**A. Bigraded differential Hopf algebras.** In this section we work over a perfect field $\mathbb{k}$ of characteristic $p > 0$. In defining cocommutativity, graded derivations etc. in bigraded objects $\{G_{i,j}\}$ we use the signs associated with the corresponding total gradation $G_n = \{G_{i,j}\}_{i+j=n}$.

Our objective is to refine certain results of Browder in [Br]. Thus we shall be interested in bigraded Hopf algebras $G$ satisfying the following conditions:

$$\left.\begin{array}{l} \bullet\ \ G_{0,0} = \mathbb{k};\ G = \{G_{i,j}\}_{i,j\geq 0};\ \ \text{each } G_{i,j} \text{ is finite dimensional.} \\ \bullet\ \ G \text{ is cocommutative.} \\ \bullet\ \ \text{The sub Hopf algebras } G_{*,0} \text{ and } G_{0,*} \text{ are normal.} \end{array}\right\} \qquad (\mathbf{A.1})$$

The surjections $G \longrightarrow G_{*,0}$ and $G \longrightarrow G_{0,*}$ define a surjection

$$G \longrightarrow G_{*,0} \otimes G_{0,*}$$

of bigraded Hopf algebras, whose Hopf kernel $G^M$ will be called the *mixed sub Hopf algebra* of $G$.

Next observe that a natural isomorphism

$$\alpha_G : G_{*,0} \otimes G^M \otimes G_{0,*} \xrightarrow{\cong} G$$

of bigraded Hopf algebras is given by $a \otimes b \otimes c \mapsto abc$. Indeed, $\alpha_G$ preserves the diagonal because multiplication in $G$ does. Moreover, because $G_{*,0}$, $G^M$ and $G_{0,*}$ are all normal they commute pairwise, and so $\alpha_G$ preserves products as well.

*For the rest of this section we shall fix a bigraded differential Hopf algebra $(G, d)$ in which*

- *$G$ is assumed to satisfy (A.1).*

- *$d$ has bidegree $(-r, r-1)$, some $r \geq 2$.*

**Lemma A.2.** *The bigraded homology Hopf algebra $H(G)$ also satisfies (A.1). Thus $H(G) = H_{*,0} \otimes H(G)^M \otimes H_{0,*}$.*

**proof:** We have only to check that $H_{0,*}$ and $H_{*,0}$ are normal. Homology classes in these algebras are represented by cycles $x \in G_{0,*}$ and $y \in G_{*,0}$. For any cycle $z \in G$, $(Ad\, z)x$ and $(Ad\, z)y$ are respectively cycles in $G_{0,*}$ and $G_{*,0}$, since these are normal subalgebras. Hence $(Ad\,[z])[x] \in H_{0,*}$ and $(Ad\,[z])[y] \in H_{*,0}$, as desired.

■

By definition the Hopf algebras $H_{*,0}$ and $H_{0,*}$ are given by

$$H_{*,0} = G_{*,0} \cap \ker d \qquad \text{and} \qquad H_{0,*} = G_{0,*}/d(G_{r,*}).$$

Now recall from [MSI; §2] that the *p-derived Hopf algebra* of a cocommutative graded Hopf algebra, $A$, is the Hopf kernel $\langle A, A\rangle$ of the surjection $A \longrightarrow A/I$, where $I$ is the ideal generated by commutators and $p^{\text{th}}$ powers.

**Proposition A.3.** (i) *The Hopf kernel of the surjection* $G_{0,*} \longrightarrow H_{0,*}$ *is generated by central primitive elements of* $G_{0,*}$.

(ii) *The sub Hopf algebra* $H_{*,0}$ *contains* $\langle G_{*,0}, G_{*,0}\rangle$. *In particular it is normal in* $G_{*,0}$.

**proof:** (i) The ideal $d(G_{r,*})$ in $G_{0,*}$ is generated by $d(A_{r,*})$, where $A = G_{*,0} \otimes G^M$. Since $A$ is a subcoalgebra and $A_{0,*} = \mathbb{k}$, $dx$ is primitive for $x \in A_{r,*}$. Similarly for $y \in G_{0,*}$ we have $[x, y] = 0$, so that $0 = d[x, y] = [dx, y]$. Thus $dx$ is central in $G_{0,*}$.

(ii) Let $b \in \langle G_{*,0}, G_{*,0}\rangle$ be an element of least degree for which $db \neq 0$. Then $db$ is primitive of bidegree $(*, r-1)$. Since $\alpha_G$ is an isomorphism of coalgebras, $db \in G^M \otimes G_{0,*}$. But $b$ is decomposable in $G_{*,0}$, so that is impossible.

