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Lorentz Invariance and Normal Product in an Interacting System

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It is shown that from requirement of the Lorentz invariance any vector and tensor operators composed of interacting field variables should be written in a normal product, in the sense of zero vacuum expectation value, except for the term constant $\times \delta_{\mu\nu}$ for the tensor operator.

§ 1. Introduction and results

In the preceding paper, we have shown that the energy-momentum tensor of a free complex scalar field with a correct transformation property about the Lorentz group should be written in a normal product.[1] In this paper we shall extend this result to interacting cases and to more general local operators.

First we shall discuss the vector operators given by

$$j_\mu^{(0)}(x) = V_\mu(x) S(x) \tag{1·1}$$

and

$$J_\mu^{(0)}(x) = \bar{\varphi}(x) \gamma_\mu \psi(x), \tag{1·2}$$

where V_μ, S and $\varphi(\psi)$ are arbitrary vector, scalar and spinor operators. Our conclusion on the vector operators is that, for (1·1) and (1·2) to have a correct transformation property as a vector, the product must be a normal product in the sense of zero vacuum expectation value.

Next we shall discuss the tensor operators given by

$$t_{\mu\nu}^{(0)}(x) = V_\mu(x) \frac{\partial}{\partial x_\nu} S(x) \tag{1·3}$$

and

$$T_{\mu\nu}^{(0)}(x) = \bar{\varphi}(x) \gamma_\mu \frac{\partial}{\partial x_\nu} \psi(x). \tag{1·4}$$

We shall prove that the vacuum expectation values of (1·3) and (1·4) must be zero except for the term constant $\times \delta_{\mu\nu}$. It is obvious that we cannot exclude this term from the Lorentz invariance alone.

We do not discuss the scalar operators given by the product of two local operators, since we cannot exclude the constant term. We also do not discuss the tensor operator given by

$$\bar{\varphi}(x)\gamma_\mu\gamma_\nu\psi(x). \tag{1·5}$$

The vacuum expectation value of (1·5) is constant $\times\delta_{\mu\nu}$, then as far as the vacuum expectation value is concerned, there is no essential difference between the product (1·5) and Lorentz scalar.

§ 2. Proof

The course of the proof is exactly the same as that of the preceding paper.[1] Instead of the products (1·1)$\sim$(1·4) we define

$$j_\mu^{(1)}(x)=\lim_{\Delta\to 0} V_\mu(x)S(x+\Delta), \tag{2·1}$$

$$J_\mu^{(1)}(x)=\lim_{\Delta\to 0} \bar{\varphi}(x)\gamma_\mu\psi(x+\Delta), \tag{2·2}$$

$$t_{\mu\nu}^{(1)}(x)=\lim_{\Delta\to 0} V_\mu(x)\frac{\partial}{\partial x_\nu}S(x+\Delta), \tag{2·3}$$

$$T_{\mu\nu}^{(1)}(x)=\lim_{\Delta\to 0} \bar{\varphi}(x)\gamma_\mu\frac{\partial}{\partial x_\nu}\psi(x+\Delta), \tag{2·4}$$

Δ being the vector having only space components. To avoid the complications due to the possible dependence on the direction of Δ, first we only discuss the transformation properties under the infinitesimal Lorentz group without rotation

$$a_{\mu\nu}=\delta_{\mu\nu}+\epsilon_{\mu\nu}. \tag{2·5}$$

Corresponding quantities in the new reference are

$$j_\mu^{(1)\prime}(x')=\lim_{\Delta\to 0} V_\mu'(x')S'(x'+\Delta), \tag{2·6}$$

$$J_\mu^{(1)\prime}(x')=\lim_{\Delta\to 0} \bar{\varphi}'(x')\gamma_\mu\psi'(x'+\Delta), \tag{2·7}$$

$$t_{\mu\nu}^{(1)\prime}(x')=\lim_{\Delta\to 0} V_\mu'(x')\frac{\partial}{\partial x_\nu'}S'(x'+\Delta), \tag{2·8}$$

$$T_{\mu\nu}^{(1)\prime}(x')=\lim_{\Delta\to 0} \bar{\varphi}'(x')\gamma_\mu\frac{\partial}{\partial x_\nu'}\psi'(x'+\Delta), \tag{2·9}$$

where

$$x_\mu'=a_{\mu\nu}x_\nu, \tag{2·10}$$

$$S'(x')=S(x), \tag{2·11}$$

$$V_\mu'(x')=a_{\mu\nu}V_\nu(x) \tag{2·12}$$

and

$$\varphi'(x') = \Lambda\varphi(x), \tag{2·13}$$

the same for ψ, with

$$\Lambda^{-1}\gamma_\mu\Lambda = a_{\mu\nu}\gamma_\nu . \tag{2·14}$$

At this stage it is important to notice that in (2·6)$\sim$(2·9) we must use Δ (not Δ') because otherwise the definition of those quantities depends on the Lorentz frames.

