$, $z_{C_1}^d(0)$, $x_{C_3}^d(0)$, $z_{C_2}^d(0)$ and $y_{C_3}^d(0)$ are initial conditions. Using these expressions together with the geometric constraints (15) and the expression of $[\omega^o\times]$ in the RPY representation (79), the remaining constraints read:

$$\begin{array}{l} 0 = \dot{y}_{C_1}^d - R(\dot{\varphi} - \dot{\psi}\sin\theta) \\ 0 = \dot{z}_{C_1}^d + R(\dot{\theta}\cos\varphi + \dot{\psi}\sin\varphi\cos\theta) \\ 0 = \dot{z}_{C_2}^d - R(\dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\cos\theta) \end{array} \tag{19}$$

■

Lemma 2. *The Pfaffian system obtained from the constraints* (19)*, namely:*

$$\begin{array}{l} \varpi_1 = -Rd\varphi + R\sin\theta d\psi + dy_{C_1}^d = 0 \\ \varpi_2 = R\cos\varphi d\theta + R\sin\varphi\cos\theta d\psi + dz_{C_1}^d = 0 \\ \varpi_3 = -R\sin\varphi d\theta + R\cos\varphi\cos\theta d\psi + dz_{C_2}^d = 0 \end{array} \tag{20}$$

is fully non holonomic.

Proof. After elimination of variables using the geometric and the integrated kinematic constraints, the kinematic constraints must be satisfied on a nine-dimensional manifold given by the variables $y_{C_1}^d$, $z_{C_1}^d$, $z_{C_2}^d$, x_1^d, y_2^d, z_3^d, and $\phi^o = (\varphi, \theta, \psi)$. The one-forms ϖ_1, ϖ_2, and ϖ_3 are linearly independent because of their components in $dy_{C_1}^d$, $dz_{C_1}^d$, and $dz_{C_2}^d$. The derived flag of the Pfaffian system $I = \{\varpi_1, \varpi_2, \varpi_3\}$ has dimensions $\{3, 0\}$. Denoting by Δ the annihilating distribution of I, we have that $\dim\Delta = 9 - 3 = 6$. Computing the dimension of its involutive closure using the derived flag, we have: $\dim\bar{\Delta} = 9 - 3 + 3 = 9$. Since 9 is the dimension of the configuration manifold after elimination, the lemma follows. □

2.2 Dynamic equations

The dynamic equations are obtained separately for each finger and the object. These dynamics are connected by the contact forces. Denote by f_i the contact force applied to the object by the ith fingertip. The vectors f_i are expressed in the basis of K^b.

The Newton-Euler equations of the object read:

$$m\ddot{p}^o = f_g + \sum_{i=1}^{3} f_i \tag{21}$$

$$\Theta\dot{\omega}^o = \sum_{i=1}^{3} \Omega(\phi^o) p^o_{C_i} \times f_i - \omega^o \times \Theta\omega^o, \tag{22}$$

where Θ is the inertia matrix of the object, m is its mass and f_g the gravitational force. The dynamic equation on the ith finger is

$$H_i(q_i)\ddot{q}_i + h_i(q_i, \dot{q}_i) = \tau_i - \tau_{i,ext}, \tag{23}$$

where τ_i is the vector of joint torques, $\tau_{i,ext}$ is the vector of external joint torques, H_i is the inertia matrix of the ith finger and h_i contains the quadratic (w.r.t. the joint velocities) and gravitational terms. The relation between the forces f_i and torques $\tau_{i,ext}$ is given by the transformation:

$$\tau_{i,ext} = J_i^T G_i^T \cdot f_i \tag{24}$$

where J_i is the Jacobian matrix of the ith finger (i.e. the derivative of its direct geometry function) and G_i^T is the linear mapping transforming the contact force f_i to an equivalent wrench (force and torque taken together) at the origin of the frame K_i^d.

2.3 Inequality constraints

For the sake of completeness, let us state the inequality constraints implied by the rolling without slipping contact assumptions. The contact forces applied by the fingers must be oriented inwards the object

$$\Omega(\phi^o)^T Dc^o(p^o_{C_i}) \cdot f_i \ < 0 \tag{25}$$

and because of the rolling contacts they cannot leave the friction cone described by the inequality

$$\frac{\|f_{it}\|}{\|f_{in}\|} < \mu$$

where f_{in} (resp. f_{it}) is the normal (resp. tangent) component of f_i and μ is the Coulomb friction coefficient. Other inequality constraints may also be used to characterize the reachable space of the finger joints, ensure collision free trajectories between the fingers, etc. Since they are not directly used in the sequel they are omitted.

2.4 Modelling of planar HOSs

The modelling hypotheses introduced at the beginning of Section 2 remain unchanged for the planar case. We proceed in the same way as for the modelling of three-dimensional HOSs, namely we start with the equations defining the geometry of the surfaces, contacts and fingers, then we continue with the kinematic constraints and with the dynamics.

Without loss of generality, we may assume that planar HOSs evolve in the xz-plane of the reference frame K^b. Hence all notations used so far remain unchanged, it is enough to omit the y-coordinates and the φ, ψ orientation angles from the RPY representation. Thus the planar version of the general RPY orientation matrix, defined by (79), simply reads

$$\Omega(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

According to the notations of Section 2.1, the only orientation angle is denoted by ϕ^o for the object and by ϕ_i^d for the finger i.

The relative position and orientation of the finger i and the object is defined by the pair

$$(p_i^r, \phi_i^r) = \left(\Omega(\phi_i^d)^T(p^o - p_i^d), \phi^o - \phi_i^d\right). \tag{26}$$

The geometric constraints (2)-(5) here read, for every finger i:

$$p_{C_i}^d - p_i^r - \Omega(\phi_i^r)p_{C_i}^o = 0 \tag{27}$$

$$c_i^d(p_{C_i}^d) = 0 \tag{28}$$

$$c^o(p_{C_i}^o) = 0 \tag{29}$$

$$\det\begin{pmatrix} Dc_i^d(p_{C_i}^d) \\ Dc^o(p_{C_i}^o)\cdot\Omega(\phi_i^r)^T \end{pmatrix} = 0, \tag{30}$$

where the last equation results from (5) expressing the colinearity of the normal vectors. Here attention has to be paid to the directions of the normal vectors. The direct geometry function of the ith finger is similar to (6), but with straightforward adaptation of its range space:

$$d_i : U \subset \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z}, \qquad d_i(q_i) = \left(p_i^d, \phi_i^d\right). \tag{31}$$

Consider next the kinematic constraints. First, note that $\ker Dc_i^d(p_{C_i}^d)$ is one-dimensional, and spanned by

$$w_{C_i}^d = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} Dc_i^d(p_{C_i}^d)^T.$$

Taking into account the fact that the relative velocity of the object and finger at the contact point is

$$v_{C_i}^r = \dot{p}_i^r + [\dot{\phi}_i^r\times]\left(p_{C_i}^d - p_i^r\right),$$

the kinematic constraint (7) reads

$$(w_{C_i}^d)^T\left(\dot{p}_i^r + [\dot{\phi}_i^r\times]\left(p_{C_i}^d - p_i^r\right)\right) = 0 \quad \text{with} \quad [\dot{\phi}_i^r\times] = \begin{bmatrix} 0 & -\dot{\phi}_i^r \\ \dot{\phi}_i^r & 0 \end{bmatrix}. \tag{32}$$

The adaptation of the dynamical equations of the object and fingers for the two-dimensional case is straightforward:

$$m\ddot{p}^o = f_g + \sum_{i=1}^m f_i \tag{33}$$

$$\Theta\ddot{\phi}^o = \sum_{i=1}^m \Omega(\phi^o)p_{C_i}^o \times f_i \tag{34}$$

$$H_i(q_i)\ddot{q}_i + h_i(q_i, \dot{q}_i) = \tau_i - J_i^T G_i^T \cdot f_i. \tag{35}$$

The next example presents a simple planar HOS.

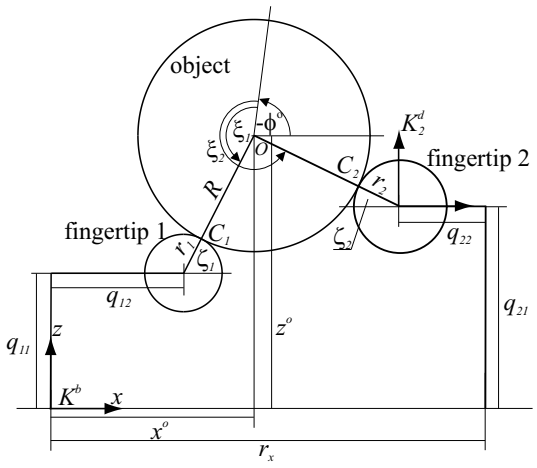

Figure 5: Planar HOS.

Example 3. Consider the HOS given in Figure 5. The hand has two fingers with prismatic joints and fingertips of radius r_i, $i = 1, 2$. The manipulated object is a disc of radius R. The position of the object is given by the coordinates x^o and z^o of the point O and its orientation is determined by the angle ϕ^o. Using (26), the geometric constraints (27)-(30) read, for $i = 1, 2$,

$$p_i^d - \begin{bmatrix} x^o - x_{C_i}^d \\ z^o - z_{C_i}^d \end{bmatrix} - \begin{bmatrix} \cos\phi^o & \sin\phi^o \\ -\sin\phi^o & \cos\phi^o \end{bmatrix} \begin{bmatrix} x_{C_i}^o \\ z_{C_i}^o \end{bmatrix} = 0 \tag{36}$$

$$(x_{C_i}^d)^2 + (z_{C_i}^d)^2 - r_i^2 = 0 \tag{37}$$

$$(x_{C_i}^o)^2 + (z_{C_i}^o)^2 - R^2 = 0 \tag{38}$$

$$\det \begin{bmatrix} x_{C_i}^d & z_{C_i}^d \\ x_{C_i}^o \cos\phi^o + z_{C_i}^o \sin\phi^o & x_{C_i}^o \sin\phi^o - z_{C_i}^o \cos\phi^o \end{bmatrix} = 0. \tag{39}$$

The direct geometry of the fingers read

$$\left(\left[q_{12}, q_{11} \right]^T, 0 \right) = \left(p_1^d, \phi_1^d \right) \qquad \left(\left[r_x - q_{22}, q_{21} \right]^T, 0 \right) = \left(p_2^d, \phi_2^d \right). \tag{40}$$

The kinematic constraint (32) reads

$$\begin{bmatrix} -z_{C_i}^d & x_{C_i}^d \end{bmatrix} \left(\begin{bmatrix} \dot{x}^o \\ \dot{z}^o \end{bmatrix} - \dot{p}_i^d + [\dot{\phi}^o \times] \left(\begin{bmatrix} x_{C_i}^d - x^o \\ z_{C_i}^d - z^o \end{bmatrix} + p_i^d \right) \right) = 0. \tag{41}$$

The dynamic equation of the object are given by (33)-(34), while the dynamics of the finger i, corresponding to (35) reads

$$\begin{bmatrix} m_{i1} + m_{i2} & 0 \\ 0 & m_{i2} \end{bmatrix} \begin{bmatrix} \ddot{q}_{i1} \\ \ddot{q}_{i2} \end{bmatrix} + \begin{bmatrix} -(m_{i1} + m_{i2})g \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_{i1} \\ \tau_{i2} \end{bmatrix} - \begin{bmatrix} f_{i,y} \\ (-1)^{i-1} f_{i,x} \end{bmatrix}, \tag{42}$$

where m_{ij} gives the mass of the jth segment of the finger i. ■

2.5 HOSs with special morphology

Consider a HOS with 3 fingers, each with 5 joints. Suppose that the following properties are satisfied (see Figure 6).

