# The Anatomy of a Step Supply Function*

MARVIN KOTTKE

**A close examination is made of the conceptual basis underlying a step function in order to identify the parts of steps and relate them to an economic interpretation. Emphasis is placed on the relevance of (1) the "relation," "function," and "discontinuity" definitions, (2) the identity of marginal cost in the steps, and (3) the MR $\geq$ MC profit-maximization criterion to the interpretation. It is contended that, if producers conform to linear programming specifications, their step supply functions should be interpreted as representing a sporadic response which calls for a modified applicability of elasticity, reversibility, and length-of-run concepts.**



TEN years ago, Candler [6, pp. 416–417] raised the question, "Should we think of a firm's supply function as being smooth or stepped?"[1] Since then, step functions derived by linear programming have matured into full bloom in supply studies. Now, questions of interpretation are being raised. Does a step supply function have the same conceptual basis as a smooth supply function? Is elasticity a meaningful measure of responsiveness for step supply functions [2, 10]? Can a step function be converted to a smooth function and retain its original interpretation of supply behavior [5, 12]? The position taken in this article is that the interpretation of a step supply function follows the conventional explanation of a smooth supply function, except that the steps modify and accentuate parts of the interpretation. The purpose of this article is to review the basic formulation and relate it to the use of step functions for explaining and quantifying supply behavior.

## The Definition of a Step Function

The basic difference, in implied supply behavior, between smooth and stepped functions originates in the definition of a function. The rigorous definition of a function can be stated thus: A function is a subset of ordered pairs such that to each element x in the domain there corresponds a unique element y in the range.[2] A smooth, continuous supply curve usually can clearly meet the definition. A step relation can also meet the rigorous definition if specific pairings are considered. For example, the pairings

* Scientific contribution No. 214, Agricultural Experiment Station, University of Connecticut, Storrs. Some of the ideas for this paper grew out of research efforts in conjunction with the Northeast Dairy Adjustment and Supply Response Project. Appreciation is extended to Ian Hardie and Stanley Seaver for their helpful comments.

[1] Candler's was one of the earliest writings on the subject, although his reference to a step function was a digression from his main topic of variable price programming. Other early published appearances of supply step functions were by Knutson and Cochrane [11] in Febru-

MARVIN KOTTKE *is professor of agricultural economics at the University of Connecticut.*

$$(1) \qquad \begin{aligned} f(x) &= 1 \quad \text{when} \quad 1 < x \leq 3 \\ &= 2 \quad \text{when} \quad 3 < x \leq 5 \\ &= 3 \quad \text{when} \quad 5 < x \leq 8 \end{aligned}$$

form a step function because a single y in the range corresponds to an x in the domain. Notice that there can be more than one x element for each y. Notice also that the step function is discontinuous at $x=3$ and $x=5$.[3] If the pairings in (1) were connected, then at $x=3$ and at $x=5$ the range would have more than a single y element for each x element and, in that case, (1) would be a relation rather than a function.[4]

By definition, therefore, a step function is always discontinuous, whereas a smooth function is usually continuous. The relevance and implications of the discontinuous step function will be the central theme of the discussion that follows.

## The Basic Formulation of the Steps

Conventional supply theory states that the willingness of producers to supply a certain quantity at a certain price depends upon the marginal cost of producing the marginal unit of that certain quantity. The smooth, continuous supply curve, accordingly, represents a continuous series of marginal costs over a range of produceable quantities. A stepped supply function, constructed by linear programming, is based on the same marginality concepts. As Dorfman, Samuelson, and Solow explain: "It would be misleading to contrast the linear programming model with marginal analysis in general. Linear programming *is* marginal analysis, appropriately tailored to the case of a finite number of activities" [7, p. 133].[5]

While the stepped and smooth supply functions have a close kinship because of their marginal analysis parentage, they are not identical twins in terms of their structure and the kind of behavior they represent. In other words, both have the same theoretical basis, but they differ in the combining nature of the relevant variables.

To identify and describe the anatomy of a step supply function, it is

---

ary 1958, Toussaint [18] in June 1958, McPherson and Faris [14] in November 1958, and Ladd and Easley [12] in May 1959.

[2] See Yamane [20, pp. 12–20] or almost any college mathematics textbook.