■

The next step is to consider the mixed sub Hopf algebra $H(G)^M$. To do so we introduce the Hopf algebras

$$Z = Hopf\ker(G_{0,*} \longrightarrow H_{0,*}) \qquad \text{and} \qquad W = G_{*,0}/\!/H_{*,0},$$

which will play a fundamental role in the sequel. Now the sequence of Hopf algebra morphisms,

$$\mathbb{k} \longrightarrow H_{*,0} \xrightarrow{\sigma} G \xrightarrow{\kappa} H_{0,*} \longrightarrow \mathbb{k},$$

is exact at $H_{*,0}$ and $H_{0,*}$, and the failure of exactness at $G$ is measured by the quotient Hopf algebra,

$$(Hopf\ker\kappa)/\!\!/ Im\,\sigma = W \otimes G^M \otimes Z.$$

Since $H(\sigma)$ and $H(\kappa)$ are evidently respectively injective and surjective, it follows (eg [Tors; Lemma 2.1]) that

$$H(G)^M = Hopf\ker H(\kappa)/\!\!/ Im\,H(\sigma) = H(W \otimes G^M \otimes Z). \qquad \textbf{(A.4)}$$

Finally in the spirit of [Br] we analyze the structure of $H(G)^M$ under the additional hypothesis that

$$G^M \text{ is an exterior algebra, } \Lambda U \text{, on a space } U \text{ of primitive elements of odd degrees.} \qquad \textbf{(A.5)}$$

**Proposition A.6.** *If $G^M$ satisfies (A.5) then so does $H(G)^M$ : $H(G)^M = \Lambda V$, $V$ an oddly graded space of primitives.*

**Remark A.7.** In [Br; Lemma 5.7; Theorem 5.8] Browder establishes a coalgebra isomorphism $H(G) \cong H_{0,*} \otimes \Lambda V \otimes H_{0,*}$. Combining Propositions A.2 and A.6 we now have this as an isomorphism of Hopf algebras. In fact, given (A.2), we could deduce (A.6) directly from Browder's result. The purpose of the independent proof we give below is to exhibit the precise interdependence between $V$ and the Hopf algebra structures in $Z$ and $W$. This interdependence will play an important role in the applications.

**proof of A.6:** Denote by $\mathbb{k}(x)$ the Hopf algebra generated by a single primitive element $x$. We write $\mathbb{k}(x) = \Lambda x$ if $x^2 = 0$; in this case $x$ has odd degree unless $p = 2$. Recall the classical and elementary fact that

$$H(\Lambda x \otimes \mathbb{k}(dx)) = \begin{cases} \mathbb{k} & \text{if } \mathbb{k}(dx) = \mathbb{k}[dx]. \\ \mathbb{k} \oplus \mathbb{k}x(dx)^{n-1} & \text{if } \mathbb{k}(dx) = \mathbb{k}[dx]/\!\!/(dx)^n. \end{cases}$$

If $p = 2$ then $n = 2^k$ and $n-1$ is odd; otherwise $x$ has odd degree and $dx$ has even degree. Thus in all cases $x(dx)^{n-1}$ has odd degree.

We shall decompose $W \otimes \Lambda U \otimes Z$ as the tensor product of chain complexes of the form $\Lambda x$, $\Lambda x \otimes I\!k(dx)$ or $[\Lambda x \otimes I\!k(dx)]^\vee$, where

$$X^\vee = \text{(bigraded) dual vector space of } X.$$

It will follow that the Hilbert series $H(W \otimes \Lambda U \otimes Z)(t) \overset{\text{def}}{=} \sum_i [\dim H_i(W \otimes \Lambda U \otimes Z] t^i$ has the form

$$H(W \otimes \Lambda U \otimes Z)(t) = \prod_i (1 + t^{2n_i+1}).$$

This implies that $H(W \otimes \Lambda U \otimes Z)$ has the form $\Lambda V$, where $V$ is the subspace of primitive elements, and is concentrated in odd degrees.

It remains to construct the decomposition of $W \otimes \Lambda U \otimes Z$. We begin with some observations that follow directly from (A.5) and the Borel structure theorem for commutative graded Hopf algebras.

**(i)** Elements of odd degree in $W \otimes \Lambda U$ square to zero. (When $p = 2$ this uses the fact that $p^{\text{th}}$ powers vanish in $W$.)

**(ii)** Since $d(U)$ consists of primitives of even degree,

$$d(U_{i,*}) = 0,\ i \neq r \qquad \text{and} \qquad d(U_{r,*}) \subset Z.$$

Moreover $Z$ is generated as an algebra by $d(U_{r,*})$ and $d(W_{r,0})$. Thus there are elements $\tilde{u}_j \in U_{r,*} + Z_+ \cdot (U_{r,*} + W_{r,0})$ and there is a basis $w_i$ of $W_{r,0}$ such that $d\tilde{u}_j$ is primitive and

$$Z = \left[\bigotimes_i I\!k(dw_i)\right] \otimes \left[\bigotimes_j I\!k(d\tilde{u}_j)\right] \qquad \text{as Hopf algebras.}$$

In particular the components $u_j$ of $\tilde{u}_j$ in $U_{r,*}$ are linearly independent modulo $\ker d$.