1. The last segments of the fingers are spheres such that they can be rotated around two independent symmetry axes. These rotations correspond to the joints 4 and 5.

2. For each finger, the joint coordinates corresponding to the joints number 4 and 5 are cyclic coordinates [29] (i.e. $\frac{\partial \mathcal{L}_i}{\partial q_{i,4}} = \frac{\partial \mathcal{L}_i}{\partial q_{i,5}} = 0$, for $i = 1 \dots 3$, where $\mathcal{L}_i$ is the Lagrangian of the ith finger).

3. The orientation of the last joint axis does not depend on $q_{i,4}$ $(i = 1, \dots, 3)$.

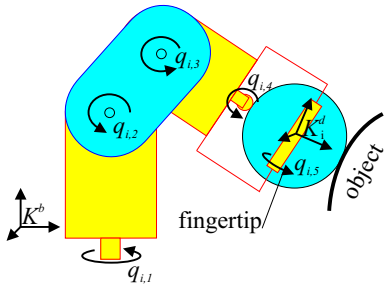

Figure 6: Finger with symmetries satisfying the conditions

Remark 1. *The last condition corresponds to special mechanical devices, roughly speaking, similar to the ones used for trackballs and for the mechanical integrator presented in [23, Chapter 1]. The results presented for fingers with three fingers, five joints each, can be easily extended to HOSs with higher number of fingers and joints. We have restricted ourselves to the simplest case for readability's sake.*

Remark 2. *Let us give a mechanical consequence of the above conditions. Assuming that the last segments of the fingers have all radius r (these radii may be different for the general case), one can virtually enlarge the volume of the convex object by constructing a new surface which is at distance r from the original one in the direction of the corresponding finger. It is easy to see that the centers of the last segments are always on this enlarged surface as far as the contact is maintained. Let this point of the enlarged surface be denoted by $\bar{C}_i$ for the finger i and let $t_i = \overrightarrow{\bar{C}_i C_i}$. (Notice that the direction of t_i is normal to both surfaces at C_i and $\bar{C}_i$). Denote the linear (resp. angular) velocity of the last segment (i.e. the sphere) of the finger i by $v^d_{4,5}$ (resp. $\omega^d_{4,5}$). Then the rolling without slipping constraint at the real contact point C_i reads*

$$v^r_{C_i} = v^d_{4,5} + [\omega^d_{4,5}\times]t_i - \left(\dot{p^o} + [\omega^o\times]\Omega(\phi^o)p^o_{C_i}\right) = 0.$$

Since the linear velocity $v^d_{4,5}$ stands for the velocity of $\bar{C}_i$ which also belongs to the enlarged surface, the relative velocity at $\bar{C}_i$ reads

$$v^r_{\bar{C}_i} = v^d_{4,5} - \left(\dot{p}^o + [\omega^o\times]\left(\Omega(\phi^o)p^o_{C_i} - t_i\right)\right) = \left[(\omega^o - \omega^d_{4,5})\times\right] t_i$$

which is different from zero as far as $\omega^o - \omega^d_{4,5} \neq 0$. This relation may be interpreted as the fact that the origin of the last segment (i.e. the center the sphere) slides on the enlarged surface of the object.

Example 4. Consider the HOS depicted in Figure 7. The manipulated object is a sphere of radius R. The orientation of the sphere is represented by the RPY angles $\phi^o = (\varphi, \theta, \psi)$. The

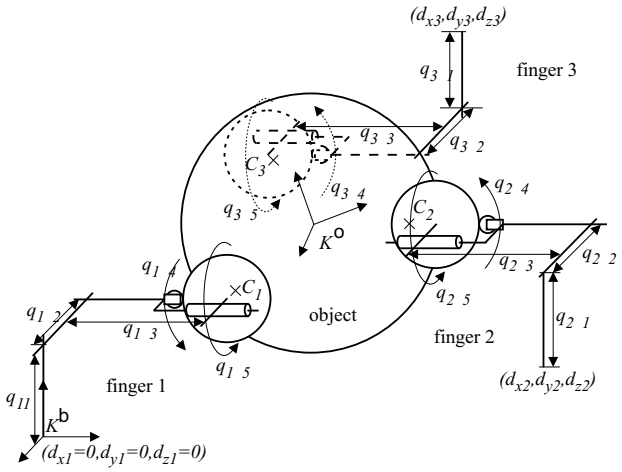

Figure 7: HOS in 3 dimensions

vectors $(d_{xi}, d_{yi}, d_{zi})^T$ $(i = 1, 2, 3)$ give the coordinates of the finger base points in the frame K^b. The frames K_i^d are fixed to the centers of the last segments of the fingers. The surface equations (3)-(4) are given by

$$(x_{C_i}^o)^2 + (y_{C_i}^o)^2 + (z_{C_i}^o)^2 - R^2 = 0 \qquad (x_{C_i}^d)^2 + (y_{C_i}^d)^2 + (z_{C_i}^d)^2 - r^2 = 0 \tag{43}$$

with derivatives

$$Dc^o(p_{C_i}^o) = 2\left(p_{C_i}^o\right)^T \qquad Dc_i^d(p_{C_i}^d) = 2\left(p_{C_i}^d\right)^T,$$

hence the constraint (5) reads

$$p_{C_i}^d = -\frac{r}{R}\Omega(\phi_i^r)p_{C_i}^o.$$

Reporting this in (2), and multiplying both sides from left by $\Omega(\phi_i^d)$ we get

$$\begin{bmatrix} q_{i,3} + d_{xi} \\ q_{i,2} + d_{yi} \\ q_{i,1} + d_{zi} \end{bmatrix} - p^o - \left(1 + \frac{r}{R}\right)\Omega(\phi^o)p_{C_i}^o = 0. \tag{44}$$

The direct geometry function of the finger 1 is given by

$$\left(\begin{bmatrix} q_{12} \\ q_{13} \\ q_{11} \end{bmatrix}, \begin{bmatrix} \cos q_{15} & 0 & \sin q_{15} \\ \sin q_{15}\sin q_{14} & \cos q_{14} & -\cos q_{15}\sin q_{14} \\ -\sin q_{15}\cos q_{14} & \sin q_{14} & \cos q_{15}\cos q_{14} \end{bmatrix}\right) = \left(p_1^d, \Omega(\phi_1^d)\right) \tag{45}$$

and has similar form for the remaining two fingers. The relative velocity at the contact point reads

$$v_{C_i}^r = \dot{p}^o + [\omega^o\times]\Omega(\phi^o)p_{C_i}^o - \begin{bmatrix} \dot{q}_{i,3} \\ \dot{q}_{i,2} \\ \dot{q}_{i,1} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \dot{q}_{15} \\ 0 & 0 & -\dot{q}_{14} \\ -\dot{q}_{15} & \dot{q}_{14} & 0 \end{bmatrix}\left(\Omega(\phi^o)p_{C_i}^o + p^o - \begin{bmatrix} q_{i,3} + d_{xi} \\ q_{i,2} + d_{yi} \\ q_{i,1} + d_{zi} \end{bmatrix}\right).$$

The vectors spanning the tangent plane at the contact point are $w_{1,C_i}^d = [-z_{C_i}^d, 0, x_{C_i}^d]^T$ and $w_{2,C_i}^d = [-y_{C_i}^d, x_{C_i}^d, 0]^T$ which allows to determine the kinematic constraint (7) using the above expression of the relative velocity. Finally, since the structure of the inertia matrices are not specified neither for the fingers nor for the object, the dynamic equations of the object and the fingers are given in their general form by (21)-(22) and by (23). ∎

3 Holonomy, flatness and Liouvillian properties

The definition of holonomy is given in the Appendix. Let us give here the definitions of flat and Liouvillian systems.

Definition 1 (flatness). *The system*

$$\dot{x} = f(x, u) \tag{46}$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is differentially flat if one can find a set of variables, called flat output,

$$y = h(x, u, \dot{u}, \ddot{u}, \ldots, u^{(r)}), \qquad y \in \mathbb{R}^m \tag{47}$$

with r finite integer, such that

$$x = \alpha(y, \dot{y}, \ddot{y}, \ldots, y^{(q)}) \qquad u = \beta(y, \dot{y}, \ddot{y}, \ldots, y^{(q+1)})$$

with q finite integer, and such that the system equations

$$\frac{d\alpha}{dt}(y, \dot{y}, \ddot{y}, \ldots, y^{(q+1)}) = f(\alpha(y, \dot{y}, \ddot{y}, \ldots, y^{(q)}), \beta(y, \dot{y}, \ddot{y}, \ldots, y^{(q+1)}))$$

are identically satisfied.

The weaker notion of Liouvillian systems is defined in [4] using differential algebra. We adopt here a slightly different definition.

Definition 2 (Liouvillian system). *The system* (46) *is said to be Liouvillian if there exists a set of variables defined as* (47)*, and a finite number of variables $\xi_1, \ldots, \xi_p$ given by*

$$\xi_i^{(\mu_i)} = \gamma_i(y, \dot{y}, \ldots, y^{(q+1)}) \qquad i = 1, \ldots, p \tag{48}$$

such that

$$x = \alpha(y, \dot{y}, \ddot{y}, \ldots, y^{(q)}, \bar{\xi}_1^{\mu_1 - 1}, \ldots, \bar{\xi}_p^{\mu_p - 1}) \qquad u = \beta(y, \dot{y}, \ddot{y}, \ldots, y^{(q+1)}, \bar{\xi}_1^{\mu_1 - 1}, \ldots, \bar{\xi}_p^{\mu_p - 1}),$$

with $\bar{\xi}_i^{\mu_i - 1} = (\xi_i, \dot{\xi}_i, \ldots, \xi_i^{(\mu_i - 1)})$, and such that α and β identically satisfy the system equations:

$$\frac{d\alpha}{dt} = f(\alpha, \beta).$$

We refer to y as a partially flat output and to $\xi_1, \ldots, \xi_p$ as integral variables.

Recall that the notion of Liouvillian systems is closely related to integrability by quadratures and thus to strictly triangular forms, also appearing in [16]. The remaining part of this section discusses the holonomy, flatness and Liouvillian properties of HOSs. General results are obtained for planar HOSs and for HOS structures with symmetries. Specific results are presented for the examples of Section 2. The solutions of the motion planning problem presented for these HOSs in Section 4 are based on the propositions of the present section.

Let us insist on the fact that the differential equations (48), satisfied by the integral variables ξ_i, only depend on the partially flat output y. Therefore, once we choose a smooth trajectory $t \to y(t)$, the RHS of (48) becomes a function of t only which justifies the fact that ξ_i are called integral variables.

3.1 Holonomy and flatness of planar structures

We prove that planar configurations are both holonomic and flat.