Using (2·10)$\sim$(2·14), we can rewrite the quantities (2·6)$\sim$(2·9) in the new Lorentz frame by those in the old Lorentz frame:

$$j_\mu^{(1)'}(x') = a_{\mu\nu}j_\nu^{(1)}(x) - \lim_{\Delta\to 0}\epsilon_{4i}\Delta_i V_\mu(x)\frac{\partial}{\partial x_4}S(x+\Delta), \tag{2·15}$$

$$J_\mu^{(1)'}(x') = a_{\mu\nu}J_\nu^{(1)}(x) - \lim_{\Delta\to 0}\epsilon_{4i}\Delta_i\bar{\varphi}(x)\gamma_\mu\frac{\partial}{\partial x_4}\psi(x+\Delta), \tag{2·16}$$

$$t_{\mu\nu}^{(1)'}(x') = a_{\mu\rho}a_{\nu\sigma}t_{\rho\sigma}^{(1)}(x) - \lim_{\Delta\to 0}\epsilon_{4i}\Delta_i V_\mu(x)\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_4}S(x+\Delta), \tag{2·17}$$

$$T_{\mu\nu}^{(1)'}(x') = a_{\mu\rho}a_{\nu\sigma}T_{\rho\sigma}^{(1)}(x) - \lim_{\Delta\to 0}\epsilon_{4i}\Delta_i\bar{\varphi}(x)\gamma_\mu\frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x_4}\psi(x+\Delta). \tag{2·18}$$

The second terms in (2·15)$\sim$(2·18) are not zero because of the singularities of[2)]

$$\langle V_\mu(x)S(y)\rangle_0 = \frac{1}{2}\int_0^\infty dm^2\rho(m^2)\frac{\partial}{\partial x_\mu}\Delta^{(1)}(x-y, m^2), \tag{2·19}$$

$$\langle\bar{\varphi}(x)\psi(y)\rangle_0 = \frac{1}{2}\int_0^\infty dm^2\left[\rho_1(m^2)+\rho_2(m^2)\gamma_\mu\frac{\partial}{\partial x_\mu}\right]\Delta^{(1)}(x-y, m^2) \tag{2·20}$$

at $x-y\to 0$, where $x-y$ is a space-like vector. Therefore we must add to (2·1)$\sim$(2·4) some counter terms so as to assure the correct transformation properties as the vector or tensor operators.

After the straightforward calculations (details are seen in the preceding paper[1)]) we can easily see that

$$j_\mu^{(2)}(x) = \lim_{\Delta\to 0}V_\mu(x)\left[1+\frac{5}{3}\Delta\cdot\nabla+\frac{1}{3}(\Delta\cdot\nabla)^2\right]S(x+\Delta), \tag{2·21}$$

$$J_\mu^{(2)}(x) = \lim_{\Delta\to 0}\bar{\varphi}(x)\gamma_\mu\left[1+\frac{5}{3}\Delta\cdot\nabla+\frac{1}{3}(\Delta\cdot\nabla)^2\right]\psi(x+\Delta), \tag{2·22}$$

$$t_{\mu\nu}^{(2)}(x) = \lim_{\Delta\to 0}\left[V_\mu(x)\left(1+\frac{1}{4}\Delta\cdot\nabla\right)\frac{\partial}{\partial x_\nu}S(x+\Delta) - \frac{1}{4}\frac{\Delta_\mu\Delta_\nu}{\Delta^2}V_\lambda(x)\Delta\cdot\nabla\frac{\partial}{\partial x_\lambda}S(x+\Delta)\right], \tag{2·23}$$

$$T_{\mu\nu}^{(2)}(x)=\lim_{\Delta\to 0}\bar{\varphi}(x)\left[\gamma_\mu\frac{\partial}{\partial x_\nu}\left(1+\frac{1}{4}\Delta\cdot\nabla\right)\right.$$
$$\left.-\frac{1}{4}\gamma_\lambda\frac{\partial}{\partial x_\lambda}\frac{\Delta_\mu\Delta_\nu}{\Delta^2}\Delta\cdot\nabla\right]\psi(x+\Delta) \qquad (2\cdot 24)$$

have the correct transformation properties. The vacuum expectation values of $(2\cdot 21)\sim(2\cdot 24)$ are

$$\langle j_\mu^{(2)}(x)\rangle_0=0\,, \qquad (2\cdot 25)$$
$$\langle J_\mu^{(2)}(x)\rangle_0=0\,, \qquad (2\cdot 26)$$
$$\langle t_{\mu\nu}^{(2)}(x)\rangle_0=-\delta_{\mu\nu}\frac{1}{32\pi^2}\lim_{\Delta\to 0}\int_0^\infty dm^2\rho(m^2)m^4$$
$$\times\left[\frac{2}{m^2\Delta^2}+\log\frac{m\Delta}{2}-\frac{1}{2}+\gamma\right], \qquad (2\cdot 27)$$
$$\langle T_{\mu\nu}^{(2)}(x)\rangle_0=-\delta_{\mu\nu}\frac{1}{32\pi^2}\lim_{\Delta\to 0}\int_0^\infty dm^2\rho_2(m^2)m^4$$
$$\times\left[\frac{2}{m^2\Delta^2}+\log\frac{m\Delta}{2}-\frac{1}{2}+\gamma\right], \qquad (2\cdot 28)$$

γ being the Euler constant. Thus the vacuum expectation values of these operators are zero except for the term constant $\times\delta_{\mu\nu}$ for the tensors. Here we emphasize that the coefficients of $\delta_{\mu\nu}$ in $(2\cdot 27)$ and $(2\cdot 28)$ have no special meaning. These coefficients depends on a special choice of the counter terms in $(2\cdot 23)$ and $(2\cdot 24)$. The only important conclusion is that $\langle t_{\mu\nu}^{(2)}(x)\rangle_0$ and $\langle T_{\mu\nu}^{(2)}(x)\rangle_0$ are in the form constant $\times\delta_{\mu\nu}$. Since the Δ dependences of these quantities appear only in the vacuum expectation values and $(2\cdot 25)\sim(2\cdot 28)$ explicitly show that there are no dependence on the direction of Δ, $j_\mu^{(2)}(x)$, $J_\mu^{(2)}(x)$, $t_{\mu\nu}^{(2)}(x)$ and $T_{\mu\nu}^{(2)}(x)$ defined by $(2\cdot 21)\sim(2\cdot 24)$ have the correct transformation properties under the full Lorentz transformation.

References

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2) G. Källén, Helv. Phys. Acta **25** (1952), 417.
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