**(iii)** There are cycles $\tilde{u}'_k \in U_{r,*} + Z_+ \cdot (U_{r,*} + W_{r,0})$ whose components $u'_k$ in $U_{r,*}$, together with the $u_j$ provide a basis for $U/U \cap Im\, d$.

**(iv)** The differential $d$ maps $W$ injectively into $W \otimes (U \cap d(W) \oplus Z_{0,r-1})$. Thus

$$A = W \otimes \Lambda(U \cap Im\, d) \otimes (\bigotimes_i I\!k(dw_i))$$

is a sub differential Hopf algebra of $W \otimes \Lambda U \otimes Z$, and

$$W \otimes \Lambda U \otimes Z = A \otimes \left(\bigotimes_j \Lambda\tilde{u}_j \otimes I\!k(d\tilde{u}_j)\right) \otimes \left(\bigotimes_k \Lambda\tilde{u}'_k\right) \qquad \textbf{(A.9)}$$

as $DGA$'s.

Next, consider the dual differential Hopf algebra, $(A^\vee, \delta)$. To simplify notation we shall put

$$X = U \cap Im\, d \qquad \text{and} \qquad Z(1) = \bigotimes_i I\!k(dw_i),$$

so that $A^\vee = W^\vee \otimes \Lambda X^\vee \otimes Z(1)^\vee$. The differential Hopf algebra $A^\vee$ behaves exactly as $W \otimes \Lambda U \otimes Z$, with $Z(1)^\vee$ playing the role of $W$ and $W^\vee$ playing the role of $Z$. Thus, as above, we have

**(v)** There are elements $\tilde{x}_j \in X^\vee_{*,r-1} + W^\vee_+ \cdot (X^\vee_{*,r-1} + Z^\vee_{0,r-1})$ and there is a basis $\tilde{z}_\ell$ of $Z^\vee_{0,r-1}$ such that $\delta\tilde{x}_j$ is primitive and

$$W^\vee = \left[\bigotimes_\ell I\!k(\delta\tilde{z}_\ell)\right] \otimes \left[\bigotimes_j I\!k(\delta\tilde{x}_j)\right] \qquad \text{as Hopf algebras.}$$

In particular the components $x_j$ of $\tilde{x}_j$ in $X^\vee$ are linearly independent modulo $\ker\delta$.

**(vi)** Let $W(1) = \bigotimes_\ell I\!k(\delta\tilde{z}_\ell)$. Then $(W(1) \otimes Z(1)^\vee, \delta)$ is a sub differential Hopf algebra of $A^\vee$.

**(vii)** There are cycles $\tilde{x}'_k \in X^\vee + W^\vee_+ \cdot (X^\vee + Z^\vee_{0,r-1})$ whose components $x'_k$ in $X^\vee$, together with the $x_j$, are a basis for $X^\vee$.

**(viii)** As $DGA$'s,

$$(A^\vee, \delta) = [W(1) \otimes Z(1)^\vee] \otimes \left[\bigotimes_j \Lambda\tilde{x}_j \otimes I\!k(\delta\tilde{x}_j)\right] \otimes \left[\bigotimes_k \Lambda\tilde{x}'_k\right]. \qquad \textbf{(A.9)}$$

Finally, we consider $(W(1) \otimes Z(1)^\vee, \delta)$ and its dual, distinguishing two cases:

*Case (a): $p$ is odd.* When $r$ is odd, $W(1)$ is the exterior algebra on $W^\vee_{r,0}$ and so $W(1)^\vee$ is the exterior algebra on $W_{r,0}$. Thus

$$W(1)^\vee \otimes Z(1) = \bigotimes_i (\Lambda w_i \otimes I\!k(dw_i)).$$

Dually, when $r$ is even,

$$W(1) \otimes Z(1)^\vee = \bigotimes_\ell (\Lambda\tilde{z}_\ell \otimes I\!k(\delta\tilde{z}_\ell)).$$

*Case (b): $p = 2$.* Let $W(1)^\vee = \bigotimes_\ell A(\ell)$ be the tensor decomposition dual to the decomposition $W(1) = \bigotimes_\ell I\!k(\delta z_\ell)$. We may suppose that $(\delta z_\ell)^2 \neq 0$, $1 \leq \ell \leq q$ and $(\delta z_\ell)^2 = 0$, $\ell > q$. Let $y_{\ell,k}$ be the basis of $A(\ell)$ dual to the basis $\{(\delta z_\ell)^k\}$. Then $\Delta y_{\ell,k} = y_{\ell,k} \otimes 1 + \sum_{i=1}^{k-1} y_{\ell,i} \otimes y_{\ell,k-i} + 1 \otimes y_{\ell,k}$, while $dy_{\ell,k} = \sum_j y_{j,k-1} a_j$ for some elements $a_j \in Z_{0,r-1}$. Comparing these relations we find by induction that $dy_{\ell,k} = y_{\ell,k-1} dy_{\ell,1}$. Thus $A(\ell) \otimes I\!k(dy_{\ell,1})$ is a differential sub Hopf algebra of $W(1)^\vee \otimes Z(1)$.