[3] A rigorous definition of continuity is as follows: A function is continuous at $x=a$ if (1) $f(a)$ exists, and (2) $f(x) \to f(a)$ as $x \to a$. See Richmond [15, pp. 10–12]. A step function meets the first requirement (e.g., at $x=3$ in the above function, $f(3)=1$). But it does not meet the second requirement (e.g., $f(x) \to 2$ as $x \to 3^+$). Therefore, step functions have finite discontinuities.

[4] A relation is defined as a subset of a Cartesian product of ordered pairs, but is not restricted to the condition that the corresponding y element in the range must be unique. Thus, a function is always a relation, but a relation may not be a function.

[5] The incremental units of change differ in size and composition between step and smooth supply functions, but the concept of marginal change applies to both.

helpful to begin with the typical underlying physical relationship of a linear programming problem.[6]

Suppose a firm can produce $x_1$ and $x_2$, subject to

$$b_1 \geq a_{11}x_1 + a_{12}x_2, \tag{2}$$

$$b_2 \geq a_{21}x_1 + a_{22}x_2, \text{ and} \tag{3}$$

$$x_1 \geq 0,\ x_2 \geq 0, \tag{4}$$

where

$x_1$ and $x_2$ are units of production activities (the units may be either outputs or inputs),

$b_1$ and $b_2$ are quantities of fixed resources, and

$a_{11}$, $a_{12}$, $a_{21}$, and $a_{22}$ are resource requirements in units per activity.

Let $x_2$ be the product for which the step function will be derived and assume that the substitution ratios and the feasible production combinations of $x_1$ and $x_2$ are as depicted in Figure 1. Then the possibility of producing various quantities of $x_2$ by substituting for $x_1$ is expressed in the linear equations,

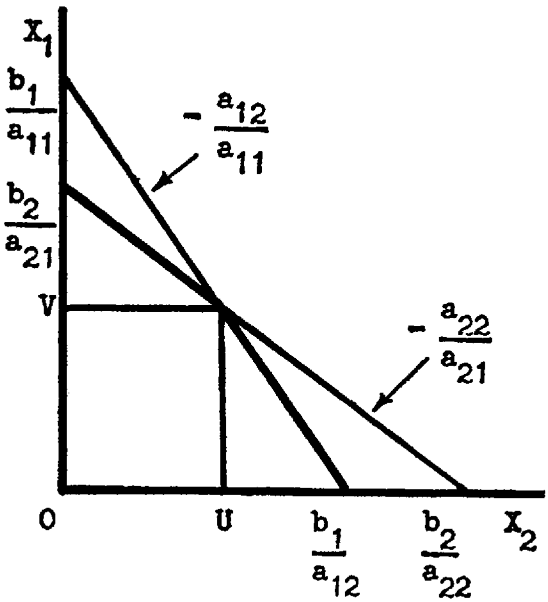


**Figure 1. Production possibilities of a firm**

[6] For a comprehensive, analytical discussion involving the marginal cost approach within the context of a linear programming model, see Wu and Kwang [19]. For a textbook discussion of the construction of a marginal cost function based on a linear programming model, see Baumol [3, pp. 288–291].

(5) $$x_1 = \frac{b_2}{a_{21}} - \frac{a_{22}}{a_{21}} x_2, \qquad 0 \leq x_2 \leq U$$

and

(6) $$x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2.^7 \qquad U \leq x_2 \leq \frac{b_1}{a_{12}}$$

As shown in Figure 1, equations (5) and (6) indicate that a producer could increase his output of $x_2$ up to U level of output at a given marginal substitution rate and that he could further increase output of $x_2$ beyond U to a higher level of output ($b_1/a_{12}$) at a less favorable marginal substitution rate.

The cost of producing $x_2$ includes its own variable cost, plus the opportunity cost of foregoing $x_1$ production. Such combined cost is termed "effective" cost here and written as

(7) $$C_2 = c_2x_2 - (p_1 - c_1)x_1,$$

where

$C_2$ is the "effective" cost of producing $x_2$,
$c_2$ and $c_1$ are the variable costs of producing $x_2$ and $x_1$, respectively, and
$p_1$ is the price of $x_1$.