Proposition 1. *In the planar case, the kinematic constraints between every finger and the object are holonomic. Consequently, planar HOSs, with an arbitrary number of fingers, are holonomic.*

Proof. Since we focus attention on a single finger in contact with the object, the subscript i corresponding to the finger's index is omitted in the proof. The one-forms corresponding to the geometric and kinematic constraints read

$$\begin{bmatrix}\varpi_1\\ \varpi_2\end{bmatrix} = d\left(p_C^d - p^r - \Omega(\phi^r)p_C^o\right) = 0 \tag{49}$$

$$\varpi_3 = d(c^d(p_C^d)) = 0 \tag{50}$$

$$\varpi_4 = d(c^o(p_C^o)) = 0 \tag{51}$$

$$\varpi_5 = d\left(\det\begin{pmatrix} Dc^d(p_C^d)\\ Dc^o(p_C^o)\cdot\Omega(\phi^r)^T\end{pmatrix}\right) = 0 \tag{52}$$

$$\varpi_6 = (w_{C_i}^d)^T\left(dp^r + [d\phi^r\times]\left(p_C^d - p^r\right)\right) = 0, \tag{53}$$

where (49)-(52) are the exterior derivatives of the constraints (27)-(30). The one-form ϖ_6 comes from the kinematic constraint (32). This Pfaffian system is made of 6 independent one-forms as can be checked by lengthy calculations that are omitted here. Nevertheless, this property is easily interpreted in geometric terms by the fact that the independence of the first five equations is a consequence of the opposite convexities of the boundaries of the object and the finger with uniqueness of the contact point and thus the coincidence of the common tangents. Moreover, the last equation is the only one involving the covectors dp^r, $d\phi^r$ without dependence on dp_C^d, dp_C^o.

Since the number of variables involved in these six constraints is seven (namely p^r, p_C^d, p_C^o, ϕ^r) the Pfaffian system ϖ_1-ϖ_6 is always locally integrable which proves holonomy. □

Proposition 2. *Planar HOSs with at least two fingers, each one having at least two joints, are differentially flat.*

Proof. Let m denote the number of fingers ($m \geq 2$) and ν_i ($\nu_i \geq 2$), $i = 1, \ldots, m$, the number of joints of finger i. We thus have $\sum_{i=1}^m \nu_i$ independent inputs, namely the joint torques τ_i of the fingers. We claim that a flat output is given by the following $\sum_{i=1}^m \nu_i$ variables: p^o, ϕ^o, $q_{i,3}, \ldots, q_{i,\nu_i}$ ($i = 1, \ldots, m$) and

$$y_i = g_i(f_1, \ldots, f_m) \qquad i = 1, \ldots, 2m-3,$$

with g_i arbitrary but fixed combinations of the contact forces. Let us denote by Y the corresponding $\sum_{i=1}^m \nu_i$ dimensional vector:

$$Y = (p^o, \phi^o, q_{1,3}, \ldots, q_{1,\nu_1}, \ldots, q_{m,3}, \ldots, q_{m,\nu_m}, y_1, \ldots, y_{2m-3}). \tag{54}$$

Using Proposition 1, the kinematic equation can be integrated to get rid of $\dot{p}_i^r$ and $\dot{\phi}_i^r$, thus (27)-(30) and (32) become a set of 6 algebraic independent equations depending on 7 variables for each finger. Using these six equations, the coordinates $p_{C_i}^d$, $p_{C_i}^o$ of the contact points and the relative positions p_i^r of the object w.r.t. to the fingers can be calculated as function of the relative orientations ϕ_i^r.

Let us show how to determine ϕ_i^r as function of the variables of the flat output. Using the direct geometry of the fingers (31) and Equation (26), we have

$$d_i(q_i) = \left(p_i^d, \phi_i^d\right) = \left(p^o - \Omega(\phi_i^d)p_i^r, \phi^o - \phi_i^d\right) \qquad i = 1, \ldots, m. \tag{55}$$

We thus have $3m$ independent equations with the $3+m+\sum_{i=1}^{m}\nu_i$ variables: ϕ^o, p^o, $q_{i,1}, \ldots, q_{i,\nu_i}$, $(i = 1, \ldots, m)$, $\phi_1^r, \ldots, \phi_m^r$. This shows that the knowledge of ϕ^o and p^o, the orientation and position of the object, together with $q_{i,3}, \ldots, q_{i,\nu_i}$ allows to calculate the $3m$ variables ϕ_i^r, $q_{i,1}$, $q_{i,2}$ for $i = 1, \ldots, m$, using (55). Hence the variables included in the flat output allow to express all the remaining variables involved in the geometry and the kinematics of the planar HOS.

The dynamic equations of the object (33)-(34) together with y_i $(i = 1, \ldots, 2m-3)$ give the contact forces f_i $(i = 1, \ldots, m)$ as functions of $Y, \dot{Y}, \ddot{Y}$. The contact forces and $Y, \dot{Y}, \ddot{Y}$ allow then to calculate the vectors of joint torques of the fingers τ_i, $(i = 1, \ldots, m)$ using (35).

Since we have shown that all system variables can be expressed as functions of Y and derivatives we have proved that Y is a flat output and the system is flat. □

3.2 The case of one finger (Example 1)

Two cases may be distinguished. First, we examine the case when pivoting motions are controlled (i.e. we can control the rolling motions of the object around the normal vector to its surface at the contact point). Then pivoting motions are prohibited by an additional constraint, which eliminates one of the inputs.

In Example 1, we make an additional assumption stating that the velocity of the finger in the x and y directions of the reference frame equals to the relative velocity at the contact point with opposite direction, namely $\dot{x}^d = -\dot{x}_C^d$ and $\dot{z}^d = -\dot{z}_C^d$. Using this assumption the variables x^d and y^d can be also eliminated since they can be calculated as functions of x_C and y_C. Since x^d and y^d are not involved in the kinematic constraints, Lemma 1 remains valid but with a manifold of dimension 5 after eliminations. The remaining part of this section considers the kinematics on this five-dimensional manifold given by x_C^d, y_C^d and ϕ^o.

3.2.1 Flatness of the 3-input case

To study the kinematics (11) of Example 1, define the matrix $\mathbf{A}$ and the vector $\mathbf{X}$ as

$$\mathbf{A} = \begin{bmatrix} 0 & R\cos\varphi & R\sin\varphi\cos\theta & -1 & 0 \\ 0 & R\sin\varphi & -R\cos\varphi\cos\theta & 0 & -1 \end{bmatrix}, \quad \mathbf{X} = \left[\varphi, \theta, \psi, x_C^d, y_C^d\right]^T. \tag{56}$$

Obviously, Equation (11) is equivalent to $\mathbf{A}\dot{X} = 0$.

We assume that the control inputs u_1, u_2, u_3 of this kinematics are the translational relative velocities of the finger w.r.t. the object along the horizontal plane and the angular velocity of the object around the vertical axis, namely: $u_1 = \dot{x}_C^d$, $u_2 = \dot{y}_C^d$, $u_3 = \omega_z^o$.

Remark 3. *The physical realization of the third input $u_3 = \omega_z^o$ that makes pivoting the object is not straightforward. One could imagine that such motion may be produced by a magnetic actuator creating a rotating field around the sphere made of suitable material.*

Proposition 3. *The kinematics* (11) *given by*

$$\mathbf{A}\dot{\mathbf{X}} = 0 \tag{57}$$

where $\mathbf{A}$ *and* $\mathbf{X}$ *are defined by* (56) *with three inputs:* $u_1 = \dot{x}_C^d$, $u_2 = \dot{y}_C^d$, $u_3 = \omega_z^o$ *is differentially flat with a flat output* Y *given by*

$$Y = \begin{bmatrix} \varphi \\ \theta \\ \zeta \end{bmatrix}$$

with

$$\zeta = \frac{1}{R} \begin{bmatrix} \dot{\varphi} \sin\varphi \cos\theta + \dot{\theta} \cos\varphi \sin\theta \\ -\dot{\varphi} \cos\varphi \cos\theta + \dot{\theta} \sin\varphi \sin\theta \end{bmatrix}^T \mathbf{AX}. \tag{58}$$

Proof. Introduce the variables y_1 and y_2 as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{AX}. \tag{59}$$

Note that y_1, y_2 and ζ are functions of the system variables (57) including the inputs $u_1 = \dot{x}_C^d$, $u_2 = \dot{y}_C^d$, and $u_3 = \omega_z^o$.

Differentiating ζ, using (58) and (59), one obtains

$$\frac{1}{R}(a y_1 + b y_2) = \dot{\zeta} + \theta \dot{\varphi}^2 \cos\theta \tag{60}$$

with

$$\begin{aligned} a &= \ddot{\varphi} \sin\varphi \cos\theta + (\dot{\varphi}^2 + \dot{\theta}^2) \cos\varphi \cos\theta - 2\dot{\varphi}\dot{\theta} \sin\varphi \sin\theta + \ddot{\theta} \cos\varphi \sin\theta \\ b &= -\ddot{\varphi} \cos\varphi \cos\theta + (\dot{\varphi}^2 + \dot{\theta}^2) \sin\varphi \cos\theta + 2\dot{\varphi}\dot{\theta} \cos\varphi \sin\theta + \ddot{\theta} \sin\varphi \sin\theta \end{aligned}$$

Observe now that (60) together with (58) constitute a linear system w.r.t. to the variables y_1 and y_2

$$\frac{1}{R} \begin{bmatrix} \dot{\varphi} \sin\varphi \cos\theta + \dot{\theta} \cos\varphi \sin\theta & -\dot{\varphi} \cos\varphi \cos\theta + \dot{\theta} \sin\varphi \sin\theta \\ a & b \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \zeta \\ \dot{\zeta} + \theta \dot{\varphi}^2 \cos\theta \end{bmatrix},$$

hence, inverting the system, y_1 and y_2 are obtained as functions of ζ, $\dot{\zeta}$, θ, $\dot{\theta}$, $\ddot{\theta}$, φ, $\dot{\varphi}$, and $\ddot{\varphi}$. In order to obtain ψ as function of ζ, φ, θ and their successive time derivatives, we differentiate Equation (59) with (57):

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \frac{d}{dt}(\mathbf{AX}) = \left(\frac{d}{dt}\mathbf{A}\right) \cdot \mathbf{X} \quad = R \begin{bmatrix} -\theta\dot{\varphi} \sin\varphi + \psi \left(\dot{\varphi} \cos\varphi \cos\theta - \dot{\theta} \sin\varphi \sin\theta\right) \\ \theta\dot{\varphi} \cos\varphi + \psi \left(\dot{\varphi} \sin\varphi \cos\theta + \dot{\theta} \cos\varphi \sin\theta\right) \end{bmatrix}$$

thus yielding ψ:

$$\psi = \frac{\dfrac{\dot{y}_1}{R} + \theta\dot{\varphi} \sin\varphi}{\dot{\varphi} \cos\varphi \cos\theta - \dot{\theta} \sin\varphi \sin\theta} = \frac{\dfrac{\dot{y}_2}{R} - \theta\dot{\varphi} \cos\varphi}{\dot{\varphi} \sin\varphi \cos\theta + \dot{\theta} \cos\varphi \sin\theta}. \tag{61}$$

Since we have already shown that y_1 and y_2 are functions of Y and its derivatives, the same holds true for ψ.