Now suppose $1 \leq \ell \leq q$. Then $d(y_{\ell,1})^2 = d^2(y_{\ell,2}) = 0$. It follows that we may, in our original basis $\{W_i\}$ for $W_{r,0}$, take $W_\ell = y_{\ell,1}$, $1 \leq \ell \leq q$. Since (observation (i)) each $w_\ell^2 = 0$, we have an isomorphism of differential Hopf algebras:

$$W(1)^\vee \otimes Z(1) = \left[\bigotimes_{\ell=1}^{q} A(\ell) \otimes \Lambda(dw_\ell)\right] \otimes \left[\bigotimes_{\ell > q} \Lambda w_\ell \otimes I\!k(dw_\ell)\right] \qquad (\mathbf{A.11})$$

Since $A(\ell) \otimes \Lambda(dw_\ell) = [I\!k(d\tilde{z}_\ell) \otimes \Lambda \tilde{z}_\ell]^\vee$ this, together with (A.9) – (A.11), provides the desired decomposition of $W \otimes \Lambda U \otimes Z$.

■

**Corollary (of proof) A.12.** *With the notation above,*

*(i)* $V_{i,j} \cong U_{i,j}$ *unless* $i = r$ *or* $j = r-1$.

*ii)* $\dim V_{r,*} \leq \dim U_{r,*} + \dim W_{r,0} \leq \dim U_{r,*} + \dim G_{r,0}$.

*(iii)* $\dim V_{*,r-1} \leq \dim U_{*,r-1} + \dim Z_{0,r-1} \leq \dim U_{*,r-1} + \dim G_{0,r-1}$.

*iv)* $\dim V_{r,j} \leq \dim U_{r,j}$, $j \leq r-2$.

*v)* $\dim V_{i,r-1} \leq \dim U_{i,r-1}$, $i \leq r-1$.

■

**Remark A.13.** As observed in A.12, the bigraded spaces $U$ and $V$ differ only in the row $(*, r-1)$ and the column $(r, *)$. The proof of (A.7) yields the following recipe for constructing a basis of $V$ from a basis of $U$. (Recall that $I\!k(x)$ denotes the Hopf algebra generated by a single primitive element $x$.)

First, the Hopf algebra structure of $Z$ and $W$ is given by

$$Z = \left[\bigotimes_{i=1}^{s} \mathbb{k}(z_i)\right] \otimes \left[\bigotimes_{\alpha\in y} \mathbb{k}(a_\alpha)\right],$$

and

$$W^\vee = \left[\bigotimes_{i=1}^{s} \mathbb{k}(\tilde{w}_i)\right] \otimes \left[\bigotimes_{\beta\in y} \mathbb{k}(\tilde{b}_\beta)\right].$$

The tensor product decomposition of $W^\vee$ defines a vector space basis, consisting of elements of the form $(\tilde{w}_i)^k$, $(\tilde{b}_\beta)^\ell$ and their products. In the dual basis let $w_i$, $b_\beta$ be the elements dual to $\tilde{w}_i$, $\tilde{b}_\beta$, and let $w_i^{(k)}$, $b_\beta^{(\ell)}$ be the duals of $(\tilde{w}_i)^k$, $(\tilde{b}_\beta)^\ell$.

Second, $d$ maps $W_{r,0}$ isomorphically onto $Z_{0,r-1}$ and we may suppose

$$z_i = dw_i, \qquad 1 \le i \le s.$$

Furthermore, $U$ has a vector space basis of the form $\{x_\alpha, y_\beta, v_\gamma\}$, where:

$$dv_\gamma = 0, \quad d(x_\alpha + \Phi_\alpha) = a_\alpha \qquad (\Phi_\alpha \text{ decomposable}), \quad \text{and} \quad db_\beta = y_\beta.$$

Thus $x_\alpha \in U_{r,*}$, $y_\beta \in U_{*,r-1}$ and $a_\alpha$, $b_\beta$ have even degrees satisfying

$$deg\, a_\alpha = deg\, x_\alpha - 1 \quad \text{and} \quad deg\, b_\beta = deg\, y_\beta + 1.$$

Third, the elements $v_\gamma$ survive to form part of a basis $\{v_\alpha, e_\alpha, f_\beta, g_i\}$ for $V$, constructed as follows:

- If $\mathbb{k}(a_\alpha)$ is a polynomial algebra, then $x_\alpha$ simply disappears. If $\mathbb{k}(a_\alpha) = \mathbb{k}[a_\alpha]/a_\alpha^{p^k}$ then $x_\alpha$ disappears and an element $e_\alpha$ is created, represented by $(x_\alpha + \Phi_\alpha)a_\alpha^{p^k-1}$. We write

$$e_\alpha \sim x_\alpha a_\alpha^{p^k-1}.$$

- If $\mathbb{k}(\tilde{b}_\beta)$ is a polynomial algebra, then $y_\beta$ simply disappears. If $\mathbb{k}(\tilde{b}_\beta) = \mathbb{k}[\tilde{b}_\beta]/\tilde{b}_\beta^{p^\ell}$ then $y_\beta$ disappears and an element $f_\beta$ is created, with

$$f_\beta \sim y_\beta {\cdot} b_\beta^{(p^\ell-1)}.$$

To describe the $g_i$ we distinguish two cases.

*Case (a): p is odd.*

- If $r$ is odd then either $I\!k(dw_i)$ is a polynomial algebra and no basis element of $V$ is created, or else $I\!k(dw_i) = I\!k[dw_i]/(dw_i)^{p^k}$ and $w_i(dw_i)^{p^k-1}$ represents a basis vector $g_i$ of $V$. We write

$$g_i \sim w_i(dw_i)^{p^k-1}.$$

- If $r$ is even then either $I\!k(\tilde{w}_i)$ is a polynomial algebra and no basis element of $V$ is created, or else $I\!k(\tilde{w}_i) = I\!k[\tilde{w}_i]/\tilde{w}_p^{p^\ell}$ and a basis vector $g_i$ is created with

$$g_i \sim w_i^{(p^\ell-1)} dw_i.$$

*Case (b): $p = 2$.* In this case for each $i$ either $w_i^2 = 0$ or $\tilde{z}_i^2 = 0$, $\tilde{z}_i$ the dual of $z_i$. Either or both possibilities may occur, independent of the parity of $r$.

- If $w_i^2 = 0$ then either $I\!k(dw_i)$ is a polynomial algebra and no basis element of $V$ is created, or else $I\!k(dw_i) = I\!k[dw_i]/(dw_i)^{2^k}$ and $w_i(dw_i)^{2^k-1}$ represents a basis vector $g_i$ of $V$. We write

$$g_i \sim w_i(dw_i)^{2^k-1}.$$

- If $\tilde{z}_i^2 = 0$ then either $I\!k(d\tilde{z}_i)$ is a polynomial algebra and no basis element of $V$ is created, or else $I\!k(d\tilde{z}_i) = I\!k[d\tilde{z}_i]/(d\tilde{z}_i)^{2^\ell}$ and a basis vector $g_i \in V$ is created with

$$g_i \sim w_i^{(2^\ell-1)} dw_i.$$

The remarks above are summarized in

**Proposition A.14.** *The elements $v_\gamma$, $e_\alpha$, $f_\beta$, $g_i$ described above constitute a basis for $V$.*

■

Finally, the proof of (A.6) and the remarks above yield

**Proposition A.15.** *Suppose the rows $U_{*,j}$ and the columns $U_{i,*}$ of $U$ are all finite dimensional. Then:*

*(i) The Hopf algebras $Z$ and $(W^\vee)$ are the finite tensor products of monogenic Hopf algebras;*

*(ii) The rows and columns of $V$ are finite dimensional. If, moreover, $U$ is finite dimensional, then so is $V$.*

■

**B. Spectral sequences of Hopf algebras.** In this section we consider a spectral sequence

$$(E^r_{*,*}, d^r) \quad r \geq 2$$

of bigraded differential Hopf algebras, satisfying the following conditions:

$$\left.\begin{array}{l} \bullet\ E^2_{0,0} = k;\ E^2 = \{E^2_{i,j}\} \quad i,j \geq 0; \text{each } E^2_{i,j} \text{ is finite dimensional.} \\ \bullet\ E^2 \text{ is cocommutative.} \\ \bullet\ E^2 = E^2_{*,0} \otimes E^2_{*,0} \text{ as bigraded Hopf algebras.} \\ \bullet\ d^r \text{ has bidegree } (-r, r-1). \end{array}\right\} \qquad (\mathbf{B.1})$$

During the entire section the spectral sequence will be fixed and assumed to satisfy (B.1). As usual the bigraded Hopf algebra $E^\infty_{*,*}$ associated with the spectral sequence is defined by

$$E^\infty_{i,j} = E^r_{i,j} \qquad r > \max(i, j+1).$$

It is immediate from Lemma A.2 and Proposition A.6 that each term $(E^r, d^r)$ satisfies both conditions (A.1) and (A.5) of the preceding section. Thus we have the following refinement of [Br; Theorem 5.8].