By substituting[8] (5) and (6) separately into (7) and then differentiating $C_2$ with respect to $x_2$, we obtain the respective marginal costs for each segment:

(8) $$MC_a = c_2 + (p_1 - c_1)\frac{a_{22}}{a_{21}}, \qquad 0 < x_2 \leq U$$

and

(9) $$MC_b = c_2 + (p_1 - c_1)\frac{a_{12}}{a_{11}}.^9 \qquad U < x_2 \leq \frac{b_1}{a_{12}}$$

[7] Equations (5) and (6) are rearrangements of (3) and (2), respectively. The value U is a solution for $x_2$ when (5) and (6) are solved by simultaneous equations to obtain the respective ranges in $x_2$ to which equations (5) and (6) pertain.

[8] The substitution results in the following equations:

$$C_2 = c_2x_2 - (p_1 - c_1)\left[\frac{b_2}{a_{21}} - \frac{a_{22}}{a_{21}} x_2\right], \qquad 0 \leq x_2 \leq U$$

and

$$C_2 = c_2x_2 - (p_1 - c_1)\left[\frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2\right]. \qquad U \leq x_2 \leq \frac{b_1}{a_{12}}$$

[9] If both $x_1$ and $x_2$ were input activities to produce the same product, then $p_1$ would not be involved and equations (8) and (9) would be rewritten with $MC_a$ replacing $p_1$. Then by rearrangement of terms,

$$MC_a - \frac{a_{22}}{a_{21}} MC_a = c_2 - \frac{a_{22}}{a_{21}} c_1. \qquad 0 < x_2 \leq U$$

As indicated by equations (8) and (9), marginal costs are segments of constant values over specific ranges of output. Note that the segment designations now have a discontinuity at U because corner points do not have derivatives. Thus, one part of the anatomy of the steps, the horizontal segment depicted in Figure 2, is identified as marginal cost. The reason for showing the marginal costs as broken lines is that they only provide the skeletal framework underlying a step supply function.

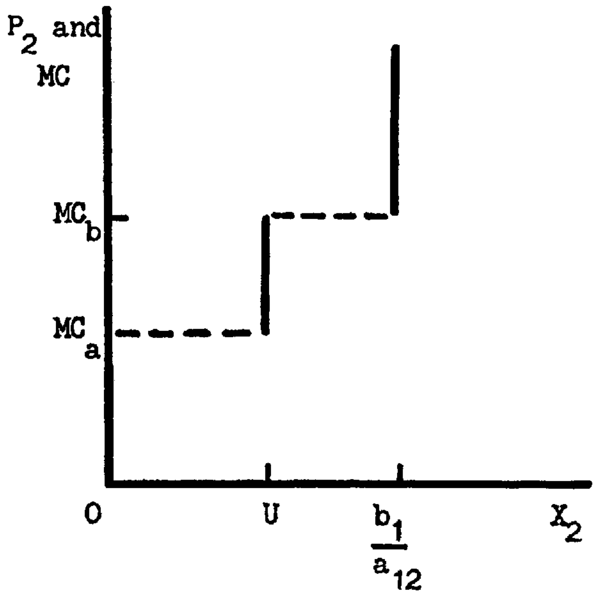


**Figure 2. Step supply function of a firm**

The vertical segment of the steps is identified by introducing $p_2$ (the price of $x_2$) and applying the profit-maximizing criterion of $MR \geq MC$. If a producer's goal is to maximize profits, for the first step he should be willing to produce those quantities where

$$p_2 \geq c_2 + (p_1 - c_1)\frac{a_{22}}{a_{21}} \cdot \qquad 0 < x_2 \leq U \tag{10}$$

Thus, the vertical segment at U output is an optimum solution for a range of $p_2$ prices (Fig. 2). More specifically, the optimum solutions and ranges in prices for all of the steps form the following step function:

$$\begin{aligned} f(p_2) &= 0 && \text{when} \quad 0 \leq p_2 \leq MC_a, \\ &= U && \text{when} \quad MC_a < p_2 \leq MC_b, \\ &= b_1/a_{12} && \text{when} \quad MC_b < p_2. \end{aligned} \tag{11}$$

In this final form, the step supply function consists of vertical segments with horizontal discontinuities. It can meet the definition of a function if its domain and range designations are reversed. The omission of the horizontal segments in (11) is based on an economic interpretation. For example, in (10) the optimum quantity could be $0<x_2\leq U$ when $p_2=MC$; however, it seems reasonable from a resource conservation standpoint that a producer would not employ more resources as long as profit remained constant over a range of output. In other words, producers can maximize profits by operating at the corner points of their feasible production solutions.[10]

The anatomy of stepped supply, therefore, consists basically of two parts. One is the marginal cost function which forms the horizontal segments and the other is the profit-maximizing supply function which forms the vertical segments. Taken together, these two parts form a discontinuous step relation rather than a discontinuous step function. However, economic interpretation stresses the vertical part, with MC camouflaged as an "under the surface" part of the anatomy.