Finally, by (59), we have

$$\begin{aligned} x_C^d &= y_1 - R(\theta \cos\varphi + \psi \sin\varphi \cos\theta) \\ y_C^d &= y_2 - R(\theta \sin\varphi - \psi \cos\varphi \cos\theta) \end{aligned}$$

proving that x_C^d and y_C^d are also functions of Y and its time derivatives. Since $u_1 = \dot{x}_C^d$, $u_2 = \dot{y}_C^d$, $u_3 = \omega_z^o = \dot{\varphi} - \dot{\psi} \sin\theta$, the inputs are also functions of Y and derivatives which achieves the proof. □

Remark 4. *The third component ζ of Y appears quite involved and its physical interpretation is far from obvious. It has been obtained by integration techniques which are not reported here and its expression explicitly contains the input variables, which is quite unusual: in* (58)*, $\dot\varphi$ and $\dot\theta$ are present and, according to the fact that $u_1 = \dot x_C^d$, $u_2 = \dot y_C^d$ and $u_3 = \omega_z^o = \dot\varphi - \dot\psi\sin\theta$,* (57) *reads*

$$\begin{bmatrix} 0 & \cos\varphi & \sin\varphi\cos\theta \\ 0 & \sin\varphi & -\cos\varphi\cos\theta \\ 1 & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} \dot\varphi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} \frac{u_1}{R} \\ \frac{u_2}{R} \\ u_3 \end{bmatrix} \tag{62}$$

or, after inversion,

$$\begin{aligned} \dot\varphi &= \frac{u_1}{R}\sin\varphi\tan\theta - \frac{u_2}{R}\cos\varphi\tan\theta + u_3 \\ \dot\theta &= \frac{u_1}{R}\cos\varphi + \frac{u_2}{R}\sin\varphi \\ \dot\psi &= \frac{u_1}{R}\frac{\sin\varphi}{\cos\theta} - \frac{u_2}{R}\frac{\cos\varphi}{\cos\theta}. \end{aligned} \tag{63}$$

Thus, reporting these expressions in (58)*, yields the above mentioned expression of ζ involving the inputs.*

A simpler flat output is not known to the authors at present. Notice that since (57) *is equivalent to the driftless system with five states and three inputs given by* (63) *together with*

$$\dot x_C^d = u_1 \qquad\qquad \dot y_C^d = u_2,$$

its flatness property is a consequence of [17]. Note that this reference doesn't provide direct means to determine the flat output.

3.2.2 Liouvillian property of the 2-input case

Let us now examine the case when pivoting motions are eliminated by the constraint (12) of Example 1. It corresponds to the elimination of the input $u_3 = \omega_z^o$ by setting it identically to zero. Defining the matrix A' as

$$\mathbf{A}' = \begin{bmatrix} & & \mathbf{A} & & \\ 1 & 0 & -\sin\theta & 0 & 0 \end{bmatrix},$$

the kinematics of this system reads

$$\mathbf{A}'\dot{\mathbf{X}} = 0.$$

Again, the corresponding driftless system, similar to (63), is obtained:

$$\begin{cases} \dot\varphi = \dfrac{u_1}{R}\sin\varphi\tan\theta - \dfrac{u_2}{R}\cos\varphi\tan\theta \\ \dot\theta = \dfrac{u_1}{R}\cos\varphi + \dfrac{u_2}{R}\sin\varphi \\ \dot\psi = \dfrac{u_1}{R}\dfrac{\sin\varphi}{\cos\theta} - \dfrac{u_2}{R}\dfrac{\cos\varphi}{\cos\theta} \\ \dot x_C^d = u_1 \\ \dot y_C^d = u_2 \end{cases}. \tag{64}$$

Proposition 4. *The system* (64) *is Liouvillian with $Y = (\varphi, \theta)$ as partially flat output and x_C^d, y_C^d, ψ as integral variables.*

Proof. By the first two equations of (64), the variables u_1, u_2 are functions of Y and $\dot{Y}$. In addition, the variables x_C^d, y_C^d, and ψ can be obtained from the last three equations of (64) by simple integrals provided that their initial conditions are known, which proves that Y is a partially flat output and that x_C^d, y_C^d, ψ are integral variables. The system is thus proven to be Liouvillian. □

3.3 Liouvillian property of the case with three fingers (Example 2)

The kinematics (19) of the HOS presented in Example 2 (ball with three fingers) can be transformed to the following driftless system

$$\begin{cases} \dot{\varphi} = -u_1 + u_2 \tan\theta \sin\varphi + u_3 \tan\theta \cos\varphi \\ \dot{\theta} = u_2 \cos\varphi - u_3 \sin\varphi \\ \dot{\psi} = u_2 \dfrac{\sin\varphi}{\cos\theta} + u_3 \dfrac{\cos\varphi}{\cos\theta} \\ \dot{y}_{C_1}^d = R u_1 \\ \dot{z}_{C_1}^d = R u_2 \\ \dot{z}_{C_2}^d = R u_3, \end{cases} \tag{65}$$

with six states and three inputs.

Proposition 5. *The system* (65) *is Liouvillian with* $Y = (\varphi, \theta, \psi)$ *as partially flat output and* $y_{C_1}^d$, $z_{C_1}^d$, $z_{C_2}^d$ *as integral variables.*

Proof. By the first three equations of (65), the variables $u_1 = \frac{1}{R}\dot{y}_{C_1}^d$, $u_2 = \frac{1}{R}\dot{z}_{C_1}^d$, $u_3 = \frac{1}{R}\dot{z}_{C_2}^d$ are functions of Y and $\dot{Y}$. In addition, the variables $y_{C_1}^d$, $z_{C_1}^d$, $z_{C_2}^d$ can be obtained from the last three equations of (65) by simple integrals provided that their initial conditions are known, which proves that Y is a partially flat output and that $y_{C_1}^d$, $z_{C_1}^d$, $z_{C_2}^d$ are integral variables. Therefore the system is indeed Liouvillian. □

3.4 Liouvillian property of the HOS with special morphology

Proposition 6. *Consider a HOS such that Conditions 1-3 hold true. This system is Liouvillian and admits a partially flat output* $Y = (p^o, \phi^o, x_{C_i}^o, y_{C_i}^o, y_1, y_2, y_3)$*, position and orientation of the object and coordinates of the contact points on the object boundary, with three combinations of the contact forces:* $y_i = g_i(f_1, f_2, f_3)$*,* $i = 1, 2, 3$*. Moreover,* $q_{i,4}$*,* $q_{i,5}$*,* $i = 1, 2, 3$*, the last two joint coordinates of the fingers, are integral variables.*

Proof. Without loss of generality, we may assume that each fingertip has radius r. Fix the frame K_i^d to the fingertip of the ith finger with its origin at the center of the sphere. Hence

$$Dc_i^d(p_{C_i}^d) = 2\left(p_{C_i}^d\right)^T. \tag{66}$$

The position p_i^d of the finger i is function of the joint coordinates $q_{i,1}, \ldots, q_{i,3}$ and doesn't depend on the last joints $q_{i,4}$ and $q_{i,5}$. Moreover, due to Condition 3, the angular velocity of the finger ω_i^d doesn't depend on $q_{i,4}$ and $q_{i,5}$ (it depends on $q_{i,1}, \ldots, q_{i,3}$ and $\dot{q}_{i,1}, \ldots, \dot{q}_{i,5}$):

$$\begin{aligned} p_i^d &= p_i^d(q_{i,1}, q_{i,2}, q_{i,3}) \\ \Omega\left(\phi_i^d\right) &= \Omega(q_{i,1}, q_{i,2}, q_{i,3}, q_{i,4}, q_{i,5}) \\ \omega_i^d &= \omega_i^d(q_{i,1}, q_{i,2}, q_{i,3}, \dot{q}_{i,1}, \dot{q}_{i,2}, \dot{q}_{i,3}, \dot{q}_{i,4}, \dot{q}_{i,5}). \end{aligned} \tag{67}$$

Thus the geometric constraints of the unique contact point between the object and the ith finger and the kinematic condition of rolling without slipping can be formulated without the variables $q_{i,4}$, $q_{i,5}$, and $p^d_{C_i}$. Using (66), the geometric constraint corresponding to Equation (2), expressed in K^b reads

$$p^d_i - p^o - \Omega(\phi^o)\left(p^o_{C_i} + r\frac{Dc^o_i(p^o_{C_i})^T}{\|Dc^o_i(p^o_{C_i})^T\|}\right) = 0. \tag{68}$$

Due to the symmetry of the spherical fingertips (see Condition 1) the geometric constraints corresponding to (3) and (5) are used to eliminate the coordinates $p^d_{C_i}$ of the contact point on the finger.

In the sequel we need the expression of the relative velocity at the contact points in the *reference frame* K^b:

$$v^r_{C,i} = v^o_{C,i} - v^d_{C,i} = \dot{p^o} + [\omega^o\times]\Omega(\phi^o)p^o_{C,i} - \dot{p}^d_i - [\omega^d_i\times]\left(\Omega(\phi^o)p^o_{C,i} + p^o - p^d_i\right),$$

where we used (68) to eliminate $p^d_{C_i}$ from the expression of $v^d_{C,i}$. Denote by w^o_{1,C_i} and w^o_{2,C_i} the vectors spanning $\ker Dc^o(p^o_{C_i})$, i.e. the tangent plane to the *object* at the contact point, expressed also in the basis of K^b. Thus the kinematic constraints are given by

$$(w^o_{1,C_i})^T v^r_{C,i} = 0 \qquad (w^o_{2,C_i})^T v^r_{C,i} = 0. \tag{69}$$

Let us prove first that $z^o_{C_i}$, p^d_i, $q_{i,1}$, $q_{i,2}$, $q_{i,3}$, $\dot{q}_{i,4}$, $\dot{q}_{i,5}$ can be expressed as functions of Y and $\dot{Y}$.

First, by (4) and the implicit function theorem, one can express $z^o_{C_i}$ as $z^o_{C_i} = Z^o_{C_i}(x^o_{C_i}, y^o_{C_i})$, $i = 1, 2, 3$. Then, replacing $z^o_{C_i}$ in (68), p^d_i can be obtained as function of Y. The first (three-dimensional) equation of (67) can be locally solved for $q_{i,1}$, $q_{i,2}$, $q_{i,3}$, giving these three variables as functions of Y.

Since the kernel of Dc^o_i is a function of Y, the same holds true for w^o_{1,C_i}, w^o_{2,C_i}. Moreover, by the last equation of (67), ω^d_i depends on $Y, \dot{Y}$ and $\dot{q}_{i,4}$, $\dot{q}_{i,5}$. Denote by $W = (w^o_{1,C_i}, w^o_{2,C_i})^T$ and by $[t\times]$ the cross product matrix with $t = \Omega(\phi^o)p^o_{C_i} + p^o - p^d_i \neq 0$, both with rank 2. Then (69) reads

$$W[t\times]\omega^d_i = -W\left(\dot{p^o} + [\omega^o\times]\Omega(\phi^o)p^o_{C_i} - \dot{p}^d_i\right).$$

Thus (69) can be locally solved for $\dot{q}_{i,4}$ and $\dot{q}_{i,5}$ since, according to the above decomposition, its Jacobian has rank 2, provided that the Jacobian of the ith finger is of full rank. It results that $\dot{q}_{i,4}$ and $\dot{q}_{i,5}$ can be expressed as functions of Y and $\dot{Y}$. Therefore, introducing $\xi = (q_{i,4}, q_{i,5})$, we have $\dot{\xi} = \gamma(Y, \dot{Y})$.