**Theorem B.2.** *The spectral sequence $(E^r, d^r)$ above satisfies:*

*(i) For $r \leq \infty$, $E^r_{*,*} = E^r_{*,0} \otimes \Lambda U^r \otimes E^r_{0,*}$ as bigraded Hopf algebras, where $U^r$ consists of primitives of odd degrees.*

*(ii) In the sequence $E^2_{*,0} \supset \cdots \supset E^r_{*,0} \supset \cdots$, each $E^{r+1}_{*,0}$ is normal in $E^r_{*,0}$. The Hopf algebra $A_r$ dual to $E^r_{*,0} /\!/ E^{r+1}_{*,0}$ is generated by finitely many primitive elements, whose degrees $i$ satisfy: $i = r$, or else $i$ is an integer in $[r+1, 2r-2]$ of the form $i = 2p^k n + 2$, some $k \geq 1$.*

*(iii) In the sequence $E^2_{0,*} \twoheadrightarrow \cdots \twoheadrightarrow E^r_{0,*} \twoheadrightarrow \cdots$, the Hopf kernel $Z^r$ of the surjection $E^r_{0,*} \longrightarrow E^{r+1}_{0,*}$ is central and generated by finitely many primitive elements, whose degrees $j$ satisfy: $j = r-1$, or else $j$ is an integer in $[r, 2r-4]$ of the form $j = 2p^k n - 2$, some $k \geq 1$.*

*(iv) For $r \leq \infty$ the bidegrees $(i,j)$ of the elements of $U^r_{*,*}$ have one of the following two forms:*

$$(i,j) = \begin{cases} (p^k m - q, q-1), & \textit{some } q \in (1, r) \\ \textit{or} & \\ (q+1, p^k m - q), & \textit{some } q \in (0, r-1), \end{cases}$$

*with* $k \geq 1$ *and* $q \leq m < 2q-1$*. In the first case* $m$ *is the degree of a generator of* $A_q$*. Thus either* $m = q$ *or* $U^q_{m-q,q-1} \neq 0$*. In the second case,* $m$ *is the degree of a generator of* $Z^q$*. Thus* $m = q$ *or* $U^q_{q+1,m-q} \neq 0$*.*

*(v) In particular, for* $r < \infty$*,* $U^r$ *is finite dimensional and concentrated in the union of the columns* $i < r$ *and the rows* $j < r-1$*. The space* $U^\infty$ *may be infinite dimensional, but will still have finite dimensional rows and columns.*

**proof:** The assertions for $r < \infty$ follow directly by induction from §A. The assertions for $r = \infty$ follows by a limit argument using Corollary A.12.

∎

In the spectral sequence (B.1) consider the short exact sequence of Hopf algebras

$$\mathbb{k} \longrightarrow K \longrightarrow E^2_{0,*} \xrightarrow{\eta} E^\infty_{0,*} \longrightarrow \mathbb{k},$$

where $K$ is the Hopf kernel of the edge homomorphism $\eta$. It is the union of the Hopf kernels $K^r$ of $\eta^r : E^2_{0,*} \longrightarrow E^r_{0,*}$, $r \geq 2$, and this identifies $Z^r = K^{r+1} /\!\!/ K^r$.

**Theorem B.3.** *If* $K$ *is commutative then there are graded algebra splittings for the surjections* $K^{r+1} \longrightarrow Z^r$*, which then define an isomorphism of graded algebras,* $K^{r+1} \cong K^r \otimes Z^r$*. In particular*

$$K \cong \bigotimes_{r \geq 2} Z^r.$$

On the other hand, if $E^\infty_{*,0}$ is normal in $E^2_{*,0}$ we have short exact sequences of Hopf algebras

$$\mathbb{k} \longrightarrow E^{r+1}_{*,0} /\!\!/ E^\infty_{*,0} \longrightarrow E^r_{*,0} /\!\!/ E^\infty_{*,0} \longrightarrow E^r_{*,0} /\!\!/ E^{r+1}_{*,0} \longrightarrow \mathbb{k}.$$

**Theorem B.4.** *If* $E^\infty_{*,0}$ *is normal in* $E^2_{*,0}$ *then there are graded coalgebra splittings for the inclusions* $E^{r+1}_{*,0} /\!\!/ E^\infty_{*,0} \longrightarrow E^r_{*,0} /\!\!/ E^\infty_{*,0}$*, which then define isomorphisms of graded coalgebras,* $E^r_{*,0} /\!\!/ E^\infty_{*,0} \cong E^{r+1}_{*,0} /\!\!/ E^\infty_{*,0} \otimes E^r_{*,0} /\!\!/ E^{r+1}_{*,0}$*. In particular,*

$$E^2_{*,0} /\!\!/ E^\infty_{*,0} \cong \bigotimes_{r \geq 2} E^r_{*,0} /\!\!/ E^{r+1}_{*,0}.$$

To establish Theorems B.3 and B.4 we need to somewhat strengthen the Borel structure theorem for commutative graded Hopf algebras. Fortunately the proof of the latter given in [MM; Proposition 7.10] can be slightly modified to give us the result we need. We include this modified proof for the convenience of the reader.