The aggregate stepped supply as formulated via the micro-to-macro approach is a summation of as many equations, in the form of (11), as there are representative firms (appropriately weighted for the firms represented). Underlying this summation concept of aggregate behavior are several restrictive assumptions. One is that each firm's supply behavior is independent of that of the others. A second assumption is that the firms act and respond in unison. All firms are assumed to face the same length of run and to respond to price in perfect unison. Of course, some variation in response may be reflected by the interlacing of several representative firms' steps. However, such interlaced steps do not represent the kind of staggered response among firms that occurs when they face different length-of-run situations. Basically, therefore, the anatomy of an aggregate step supply function has the same parts as the firm step supply function. The horizontal segments are marginal costs and the vertical segments are profit-maximizing supply solutions.

## An Empirical Application

Three steps of a milk supply function, constructed by linear programming, are presented in Table 1 and Figure 3. The three steps are a mid-section of a supply function constructed for a representative southeastern New England dairy farm.

The procedure of construction involved the use of a computer program that provides a print-out of the upper and lower price bounds for an optimum solution.[11] By variable price programming, a series of upper and lower

[10] Another example of optimizing where $MR\neq MC$ has been discussed by French et al. [8] and Brems [4]. Their discussions pertain to discontinuous cost functions of firms.

[11] The procedure used in the previous section to identify the anatomy of the steps runs into computational limits in problems involving more than two activities, but the simplex method used in the empirical application follows the same basic economic concepts.

Table 1. An empirical example of steps of a supply function[a]

| Step | The milk supply function $x_1=f(p_1)$ | | The marginal cost of milk output | |
|---|---|---|---|---|
| | Quantity of milk ($x_1$) | Price of milk per cwt. ($p_1$) | MC per cwt. | Quantity of milk ($x_1$) |
| | *pounds* | *dollars* | *dollars* | *pounds* |
| A | 709,048 | 5.03–5.18 | 5.02 | 630,531–709,048 |
| B | 750,031 | 5.19–5.21 | 5.18 | 709,049–750,031 |
| C | 777,296 | 5.22–5.41 | 5.21 | 750,032–777,296 |

[a] The steps are a portion of a supply function constructed for the Northeast Dairy Adjustment and Supply Response Project. (The function is for representative farm No. 5 in Area 3.)