Let us now prove that the vectors of joint torques τ_i, $i = 1, 2, 3$ can be obtained as functions of $Y, \dot{Y}, \ddot{Y}$. By (21)-(22) and using $y_i = g_i(f_1, f_2, f_3)$, $i = 1, 2, 3$, the vector of contact forces f_1, f_2, f_3 can be computed as functions of $Y, \dot{Y}, \ddot{Y}$. Next, by (24) the same holds true for $\tau_{i,ext}$ since J_i and G_i don't depend on $q_{i,4}$ and $q_{i,5}$ by Condition 3. Finally, since $q_{i,4}$ and $q_{i,5}$ are cyclic coordinates (Condition 2), they are not involved in the dynamic equations (23) which allow to calculate τ_i, $i = 1, 2, 3$ as claimed.

It remains to show that $p^d_{C_i}$, ϕ^d_i, $i = 1, 2, 3$, are functions of Y, its derivatives, and ξ. This is obvious from the second equation of (67) and from (2), hence ξ contains the integral variables which achieves the proof. □

Remark 5. *It can be verified that the above proposition remains valid if the number of the joints is increased or decreased (the minimal number being three joints per finger) and if the number of fingers is increased. However, decreasing the number of joints will prevent from arbitrarily modifying simultaneously the position and orientation of the object and the position of the contact points on the object boundary.*

4 Motion planning

Let us show that the properties established in the previous section can be used to find algorithms to solve the MPP.

4.1 Planar case

Recall that, given, at time t_I, an initial configuration of the HOS together with initial velocity state and, at final time t_F, a final configuration of the object together with its final velocity state, the MPP consists in finding a trajectory for the variables of the HOS including the inputs that achieves the desired final configuration and velocity. According to Proposition 2, since the orientation and the position of the object are included in the flat output (54), the solution of the MPP is reduced to an interpolation problem for the variables of the flat output. It is solved here using polynomial functions of time. For simplicity's sake, we restrict to the case where the initial and final configurations are both steady-states and the derivatives of the trajectories at $t = t_I$ and $t = t_F$ vanish up to a finite order, say κ, (implying zero velocities and accelerations if $\kappa > 1$). Such a polynomial function for a variable χ of the flat output can be given as

$$\chi(t) = \chi(t_I) + (\chi(t_F) - \chi(t_I)) \left(\frac{t - t_I}{T}\right)^{\kappa+1} \sum_{j=0}^{\kappa} a_j \left(\frac{t - t_I}{T}\right)^j, \tag{70}$$

where $T = t_F - t_I$ is the travelling duration. The coefficients a_j are independent of t_I, t_F, $\chi(t_I)$, and $\chi(t_F)$, thus they are the same for all trajectories.

Example 5 (continuation of Example 3). First, we show that the kinematic constraints are indeed integrable by Proposition 1. Introduce the following polar coordinates for the contact points (see figure 5):

$$\begin{aligned} x^d_{C_i} &= r_i \cos\zeta_i \qquad & x^o_{C_i} &= R\cos\xi_i \\ z^d_{C_i} &= r_i \sin\zeta_i \qquad & z^o_{C_i} &= R\sin\xi_i. \end{aligned}$$

Then Constraint (36) becomes

$$\begin{aligned} q_{i2} + r_i\cos\zeta_i &= x^o + R\cos(\xi_i - \phi^o) \\ q_{i1} + r_i\sin\zeta_i &= z^o + R\sin(\xi_i - \phi^o) \end{aligned} \tag{71}$$

and developing Constraint (39) one obtains:

$$\sin(\xi_i - \phi^o - \zeta_i) = 0. \tag{72}$$

Using (71)-(72) and eliminating ζ_i, x^o, q_{i2}, $\dot{x}^o$, $\dot{q}_{i2}$, z^o, q_{i1}, $\dot{z}^o$, and $\dot{q}_{i1}$ from (41) we get, after easy calculations which are omitted

$$\dot{\xi}_i - \frac{r_i}{r_i + R}\dot{\phi}^o = 0,$$

which can be integrated to obtain

$$\xi_i = \frac{r_i}{r_i + R}\phi^o + \xi_{i0} \quad i = 1, 2$$

where ξ_{i0} is the initial condition for ξ_i. Proposition 2 asserts that this HOS is flat with

$$Y = (x^o, z^o, \phi^o, y_1)$$

as a flat output, y_1 being defined as $y_1 = g_1(f_1, f_2)$ where g_1 is an arbitrary but fixed combination of the contact force components. Here, we choose y_1 as the sum of the squared norm of the net contact forces at the contact points: $y_1 = \|f_1\|^2 + \|f_2\|^2$.

Hence, if we know the initial mechanical state of the HOS ($x^o(t_I)$, $z^o(t_I)$, $\phi^o(t_I)$, $q_1(t_I)$, $q_2(t_I)$) and we are given at time t_F a desired final configuration of the object ($x^o(t_F)$, $z^o(t_F)$, $\phi^o(t_F)$), since the flat output Y includes the position and the orientation of the object, it is enough to find smooth functions of time connecting ($x^o(t_I)$, $z^o(t_I)$, $\phi^o(t_I)$) to ($x^o(t_F)$, $z^o(t_F)$, $\phi^o(t_F)$), and to choose the trajectory of the remaining component of the flat output y_1 in order to deduce the contact forces and hence the corresponding inputs τ_{ij}, $(i, j = 1, 2)$ using (42) without integrating the system:

$$\begin{bmatrix}\tau_{i1}\\ \tau_{i2}\end{bmatrix} = \begin{bmatrix} m_{i1}+m_{i2} & 0 \\ 0 & m_{i2}\end{bmatrix}\begin{bmatrix}\ddot{q}_{i1}\\ \ddot{q}_{i2}\end{bmatrix} + \begin{bmatrix}-(m_{i1}+m_{i2})g \\ 0\end{bmatrix} + \begin{bmatrix} f_{i,y} \\ (-1)^{i-1} f_{i,x}\end{bmatrix}.$$

We may also wish that the time derivatives of x^o, z^o and ϕ^o at $t = t_I$ and $t = t_F$ vanish up to a finite order, to start and stop at rest points (i.e. with zero velocities and accelerations), as shown in Figure 8. We use the polynomials given by (70).

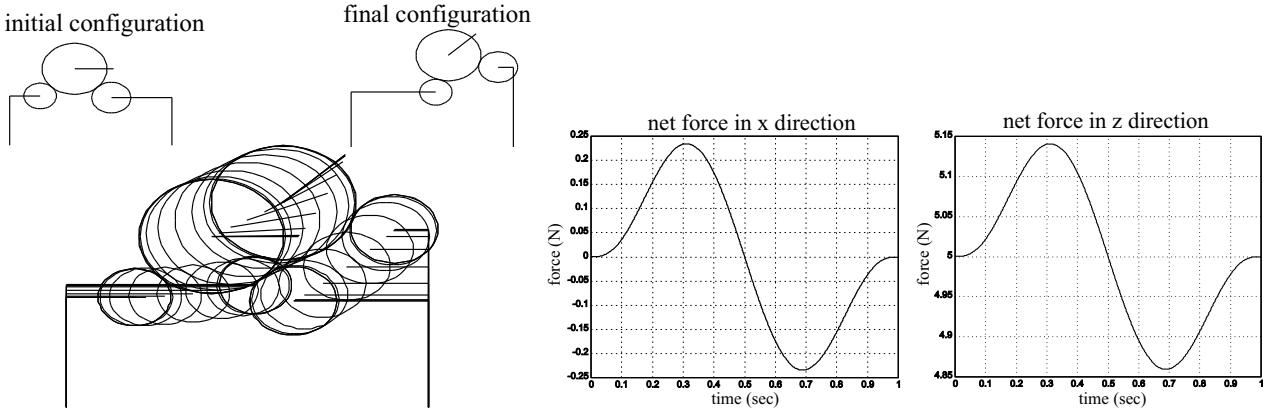

Figure 8: The solution of the MPP and the net contact force (x and z components); $t_I = 0$, $t_F = 1$.

Instead of defining an a priori trajectory of y_1 we may wish that the net contact force has minimal norm, to which case the trajectory of y_1 is deduced from (70) as follows: according to (33)-(34), the components of the contact forces satisfy

$$B\begin{bmatrix}f_1\\ f_2\end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -z^o_{C_1} & x^o_{C_1} & -z^o_{C_2} & x^o_{C_2}\end{bmatrix}\begin{bmatrix}f_{11}\\ f_{12}\\ f_{21}\\ f_{22}\end{bmatrix} = \begin{bmatrix} m\ddot{x}^o \\ m(\ddot{z}^o + g) \\ \Theta\ddot{\phi^o}^o\end{bmatrix} \tag{73}$$

and therefore the minimum norm of the net contact force is given by

$$y_1 = \left\| B^T(BB^T)^{-1}\begin{bmatrix} m\ddot{x}^o \\ m(\ddot{z}^o + g) \\ \Theta\ddot{\phi^o}^o\end{bmatrix}\right\|. \tag{74}$$

The resulting configurations and net contact force for $\kappa = 4$ are shown in Figure 8. See Extension 1 for the animation showing the motion of the HOS. ∎

4.2 Case with one finger (Example 1)

The three input case being flat, the MPP can be solved similarly to the case of planar structures. We consider the Liouvillian case of two inputs (see Section 3.2.2).

By Proposition 4, the kinematics given by Equation (64) is Liouvillian. The partially flat output $Y = (\varphi, \theta)$ doesn't contain all orientation angles thus one cannot determine freely the trajectory of the orientation of the ball. Nevertheless, the following method may be used to overcome this difficulty.

Choose the trajectory of ψ as a polynomial function of θ:

$$\psi(\theta) = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3, \tag{75}$$

and we wish to obtain the trajectory of φ as a function of θ. Noting that $\dot{\varphi} = \frac{d\varphi}{d\theta}\dot{\theta}$ and $\dot{\psi} = \frac{d\psi}{d\theta}\dot{\theta}$, the non-pivoting constraint (12) can be rewritten as

$$\frac{d\varphi}{d\theta} = \sin\theta \frac{d\psi}{d\theta}. \tag{76}$$

This expression can be integrated w.r.t. θ (such that $\varphi(\theta_I) = \varphi_I$) in order to obtain the function $\varphi(\theta)$. Reporting (75) in (76), the integration on the interval $[\theta_I, \theta]$ results in

$$\begin{aligned}\varphi(\theta) = a_1(\cos\theta_I - \cos\theta) + 2a_2(\sin\theta - \sin\theta_I + \theta_I\cos\theta_I - \theta\cos\theta)\\ + 3a_3(\theta_I^2\cos\theta_I - 2\cos\theta_I - 2\theta_I\sin\theta_I - \theta^2\cos\theta + 2\cos\theta + 2\theta\sin\theta) + \varphi_I\end{aligned}$$

which is linear w.r.t. the coefficients of the polynomial (75). Hence, these coefficients have to satisfy

$$\begin{aligned}\varphi_F &= \varphi(\theta_F)\\ \psi_I &= a_0 + a_1\theta_I + a_2\theta_I^2 + a_3\theta_I^3\\ \psi_F &= a_0 + a_1\theta_F + a_2\theta_F^2 + a_3\theta_F^3,\end{aligned}$$

where φ_I, ψ_I, θ_I (resp. φ_F, ψ_F, θ_F) give the initial (resp. final) orientation of the ball. Since we have one more degree of freedom to choose a constraint, we would like the second derivative of $\psi(\theta)$ to vanish at the final point:

$$0 = a_2 + 3a_3\theta_F.$$

Around t_F, this condition guarantees that the curvature of the curve $t \to \psi(\theta)$ will not increase too fast.