**Proposition B.5.** *Suppose* $A \longrightarrow B \xrightarrow{\pi} C$ *is a short exact sequence of connected, commutative (not necessarily cocommutative) Hopf algebras such that the spaces of indecomposables satisfy:*

$$Q_n(C) \neq 0 \Longrightarrow Q_{p^k n}(A) = 0, \quad k \geq 1.$$

*Then* $\pi$ *admits a splitting* $\sigma : C \longrightarrow B$ *of graded algebras. If* $B$ *is primitively generated we can take* $\sigma$ *to be a splitting of Hopf algebras.*

**proof:** Reduce in the standard way to the case $C = \mathbb{k}[y]/y^{p^\ell}$ with $p \deg y$ even and that each $A_i$ is finite dimensional. Denote by $\xi : x \mapsto x^p$ the Frobenius morphism and consider the short exact sequence

$$A/\xi^\ell(A) \longrightarrow B/\xi^\ell(A) \longrightarrow C.$$

To simplify notation put $A' = A/\xi^\ell(A)$.

Now choose $z \in B/\xi^\ell(A)$ to represent $y$. The reduced diagonal of $z$ is in $A'_+ \otimes A'_+$ for trivial degree reasons, and so $z^{p^\ell}$ is primitive. Since $z^{p^\ell}$ maps to zero in $C$, it is even a primitive element of $A'$: $z^{p^\ell} \in A'$.

On the other hand, clearly $Q(A)$ maps onto $Q(A')$, and so $Q(A')$ vanishes in degrees of the form $p^k \deg z$, $k \geq 1$. Thus $Q(\xi^{i-1}A')$ vanishes in degrees of the form $p^k \deg z$, $k \geq i$ and so if $s = p^\ell \deg z$ then $Q_s(\xi^{i-1}A') = 0$, $i \leq \ell$. Thus from the exact sequences

$$0 \longrightarrow P(\xi^i A') \longrightarrow P(\xi^{i-1}A') \longrightarrow Q(\xi^{i-1}A')$$

[MM; Proposition 4.21] we deduce that $P_s(\xi^\ell A') \xrightarrow{\cong} P_s(A')$. In particular, $z^{p^\ell} \in \xi^\ell(A'_+) = 0$.

Let $w \in B$ represent $z$, and so the reduced diagonal of $w$ is in $A_+ \otimes A_+$. Since $z^{p^\ell} = 0$ this implies that $\Delta(w^{p^\ell}) - w^{p^\ell} \otimes 1 \in B \otimes I$, where $I = \xi^p(A_+) \cdot B$. Hence $w^{p^\ell} \in \xi^\ell(A_+)$. Write $w^{p^\ell} = a^{p^\ell}$, $a \in A_+$, and define $\sigma$ by $\sigma(y) = w - a$.

Finally, if $B$ is primitively generated so is $A$. Thus we may choose $z$ and $w$ above to be primitive. Thus $w^{p^\ell}$ is a primitive element in $\xi^\ell(A)$ and hence the $p^\ell$-th element in $A$; i.e. we may take $a \in A_+$ above to be primitive as well. In this case $\sigma$ is a splitting of Hopf algebras.

■

**proof of Theorem B.3:** By induction on $r$, the degree of a generator of $K^r$ is the degree of a generator of some $Z^q$, $q < r$. Hence by Theorem B(iii), $K^r$ is generated in degrees $\leq 2(r-3)$. On the other hand, again from Theorem B.2(iii), $Z^r$ is generated in degrees $\geq r-1$. Thus

$$Q_n(Z^r) \neq 0 \Longrightarrow n \geq r-1 \Longrightarrow p^k n > 2(r-3) \Longrightarrow Q^k_{p^n}(K^r) = 0, \quad k \geq 1.$$

Now apply Proposition B.5.

■

**proof of Theorem B.4:** Consider the exact sequence of commutative graded Hopf algebras,

$$\mathbb{k} \longrightarrow A_r \longrightarrow \left(E^r_{*,0} /\!\!/ E^\infty_{*,0}\right)^\vee \longrightarrow \left(E^{r+1}_{*,0}/E^\infty_{*,0}\right)^\vee \longrightarrow \mathbb{k},$$

dual to the exact sequence above by the very definitions, $E^{r+1}_{*,0}/E^\infty_{*,0}$ vanishes in degrees $\leq r$. On the other hand, according to Theorem B.2(ii), $A_r$ is generated in degrees $\leq 2(r-1)$. Apply Proposition B.5.