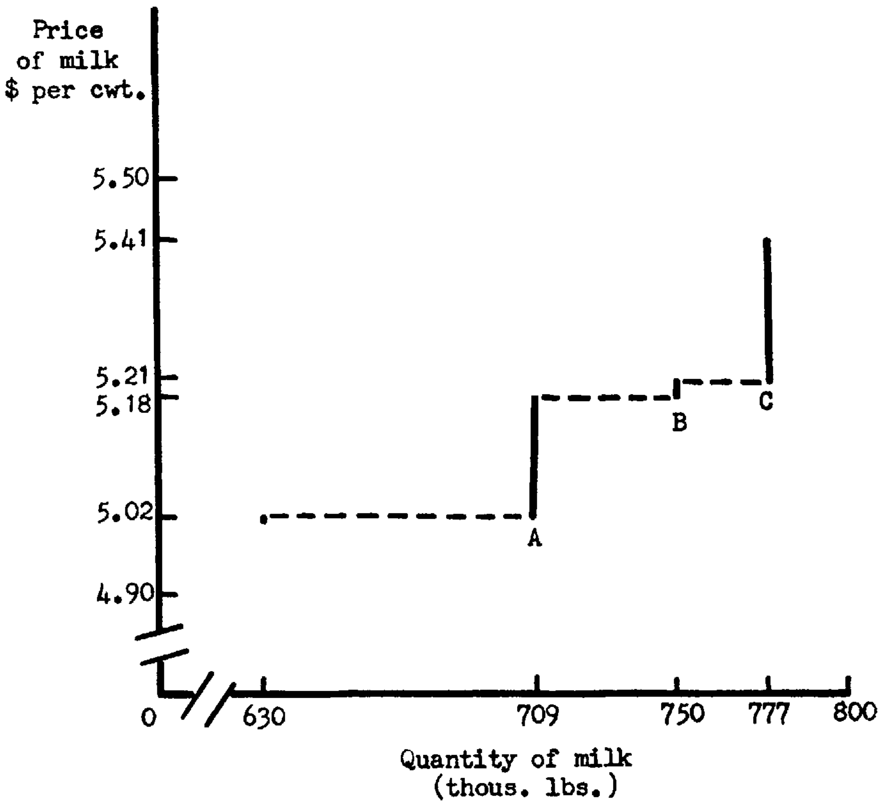


Figure 3. Empirical example of steps of a supply function

bounds was obtained. This procedure directly quantifies an equation of the form expressed in (11). The interval between the upper and lower price bounds corresponds to the vertical segment of a graph of the function. For example, the vertical segment of Step A represents an optimum quantity of

709,048 pounds of milk when the price of milk varies from \$5.03 to \$5.18. Notice that no explicit expression of MC is given in the linear programming solution. However, MC can be calculated from the data in the solution matrix and its form can be verified as being consistent with that expressed in equations (8) and (9).

For example, $MC_a$ (MC of Step A) is a change in all "effective" costs, including opportunity costs between the optimum output solutions of 630,531 and 709,048 divided by the change in quantity between those two points. Accordingly, MC is obtained by selecting the "effective" activities[12] from the linear programming solution matrix and applying the following equation:

$$(12) \quad MC_a = \frac{\sum c_j dx_j - \sum p_j dx_j}{dx_1},$$

$j = 3, 10, 11, 14, 16, 18, 22,$ and $38$ for $c_j$

$j = 42, 65, 66,$ and $67$ for $p_j$

where

$c_j$ is a variable cost of an activity,
$p_j$ is a price of an activity,
$dx_j$ is the quantity change of an activity, and
$dx_1$ is the quantity change of milk output.

Application of the appropriate data[13] to (12) gives the following result:

$$(13) \qquad MC_a = \frac{4132.08 - 187.35}{785.18} = 5.02.$$

Since activity units change in constant proportion to the change in output within the interval, the $MC_a$ is also constant within the interval: i.e., $MC_a$ equals \$5.02 for adding the 630,532nd pound, as well as all other pounds within the interval. The MC's for the other steps are given in Table 1.

Observe that the linear programming solutions indicate that the producer should stay at the beginning of the step when MC=MR and move to the "riser" of the step when MR>MC. For example, when MC=5.18 and $p_1=5.18$, the optimum quantity is 709,048, but if $p_1=5.19$, then the optimum quantity jumps to 750,031. Thus, the empirical function, as given in the first two columns of Table 1, has the same anatomy as the theoretical function presented in the preceding section.

[12] The indices represent the following activities: 3 is variable cows; 10 is grain; 11 is purchased replacements; 16 is building space; 18 is regular hired labor; 22 is interest; 38 is pasture; 42 is heifer-calf sales; 65, 66, and 67 are the reservation prices on operator's labor; and 1 is milk sales.

[13] $\Sigma c_j dx_j = (-2.5)(7.852)+81(3.815)+380(9.952)+96(-8.)+87(7.852)+5.55(26.785)+6(-.562)+.6(.31)$. $\Sigma p_j dx_j = 17.5\ (11.141)+.25(15.604)+.25(-17.176)+.25(-28.952)$.

## Interpretation of Results

The results show how a producer should respond if he (a) starts from the farm's actual situation and goes to an optimum position (from the basic resource position to a point on the function),[14] or (b) starts from one optimum position and goes to another optimum position (from one point to another on the function). Both of these interpretations are subject to the condition that only a single response is involved for the duration of the specified length of run.

The example function in Table 1 states that, at a given price of, say, $5.10, if the producer is not at an optimum output he should respond by adjusting from his actual output to 709,048 pounds. On the other hand, if optimality is assumed to prevail, then a price that starts at $5.10 and rises, say, $0.10, should prompt the producer to respond by increasing output 40,983 pounds (from 709,048 to 750,031). Note that it cannot be said, in general, that a 10-cent change in price should cause a change of 40,983 pounds in output—the starting points have to be specified.