The above system of equations is linear in the coefficients of the polynomial (75). Once the trajectory of φ and ψ are given in function of θ, it remains to find a trajectory of θ w.r.t. time, connecting θ_I to θ_F. Such a time function can be obtained using the expression (70). The input trajectories u_1 and u_2 are obtained from the first two rows of (62):

$$u_1 = R(\dot{\varphi}\cos\varphi + \dot{\theta}\sin\varphi\cos\theta) \qquad u_2 = R(\dot{\varphi}\sin\varphi - \dot{\theta}\cos\varphi\cos\theta)$$

Remark 6. *The trajectory of the contact point on the plane is given by the integral variables x_C and y_C (see Proposition 4) and can be obtained by integrating the inputs w.r.t the time. This integration can be carried out numerically. Note also that no desired final position of the contact point is prescribed, since it cannot be chosen freely. It is obtained as the result of numerical integration. This significantly contrasts with the flat case of Section 4.1.*

A sample trajectory is shown in Figure 9 connecting $\varphi_I = \frac{\pi}{3}$, $\theta_I = \frac{\pi}{4}$, $\psi_I = 0$ to $\varphi_F = \frac{\pi}{4}$, $\theta_F = 0$, $\psi_F = \frac{\pi}{3}$ in 10 seconds. The animation showing the motion of the ball on the plane is available in Extension 2.

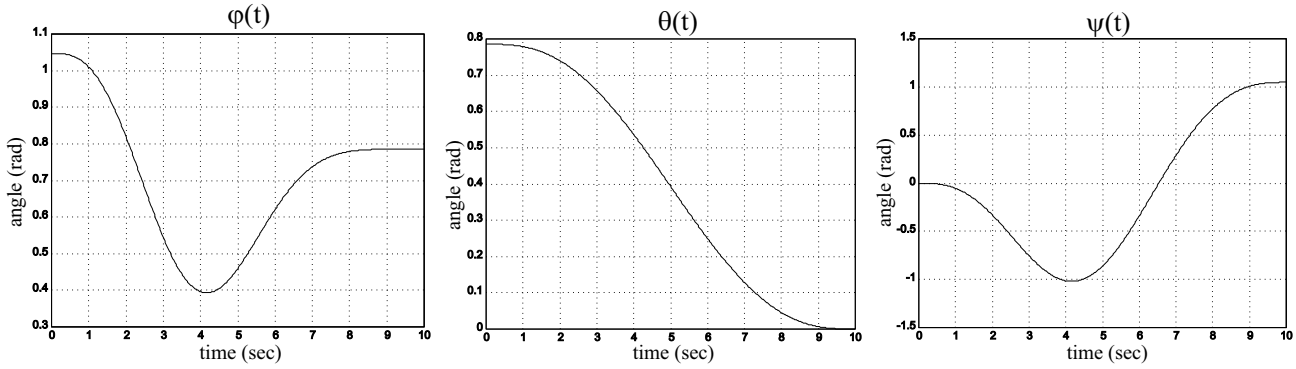

Figure 9: The solution of the MPP for the orientation angles; $t_I = 0$, $t_F = 10$.

4.3 Case with three fingers (Example 2)

For the HOS presented in Example 2, the MPP can be solved for the position and the orientation of the sphere. The variables $x^d_{C_1}$, $y^d_{C_2}$ and $z^d_{C_3}$ are obtained from the geometric constraints (15). The trajectory of the orientation allows to calculate u_1, u_2 and u_3 using the first three equations of (65). Finally we need to integrate the last three equations of (65) and use (18) and (15) in order to obtain the positions of the tangent planes. Here, once the final position and orientation of the object and the initial condition of the fingers are chosen, the final positions of the fingers are deduced by integrating the velocities.

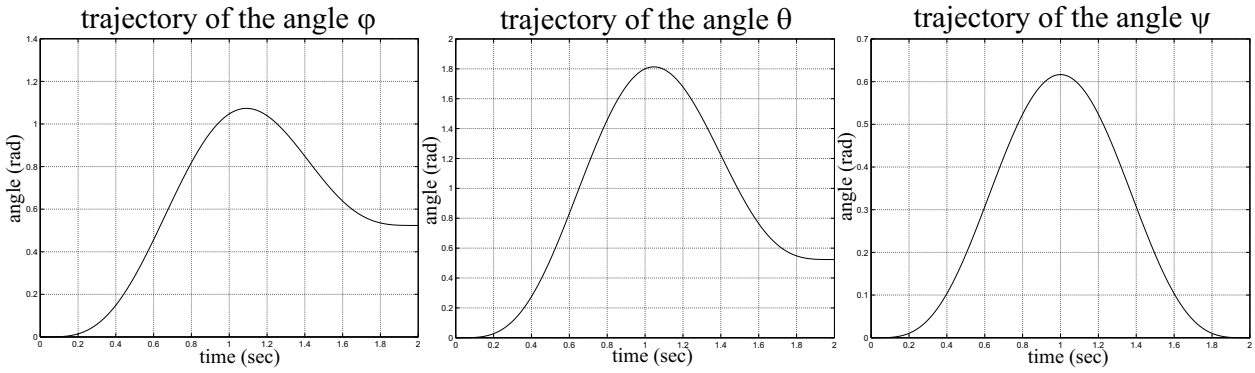

Figure 10: Trajectory of the orientation of the sphere (RPY angles: φ, θ, ψ)

The motion planning algorithm is illustrated in the case of a sphere of radius $R = 1$cm. The trajectory of the orientation (given by the RPY angles) of the sphere between its initial and desired final values is obtained using again polynomial interpolation as in the Example 3 (see Figure 10). The corresponding trajectory of the contact points on the tangent planes are shown in Figure 11. See Extension 3 for the animation of the reference trajectory.

Remark 7. *Note that, due to integration, the final values of the $y^d_{C_1}$, $z^d_{C_1}$, $z^d_{C_2}$ cannot be chosen arbitrarily (see also Remark 6). This is not a major restriction in our case since we are not particularly interested in the positions of the contact points in their respective planes.*

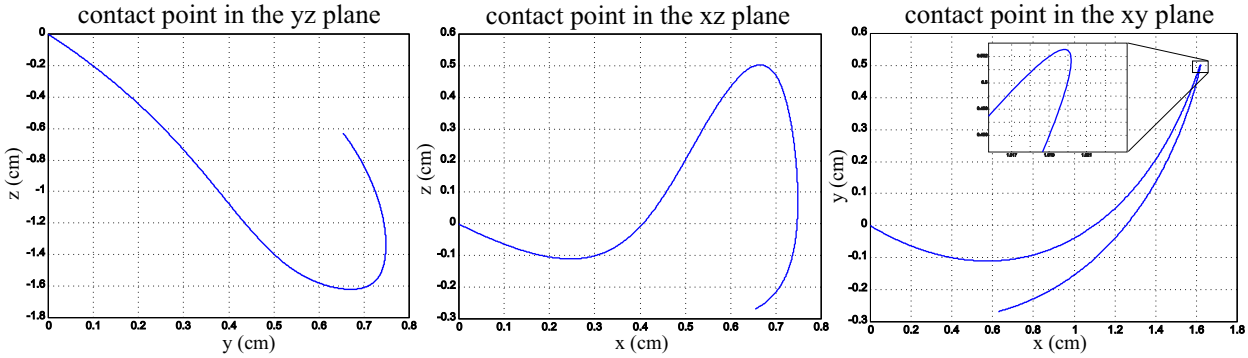

Figure 11: Trajectory of the contact points on the tangent planes

4.4 Hands with special morphology

Since HOSs with special morphology are proven to be Liouvillian by Proposition 6, the solution of the MPP is similar to the case of Example 2.

Example 6 (continuation of Example 4). Introduce the following polar coordinates for the spherical last segments of the hand presented in Section 2.5:

$$\begin{aligned} x^o_{C_i} &= R\cos\sigma_{i,1}\cos\sigma_{i,2} \\ y^o_{C_i} &= R\cos\sigma_{i,1}\sin\sigma_{i,2} \\ z^o_{C_i} &= R\sin\sigma_{i,1}. \end{aligned} \tag{77}$$

The HOS with special morphology (see Conditions 1-3), presented in Example 4 is Liouvillian according to Proposition 6 and, using the polar coordinates of the contact points defined by (77), the partially flat output is given by $Y = (x^o, y^o, z^o, \varphi, \theta, \psi, y_i, \sigma_{i,1}, \sigma_{i,2})^T$ with $i = 1, 2, 3$ and $y_i = g_i(f_1, f_2, f_3)$ where g_i are arbitrary but fixed combinations of the contact forces. (Note that we replaced by $\sigma_{i,1}$ and $\sigma_{i,2}$ the variables $x^o_{C_i}$ and $y^o_{C_i}$ in the expression of the flat output given in Proposition 6. However, this makes no difference since both pair of variables determine a surface point on the object). Here, we choose $y_i = \|f_i\|^2$. The integral variables are $q_{i,4}$ and $q_{i,5}$, $i = 1, 2, 3$.

Suppose, as in Example 3, that the initial mechanical state of the system is given at $t = t_I$ by $(x^o(t_I), y^o(t_I), z^o(t_I), \varphi(t_I), \theta(t_I), \psi(t_I), \sigma_{i,1}(t_I), \sigma_{i,2}(t_I), q_i(t_I))$, $i = 1, 2, 3$, and that we are given at time $t = t_F$ a desired final position and orientation of the object $(x^o(t_F), y^o(t_F), z^o(t_F), \varphi(t_F), \theta(t_F), \psi(t_F))$ and final position of the contact points, $(\sigma_{i,1}(t_F), \sigma_{i,2}(t_F))$, $i = 1, 2, 3$, on the object boundary.