■

In the spectral sequence (B.1) consider the infusion $E^\infty_{*,0} \xrightarrow{\lambda} E^2_{*,0}$. If $E^\infty_{*,0}$ is normal in $E^2_{*,0}$ then we may form the short exact sequence of Hopf algebras

$$\mathbb{k} \to E^\infty_{*,0} \xrightarrow{\lambda} E^2_{*,0} \to C \to \mathbb{k},$$

where $C = \mathit{coker}\,\lambda$. Denote the dual Hopf algebra by $D : D = C^\vee$. Now if $C$ is commutative then each Hopf algebra intermediate between $E^\infty_{*,0}$ and $E^2_{*,0}$ is normal in $E^2_{*,0}$. In particular, $E^r_{*,0}$ is normal in $E^2_{*,0}$, $r \geq 2$, and so we can form the commutative Hopf algebra $D_r$ dual to the quotient $E^2_{*,0} /\!\!/ E^e_{*,0}$:

$$D_r = \left(E^2_{*,0} /\!\!/ E^r_{*,0}\right)^\vee.$$

Moreover $D$ is the increasing union of the $D_r$ and this identifies the Hopf algebras $A_r$ (cf. Theorem B.2) as the quotients $D_{r+1}/\!/D_r$. The same argument as in Theorem B.3 gives

**Theorem B.6.** *Assume, in the spectral sequence (B.1), that $E^\infty_{*,0}$ is normal in $E^2_{*,0}$, and that the quotient Hopf algebra $C$ is commutative. Then the surjections $D_{r+1} \longrightarrow A_r$ admit graded algebra splittings, and these define an isomorphism*

$$D \cong \bigotimes_{r\geq 2} A_r.$$

■

**Proposition B.7.** *In the spectral sequence of Theorem B.1 the following conditions are equivalent:*

(i) $E^\infty = E^r$, *some* $r \in [2,\infty)$.

(ii) *The Hopf kernel $K$ of the edge homomorphism $E^2_{0,*} \longrightarrow E^\infty_{0,*}$ is finitely generated.*

(iii) *For some $r < \infty$, $E^r_{0,*} \xrightarrow{\cong} E^\infty_{0,*}$.*

(iv) *The algebra dual to the Hopf cohernel $C$ of the edge homomorphism $E^\infty_{*,0} \longrightarrow E^2_{0,*}$ is finitely generated.*

(v) *For some $r$, $E^\infty_{*,0} \xrightarrow{\cong} E^r_{*,0}$.*

**proof:** Clearly (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii), since the Hopf kernel of $E^{i+1}_{0,*} \longrightarrow E^i_{0,*}$ is finitely generated by Theorem B.1(ii). To show (ii) $\Longrightarrow$ (iii), note that $K$ is the increasing union of the Hopf kernels $K^r$ of the morphisms $E^2_{0,*} \longrightarrow E^r_{0,*}$, it follows that $K = K^r$, some $r$. This implies that the surjections $E^i_{0,*} \longrightarrow E^{i+1}_{0,*}$ are injective for $i \geq r$ and so $E^r_{0,*} \xrightarrow{\cong} E^\infty_{0,*}$.

**B.8.** To show (iii) $\Longrightarrow$ (i) note that $d^i : U^i \oplus E^i_{i,0} \longrightarrow E^i_{0,*}$. Hence for $i \geq r$, $d^i$ vanishes in $U^i$ and in $E^i_{i,0}$. It follows that we have a linear map $U^i \longrightarrow U^{i+1}$. Moreover from the explicit description of the creation of $U^{i+1}$ given in §A and the fact that $d^i$ vanishes in $U^i$ and $E^i_{i,0}$ we deduce that for $i \geq r$

$$U^i_{j,\ell} \longrightarrow U^{i+1}_{j,\ell} \quad \text{is surjective,} \quad j < \ell.$$

Since $U^r$ is finite dimensional (Theorem B.2(v)) we conclude that for some $q \in [r, \infty)$

$$U^i_{j,\ell} \xrightarrow{\cong} U^{i+1}_{j,\ell}, \quad j < \ell, \quad i \geq q.$$

On the other hand, observation (iv) in the proof of (A.6) gives

$$d^i : E^i_{*,0} \longrightarrow \left(d^i(E^i_{0,*}) \cap U^i_{*,0-1}\right), \ i \geq r.$$

Here $U^i_{*,0-1}$ is concentrated in bidegrees $(j, i-1)$ with $j < i-1$ by Theorem B.1(iv). The isomorphisms above imply $d^i(E^i_{*,0}) \cap U^i_{*,i-1} = 0$ for $i \geq q$ and so $d^i = 0$, $i \geq q$; i.e. the spectral sequence collapses at $E^q$.

The equivalences (i) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) follow from an identical argument using the dual spectral sequence.

■

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