The function, by itself, cannot trace out a series of price responses through a period of time even though the empirical data employed may be time-dated. The example function cannot be used to say what the producer's response should be if the price starts at $5.10, increases to $5.40, and then declines to $5.00. Instead, it applies to a single price response for the duration of the length of run. These conditional price response statements apply similarly to smooth supply functions, except that the specification of the starting point is not as crucial for smooth functions.

A degree of responsiveness can be interpreted for a step supply function, but elasticity cannot be applied as readily as it can to a conventional smooth, continuous supply function. Over a range of prices, the degree of response can vary widely, from no response to large skips or jumps in response. The relative lengths of perfectly inelastic segments and the relative gaps between the vertical segments reflect the degree of producer's responsiveness. It is difficult to generalize such responsiveness into a single elasticity measure. The horizontal segments are particularly deceptive in interpretation. At first glance, they seem to indicate perfect elasticity. However, they may be discontinuities in the function rather than perfectly elastic price responses.

The theoretical step function implies reversibility, but empirical application can introduce irreversibility for some of the steps of a function. The inclusion of a capital or durable input activity in the production matrix is usually done without differentiating between purchase and salvage (disposal) value. Yet the opportunities to acquire a capital asset usually differ

---

[14] See Baker, Langham, and Cowling [1, p. 5] for a report which adequately prefaces the results with this interpretation.

importantly from opportunities to dispose of it. Once a capital input is committed, it is not likely to be reversed unless the same cost basis prevails in the opposite direction.[15] It seems reasonable, then, to acknowledge the possibility of irreversibility for some of the steps of an empirical step function. Empirically derived smooth supply functions can also possess irreversibility, but stepped functions can pinpoint separate reversible and irreversible parts. In the empirical example, Step A can be considered irreversible, because it involves mainly a change in barn space. On the other hand, Step B can be considered reversible, because it involves a change in feeding rates, which can be adjusted easily in either direction.

The empirical steps have been interpreted here for the firm level only. The properties of responsiveness, elasticity, and reversibility (and irreversibility) have essentially the same meaning for the aggregate as they do for the firm; therefore, no numerical example of the aggregate is presented.

## Implications

What kind of supply behavior could the step function capture and approximate? The conditions of a step function imply that producers should respond sporadically rather than steadily to price changes. They should be willing to adjust output by "skips" and "jumps" rather than by "glides" and minute increments. They should operate more frequently where $MR>MC$ than where $MR=MC$. As in the conventional explanation, a price change should trigger a response, but the importance of the magnitude and the duration of the price change is accentuated.

At the aggregate level, according to the micro-to-macro approach, all producers would act largely in unison. Because of the discrete and discontinuous nature of their responses, they would have a tendency to generate market instability. On the one hand, if their aggregate supply function has relatively large output gaps (i.e., long horizontal step segments), then they would be prone to overshoot or undershoot a market demand which falls within the discontinuous output interval. On the other hand, if their aggregate supply function has extremely stable output over wide price ranges (i.e., long vertical step segments), then they would not adjust readily to a changing market demand.

Does the kind of supply behavior implicitly expressed by step functions approximate reality, or, in the normative context, is it the kind that should exist? While a verified answer to this question is beyond the scope of this article, it seems reasonable to assume that it is a valid approximation of agricultural supply behavior, particularly at the firm level. Improvements are needed in research methodology to incorporate the interdependencies of

[15] See McKee and Loftsgard [13] for a discussion of irreversibility due to capital inputs and, also, of the relevance of price duration.

firms and the diversity of length-of-run situations at the aggregate level.[16]

If the step function can be accepted as a valid approximation of supply behavior, then is it appropriate to smooth the steps for data manipulation and presentation purposes? The answer depends upon how much of the implied sporadic and discontinuous behavior needs to be retained for a valid interpretation of the results. A careful evaluation needs to be made of each case, particularly of those aggregate supply functions with long horizontal segments. Takayama and Judge [17] have proposed a procedure which avoids the necessity of smoothing for spatial equilibrium analysis. At the firm level, much of the intrinsic behavior, once captured in a step function, would be obliterated by the smoothing of the steps. Consequently, it seems advisable that steps be retained for most purposes, especially at the firm level.