Then, due to Proposition 6, the motion planning is reduced to an interpolation problem for the components of Y, using polynomials given by (70) with the same number of initial and final conditions on each component of Y. Once again, as in the Example 3, the trajectories of the flat outputs y_1, y_2, y_3 are not defined a priori, but determined such that the sum of the squared norms of the contact forces $\sum_{i=1}^{3} y_i$ be minimal: according to (21)-(22) the contact

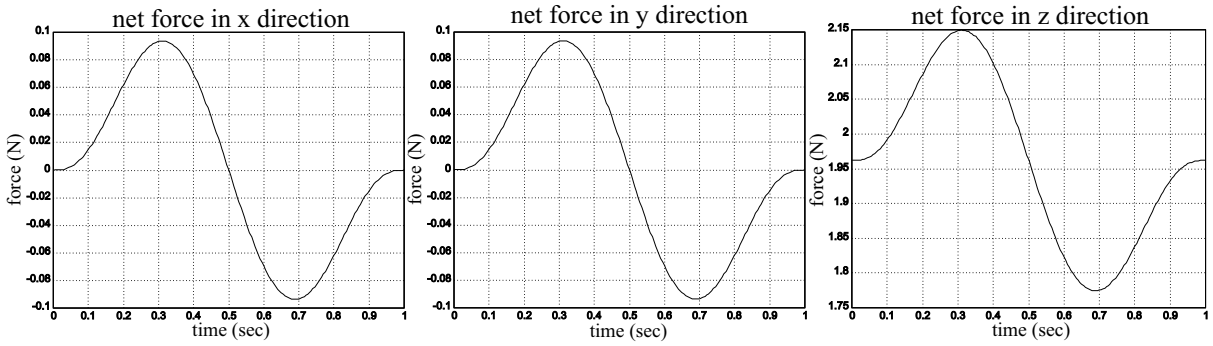

Figure 12: Components of the net force applied on the object by the fingers.

force components satisfy

$$B\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -z^o_{C_1} & y^o_{C_1} & 0 & -z^o_{C_2} & y^o_{C_2} & 0 & -z^o_{C_3} & y^o_{C_3} \\ z^o_{C_1} & 0 & -x^o_{C_1} & z^o_{C_2} & 0 & -x^o_{C_2} & z^o_{C_3} & 0 & -x^o_{C_3} \\ -y^o_{C_1} & x^o_{C_1} & 0 & -y^o_{C_2} & x^o_{C_2} & 0 & -y^o_{C_3} & x^o_{C_3} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} m\ddot{x}^o \\ m\ddot{y}^o \\ m(\ddot{z}^o + g) \\ \Theta\dot{\omega}^o + [\omega^o\times]\Theta\omega^o \end{bmatrix}$$

thus $\sum_{i=1}^3 y_i$ is minimal with the contact forces obtained by

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = B^T(BB^T)^{-1} \begin{bmatrix} m\ddot{x}^o \\ m\ddot{y}^o \\ m(\ddot{z}^o + g) \\ \Theta\dot{\omega}^o + [\omega^o\times]\Theta\omega^o \end{bmatrix}.$$

A sample trajectory is presented for the following numerical parameters: $m = 0.2$kg, $R = 0.1$m, $r = 0.02$m, $d_{x2} = 0.3$m, $d_{x3} = 0$m, $d_{y2} = 0$m, $d_{y3} = 0.3$m, $d_{z2} = 0$m, $d_{z3} = 0$m and $g = 9.81\frac{\text{m}}{\text{S}^2}$. The trajectory connects the initial point $x^o(t_I) = 0.1$m, $y^o(t_I) = 0.1$m, $z^o(t_I) = 0.1$m, $\varphi(t_I) = 0$, $\theta(t_I) = 0$, $\psi(t_I) = 0$, $\sigma_{11}(t_I) = \frac{7\pi}{6}$, $\sigma_{12}(t_I) = -\frac{\pi}{4}$, $\sigma_{21}(t_I) = \frac{11\pi}{6}$, $\sigma_{22}(t_I) = -\frac{\pi}{4}$, $\sigma_{31}(t_I) = \frac{\pi}{2}$, $\sigma_{32}(t_I) = -\frac{\pi}{4}$ to a final point $x^o(t_F) = 0.15$m, $y^o(t_F) = 0.15$m, $z^o(t_F) = 0.2$m, $\varphi(t_F) = \frac{\pi}{6}$, $\theta(t_F) = \frac{\pi}{8}$, $\psi(t_F) = -\frac{\pi}{8}$, $\sigma_{11}(t_F) = \frac{7\pi}{6}$, $\sigma_{12}(t_F) = -\frac{\pi}{2}$, $\sigma_{21}(t_F) = \frac{11\pi}{6}$, $\sigma_{22}(t_F) = -\frac{\pi}{4}$, $\sigma_{31}(t_F) = \frac{\pi}{2}$, $\sigma_{32}(t_F) = -\frac{\pi}{4}$ such that $T = t_F - t_I = 1$s. Extension 4 shows the evolution of the HOS along the reference trajectory. The components of the net contact force are given in Figure 12. The cosine of the angles between normal vectors to the object surface at the contact points and the contact forces applied by the fingers is given in Figure 13 showing that the inequality constraint (25) is satisfied along the trajectory. Note that the minimality of $\sum_{i=1}^3 y_i$ does not guarantee that the inequality constraints of Subsection 2.3 are satisfied. ■

5 Conclusions

We have investigated the motion planning problem related to robotic manipulation with rigid bodies rolling on each other without slipping.

Our approach was based on the notion of flatness and the notion of Liouvillian systems.

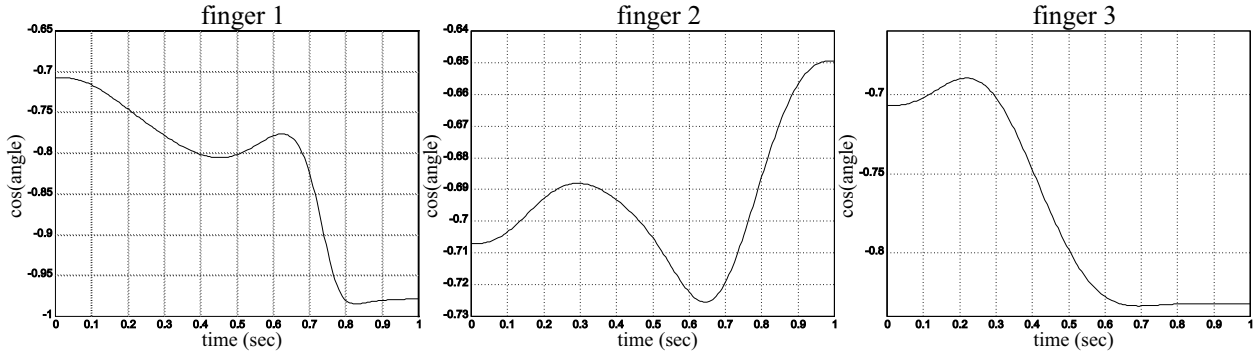

Figure 13: Cosine of angles between the contact forces and the surface normals.

All planar HOSs are proven to be flat and this also holds true for the special case of Example 1 if pivoting motions are not eliminated. Eliminating pivoting motions implies that the system becomes Liouvillian which is also the case for similar structures with multiple fingers. For the Liouvillian case, special HOSs are presented such that the partially flat output and its derivatives allows to calculate the joint torques of the fingers along the trajectory connecting the initial and desired final configuration of the object.

Let us insist on the fact that we never integrate the system differential equations or part of the associated vector fields. The reference trajectory solution of the MPP is computed once for all and integration is only required to obtain the integral variables in the Liouvillian case. Note that, in this case, the computation purely consists of integrating the functions of time obtained from the variables of the partially flat output and does not require iterations. In addition, no quasi-static approximation is used.

A Index to multi-media Extensions

The multi-media extensions to this article can be found online by following the hyperlinks from www.ijrr.org

Extension	Media type	Description
1	Video	animation of a planar hand structure (Examples 3 and 5)
2	Video	animation of the sphere with one finger (Example 1)
3	Video	animation of the sphere with three fingers (Example 2)
4	Video	animation of the hand with special morphology (Examples 4 and 6)

B Appendix

For the reader's convenience, some notions about rigid body motions and nonholonomic mechanical systems are recalled. (See [14, 20, 25, 29] for details).

Some general notational conventions are elucidated first. Notations from differential geometry are used [1]. TM denotes the tangent bundle of the manifold M, and T^*M its cotangent

bundle. $d\varpi$ denotes the exterior derivative of the one-form $\varpi \in T^*M$, and $< , >$ stands for the duality product of the pairing (T^*M, TM). The wedge product of two exterior forms ϖ_1 and ϖ_2 is denoted by $\varpi_1 \wedge \varpi_2$.

Rigid body motions

The motions of a rigid body is described by the motions of a frame fixed to it. A frame K^a in $\mathbb{R}^3$ is given by a vector p^a (its origin) and by an ordered set of three orthogonal unit vectors $\{e_1^a, e_2^a, e_3^a\}$ such that $e_1^a \times e_2^a = e_3^a$.

Consider a point P in $\mathbb{R}^3$ and a frame K^a. One can write

$$\overrightarrow{AP} = x_P^a e_1^a + y_P^a e_2^a + z_P^a e_3^a,$$

and we say that the vector $p_P^a = (x_P^a, y_P^a, z_P^a)^T$ is the position vector of the point P in the basis of K^a whose elements are the coordinates of the point P in K^a.

One distinguishes a special frame, denoted by K^b and referred to as the inertial reference frame. For any frame K^a different from K^b, we denote by p^a the position vector of its origin A in the basis of K^b.

A frame K^a is said to be fixed to a rigid body at a point A (note that A is not necessarily a point of the body itself) if all points of the rigid body have fixed coordinates w.r.t. the base e_1^a, e_2^a, e_3^a and the point A coincides with the origin of K^a.

Since the position of all points of the rigid body is fixed in the basis of K^a, their positions in the basis of K^b are given if the situation of K^a w.r.t. K^b is known. By the definition of the frame K^a, this relative situation is given by the coordinates of the vector p^a and by the vectors e_1^a, e_2^a, e_3^a, both expressed in the basis of K^b.

Introduce the matrix Ω, called *orientation matrix*, defined by

$$\Omega = \begin{pmatrix} (e_1^a)^T \cdot e_1^b & (e_2^a)^T \cdot e_1^b & (e_3^a)^T \cdot e_1^b \\ (e_1^a)^T \cdot e_2^b & (e_2^a)^T \cdot e_2^b & (e_3^a)^T \cdot e_2^b \\ (e_1^a)^T \cdot e_3^b & (e_2^a)^T \cdot e_3^b & (e_3^a)^T \cdot e_3^b \end{pmatrix}.$$

By construction, the matrix Ω satisfies $\Omega^T\Omega = \Omega\Omega^T = I$ and $\det \Omega = 1$ and thus is an element of the *special orientation Lie group* $SO(3)$. The word *orientation* comes from the fact that the scalar product $(e_i^a)^T \cdot e_j^b$ is equal to the cosine of the angle between the vectors e_i^a and e_j^b, for every $i, j = 1, 2, 3$.

Consider now a point P of the rigid body. The vector of coordinates of the same point, expressed in the basis of K^b are denoted by $p_P^b = (x_P^b, y_P^b, z_P^b)^T$. By construction of the matrix Ω and by the definition of p^a the following relation can be established for these two vectors:

$$p_P^b = p^a + \Omega \cdot p_P^a. \tag{78}$$

The vector p^a describes the *position* of the rigid body and Ω gives its *orientation* w.r.t. the inertial reference frame K^b. Therefore the possible positions and the orientations of a rigid body w.r.t. K^b form the set $\mathbb{R}^3 \times SO(3) = SE(3)$, referred to as the *special Euclidean group*, which is also a Lie group. This manifold is six-dimensional and can be locally represented by six real scalar variables (p, ϕ) such that $p \in \mathbb{R}^3$ gives the position and $\phi \in \mathbb{R}^3$ gives the orientation of the rigid body.