Undoubtedly, further improvements will be made in the methods of step supply analysis, but the basic anatomy of a step function will continue to apply and give an interpretative basis.

## References

[1] BAKER, C. B., M. R. LANGHAM, AND K. G. COWLING, *Programmed Production Responses on Dairy Farms in Northeastern Illinois*, Illinois Agr. Exp. Sta. Bul. 708, March 1965.

[2] BARKER, RANDOLPH, "Appropriate Methods for Estimating the Short-Run Elasticity of Supply for Milk," *J. Farm Econ.* 47:840–843, Aug. 1965.

[3] BAUMOL, WILLIAM J., *Economic Theory and Operations Analysis*, 2nd ed., Prentice-Hall, 1965.

[4] BREMS, HANS, "A Discontinuous Cost Function," *Am. Econ. Rev.* 42:577–586, Sept. 1952.

[5] BURT, OSCAR R., "Curve Fitting to Step Functions," *J. Farm Econ.* 46:662–672, Aug. 1964.

[6] CANDLER, WILFRED, "A Modified Simplex Solution for Linear Programming with Variable Prices," *J. Farm Econ.* 39:409–428, May 1957.

[7] DORFMAN, ROBERT, PAUL A. SAMUELSON, AND ROBERT M. SOLOW, *Linear Programming and Economic Analysis*, New York, McGraw-Hill Book Company, Inc., 1958.

[8] FRENCH, B. C., L. L. SAMMETT, AND R. G. BRESSLER, "Economic Efficiency in Plant Operations with Special Reference to the Marketing of California Pears," *Hilgardia* 24:552–554, July 1956.

[9] GATES, J. M., AND M. W. KOTTKE, *The Dynamics of Milk Supply in Southeast New England*, Connecticut Agr. Exp. Sta. Bul. 395, March 1966.

[10] KELLEY, PAUL L., AND DALE A. KNIGHT, "Short-Run Elasticities of Supply for Milk," *J. Farm Econ.* 47:93–104, Feb. 1965.

[11] KNUTSON, A. C., AND W. W. COCHRANE, "A Supply Function for Flax at the Firm Level," *J. Farm Econ.* 40:117–123, Feb. 1958.

[12] LADD, G. W., AND E. V. EASLEY, *An Application of Linear Programming to the Study of Supply Responses in Dairying*, Iowa State University Bul. 467, May 1959.

[13] MCKEE, D. E., AND L. D. LOFTSGARD, "Programming Intra-Farm Normative Supply Functions," in *Agricultural Supply Functions*, ed. Earl O. Heady et al., Ames, Iowa State University Press, 1961, pp. 159–162.

[16] The problems of aggregation have been discussed rather thoroughly recently. See Stovall [16] for a recent summary evaluation of aggregation procedures. The blending of different lengths of run by use of a combination of a Markov chain process and recursive programming has shown promise for developing a more meaningful aggregate step supply function. For an application, see Gates and Kottke [9].

[14] McPHERSON, W. W., AND J. E. FARIS, " 'Price Mapping' of Optimum Changes in Enterprises," *J. Farm Econ.* 40:821–834, Nov. 1958.
[15] RICHMOND, D. E., *Introductory Calculus*, Reading, Mass., Addison-Wesley Publishing Company, Inc., 1959.
[16] STOVALL, JOHN G., "Sources of Error in Aggregate Supply Estimates," *J. Farm Econ.* 48:477–480, May 1966.
[17] TAKAYAMA, T., AND G. G. JUDGE, "An Interregional Activity Analysis Model for the Agricultural Sector," *J. Farm Econ.* 46:349–365, May 1964.
[18] TOUSSAINT, W. D., "Programming Optimum Firm Supply Functions," in *Farm Size and Output Research—A Study in Research Methods*, Southern Cooperative Ser. Bul. 56, June 1958, pp. 62–72.
[19] WU, YUAN-LI, AND CHING-WEN KWANG, "An Analytical and Graphical Comparison of Marginal Analysis and Mathematical Programming in the Theory of the Firm," in *Linear Programming and the Theory of The Firm*, ed. Kenneth E. Boulding and W. Allen Spivey, New York, The Macmillan Company, 1960, Chap 4.
[20] YAMANE, TARO, *Mathematics for Economists*, Englewood Cliffs, N. J., Prentice-Hall, Inc., 1962.