Several local representations are used (in particular in robotics) for the $SO(3)$ group [25]. Examples are the Euler angles, *RPY* (Roll, Pitch, Yaw) angles, etc. For each choice, the elements of the vector ϕ correspond to three successive rotations allowing to make coincide the

orientations of K^b and the frame fixed to the rigid body. For instance, the orientation matrix Ω_{RPY} corresponding to the RPY angles $\phi = (\varphi, \theta, \psi)$, is given by:

$$\Omega_{RPY}(\phi) = \begin{bmatrix} \cos\varphi\cos\theta & \cos\varphi\sin\theta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\sin\theta\cos\psi + \sin\varphi\sin\psi \\ \sin\varphi\cos\theta & \sin\varphi\sin\theta\sin\psi + \cos\varphi\cos\psi & \sin\varphi\sin\theta\cos\psi - \cos\varphi\sin\psi \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi \end{bmatrix}. \tag{79}$$

Throughout the paper, the RPY representation is used, thus the subscript RPY is omitted.

Rigid bodies evolve in time w.r.t. the inertial reference frame K^b and their motions can be instantaneously decomposed into a change of position and a change of orientation.

Consider a frame K^a fixed to a rigid body at a point A. Let the trajectory of the position and orientation of the frame given by the functions $p^a(t)$ and $\Omega(t) = \Omega(\phi^a(t))$, respectively. Observe that for any curve $\Omega(t)$ in $SO(3)$, the expression $\frac{d\Omega}{dt}(\Omega)^T$ is antisymmetric (this can be seen by differentiating $\Omega(t)\Omega(t)^T \equiv I$) and introduce the following notation:

$$[\omega^a\times] = \begin{bmatrix} 0 & -\omega^a_z & \omega^a_y \\ \omega^a_z & 0 & -\omega^a_x \\ -\omega^a_y & \omega^a_x & 0 \end{bmatrix} = \frac{d\Omega}{dt}(\Omega)^T. \tag{80}$$

This notation is motivated by the fact that for any vector p, the linear transformation $[\omega^a\times]p$ is equivalent to the standard cross product $\omega^a \times p$. Let us now define $v^a = \frac{d}{dt}p^a = \dot{p}^a$ the linear velocity and ω^a the angular velocity of the frame K^a.

Consider an arbitrary point P of the rigid body. By the definition of the frame K^a, fixed to the rigid body, we have that $\dot{p}^a_P = 0$. Differentiating (78) w.r.t. t, we get the instantaneous velocity of the point P, expressed in K^b:

$$\frac{dp^b_P}{dt}(t) = \frac{dp^a}{dt}(t) + \frac{d\Omega}{dt}(t)p^a_P = \dot{p}^a(t) + \frac{d\Omega}{dt}(t)\Omega(t)^T\Omega(t)p^a_P = v^a + [\omega^a\times]\Omega(t)p^a_P \tag{81}$$

or, $\dot{p}^b_P = v^a + [\omega^a\times](p^b_P - p^a)$. Thus, to calculate the velocity of any point of the rigid body, it is enough to know the translational velocity v^a and the angular velocity ω^a of the frame fixed to it. Recall that v^a and ω^a are both expressed in K^b by definition.

Kinematic constraints and nonholonomy of mechanical systems

The mechanical state of a system composed by a collection of rigid bodies is determined by a finite number of variables, called generalized coordinates, denoted by the vector q. The admissible values of q define the configuration manifold M of the mechanical system.

Constraints on the configuration manifold of a mechanical system involving the generalized coordinates and their first order time derivatives (velocities) are referred to as kinematic constraints and assumed to be linear w.r.t. the velocities. They are given in the form:

$$A(q)\cdot\dot{q} = 0, \tag{82}$$

where we suppose that the rows of $A(q)$ are linearly independent for all q. Roughly speaking, these constraints restrict the possible directions of motion at each configuration q of the mechanical system. The admissible motions are in the right null-space of the matrix $A(q)$.

Each row of the matrix $A(q)$ span a one-form on the configuration manifold M:

$$\alpha_i = \sum_j a_{ij}(q)dq_j,$$

where $a_{ij}(q)$ is the jth element of the ith row of the matrix $A(q)$. The one-forms α_i define a codistribution on M, denoted by Π. Let m be the dimension of the manifold M, and let p be the number of the rows of $A(q)$, assumed to be linearly independent, and let us denote by $\Lambda^1(M)$ the set of all one-forms on M.

Theorem 1 (Frobenius). *A codistribution Π on a manifold M is integrable iff the exterior derivative of all one-forms ϖ in Π can be written as*

$$d\varpi = \alpha \wedge \beta \qquad \alpha \in \Pi, \quad \beta \in \Lambda^1(M).$$

Definition 3 (derived flag). *The derived flag of the codistribution Π is given by the following sequence of codistributions:*

$$I^0 = \Pi \qquad I^{k+1} \stackrel{def}{=} \left\{ \varpi \in I^k : d\varpi = \alpha \wedge \beta \text{ with } \alpha \in \Lambda^1(M), \beta \in I^k \right\}.$$

This construction terminates at some N, when $I^N = I^{N+1}$.

By definition, the sequence of the codistributions $I^0, I^1, \ldots, I^N$ is such that $I^k \supset I^{k+1}$ and by Frobenius' theorem, I^N gives the largest integrable codistribution contained in Π.

Definition 4 (holonomic and fully nonholonomic mechanical system). *Consider a mechanical system evolving on a manifold M of dimension m. Let the kinematic constraints be given by a set of independent one-forms $\{\alpha_1, \ldots, \alpha_p\}$ on M, with $p < m$, spanning the codistribution Π. Calculate the derived flag of Π using Definition 3 and denote by r the dimension of I^N. The mechanical system is said to be holonomic if $r = p$ and fully nonholonomic if $r = 0$.*

We say that a distribution Δ is the annihilator of the codistribution Π on the manifold M if for all points q of M, for all covectors $\nu \in \Pi$, and for all vectors $v \in \Delta$ we have $< \nu, v >= 0$. This is also denoted by $< \Pi, \Delta >= 0$.

Note that if the system is holonomic then there exists an $m - p$ dimensional submanifold N of M such that $\Delta = TN$. This implies that kinematic constraints involving velocities of the configuration variables can be transformed into p geometric constraints defining precisely the submanifold N of M.

References

[1] R. Abraham, J.E. Mardsen, and T. Ratiu. *Manifolds, Tensor Analysis, and Applications.* Applied Mathematical Sciences. Springer - Verlag, 1988.

[2] S. Arimoto, P.T.A. Nguyen, H.Y. Han, and Z. Doulgeri. Dynamics and control of a set of dual fingers with soft tips. *Robotica*, 18:71–80, 2000.

[3] C.C. Cheah, S. Kawamura, H.Y. Han, and S. Arimoto. Grasping and position control of multi-fingered robot hands with uncertain jacobian matrices. In *Proc. of the IEEE Int. Conf. on Robotics and Automation*, pages 2403–2408, May 1998.

[4] A. Chelouah. Extensions of differential flat fields and Liouvillian systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4268–4273, San Diego, CA. USA, December 1997.

[5] M. Cherif and K.K. Gupta. Planning quasi-static motions for re-configuring object with a multi-fingered robotic hand. In *Proceedings of the IEEE International Conference on Robotics and Automation*, pages 986–991, Albuquerque, New Mexico, USA, April 1997.

[6] A. A. Cole, P. Hsu, and S. S. Sastry. Dynamic control of sliding by robot hands for regrasping. *IEEE Transactions on Robotics and Automation*, 8(1):42–52, February 1992.

[7] M. Fliess, J. Lévine, Ph. Martin, F. Ollivier, and P. Rouchon. Controlling nonlinear systems by flatness. In C.I. Byrnes, editor, *Systems and Control in the Twenty-First Century*, pages 137–154, Boston, 1997.

[8] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *C.R. Acad. Sci. Paris*, I-315:619–629, 1992.

[9] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *International Journal of Control*, 61(6):1327–1361, 1995.

[10] M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Transactions on Automatic Control*, 38:700–716, 1999.

[11] B. Goodwine. *Control of Stratified Systems with Robotic Applications*. PhD thesis, California Institute of Technology, 1998.

[12] J.B. Kerr. *An Analysis of Multifingered Hands*. PhD thesis, Dep. Mech. Eng. Stanford University, 1984.

[13] G. Lafferriere and H.J. Sussmann. Motion planning for controllable systems without drift. In *Proceedings of the IEEE International Conference on Robotics and Automation*, pages 1148–1153, Sacramento, CA, USA, 1991.

[14] B. Lantos. *Robot Control*. Hungarian Academic Press, Budapest, 1991.

[15] Z. Li, J. F. Canny, and S. S. Sastry. On motion planning for dextrous manipulation. In *Proceedings of the IEEE Conference on Robotics and Automation*, pages 775–780, 1989.

[16] A. Marigo and A. Bicchi. Rolling bodies with regular surface: Controllability theory and applications. *IEEE Transactions on Automatic Control*, 45(9):1586–1599, September 2000.

[17] Ph. Martin and P. Rouchon. Feedback linearization and driftless systems. *Mathematics of Control, Signals, and Systems*, 7(3):235–254, 1994.

[18] M.T. Mason and J.K. Salisbury. *Robot Hands and the Mechanics of Manipulation*. MIT Press, 1985.

[19] D. Montana. The kinematics of contact and grasp. *International Journal of Robotics Research*, 7:17–31, 1988.

[20] R.M. Murray, Z. Li, and S.S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.

[21] R.M. Murray and S.S. Sastry. Nonholonomic motion planning: steering using sinusoids. *IEEE Transactions on Automatic Control*, 38(5):700–716, May 1993.

[22] T. Naniwa, S. Arimoto, and K. Wada. A learning control method for coordination of multiple manipulators holding a geometrically constrained object. *Advanced Robotics*, 13(2):132–151, 1999.

[23] Ju.I. Neĭmark and N.A. Fufaev. *Dynamics of Nonholonomic Systems*. American Mathematical Society, Providence, Rhode Island, 1972.

[24] M.A. Peshkin and A.C. Sanderson. Minimization of energy in quasi-static manipulation. *IEEE Transactions on Robotics and Automation*, 5(1):53–60, 1989.

[25] J.M. Selig. *Geometrical Methods in Robotics*. Springer, 1996.

[26] K.B. Shimoga. Robot grasp synthesis algorithms: A survey. *International Journal of Robotics Research*, 16(1):230–266, Feb 1997.

[27] H. Sira-Ramírez. On the control of the variable length pendulum. In *Proceedings of the IEEE International Conference on Decision and Control*, pages 1188–1189, 1999.

[28] G. Vass, S. Payandeh, and B. Lantos. On controlled manipulation of objects within multiple dexterous agents. In *Proceedings of the 10th Congress on the Theory of Machines and Mechanisms*, Oulu, Finland, 1999.

[29] E.T. Whittaker. *A Treatise on the Analytical Dynamics of Particles & Rigid Bodies*. Cambridge University Press, 1993.