Pseudo-Potential Games*

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preliminary: July 2004

Abstract

The notion of pseudo-potential game is due to Dubey, Haimanko and Zapechelnyuk (2002). We characterize pseudo-potential games by the absence of strict best-response cycles. The relation with other classes of potential games is discussed. We show that two-player games with strategic complements/substitutes are best-response potential games and hence pseudo-potential games. We provide a counter-example for three-players. However, we show why n-player games with strategic complements/substitutes and general aggregation of actions are pseudo-potential games. Finally, for countable games we provide a simple method for constructing a pseudo-potential function.

JEL-Classifications: C72.

Keywords: potential games, best-response potential, existence of Nash equilibrium, best-response learning, strategic complements, strategic substitutes, aggregation.

*I thank Peter Borm for drawing my attention to Mark Voorneveld's work on best-response potentials, and Jörg Oechssler and Frank Riedel for helpful discussions. Financial support by the DFG through the SFB/TR 15 C4 is gratefully acknowledged.

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1 Introduction

Many notions of potential games imply some of the following three attractive features: (i) They possess a pure strategy Nash equilibrium. (ii) Some adaptive learning processes reach a Nash equilibrium. (iii) Some adaptive learning processes reach a Nash equilibrium after finite steps. Hence, potential functions are useful for problems of existence, learning, robustness, and equilibrium selection. A potential function can be thought of a function that replaces the preferences of players in a game but preserves some of the game's structure such as Nash equilibria, best-responses, better-responses etc. Instead maximizing n preferences we may maximize just one potential function.

Rosenthal (1973) used the potential approach to show that every congestion game possesses a pure-strategy Nash equilibrium. Milchtaich (1996) studied a related class of games with the same method. Monderer and Shapley (1996) developed various notions of potentials such as exact, weighted, ordinal and generalized ordinal potential (see section 4) and provided a characterization of exact potential games by the property that payoffs along cycle sum to zero. They also showed that finite generalized ordinal potential games are characterized by the property that a sequential myopic better-response dynamics leads to Nash equilibrium after finite steps. Voorneveld and Norde (1997) showed that ordinal potential games are characterized by the absence of better-reply cycles and an order condition on the action space. Those notions of potentials concern better-responses but what matters for Nash equilibrium and some learning processes are best-responses. Consequently, Voorneveld (2000) introduced the notion of best-response potential and characterized best-response potential games by the absence of best-response cycles and an order condition on the action space. Recently, Dubey, Haimanko and Zapechelnyuk (2002) introduced the notion of pseudo-potential functions and showed that games with strategic substitutes or complements and with aggregation of opponents' actions have a pseudo-potential that can be used to show existence of equilibrium in this important class of games that comprises of Cournot oligopoly (for an application of pseudo-potentials to Cournot oligopoly see for example Schipper, 2004b), Bertrand oligopoly, rent seeking, coordination games, some public goods games, common pool resource games, search games, tournaments etc. For games with strategic complements, existence of equilibrium is conventionally shown with Tarski's fixed point theorem (e.g., see Topkis, 1998). However, games with strategic substitutes required previously a very different treatment of existence (i.e., Novshek, 1985, Kukushkin, 1994). Thus the pseudo-potential approach provides an alternative, simple, and unified route to establish existence of equilibrium in games with strategic complements or strategic substitutes, and aggregation of opponents' actions.

A result related to Dubey, Haimanko and Zapechelnyuk (2002) is Dinoš and Mezzetti (2003) who showed that a better-response dynamics in quasi-concave n-player games with aggregation converges to Nash equilibrium if actions are either strategic substitutes or complements around Nash equilibrium. Research on existence of equilibria in (infinite) generalized ordinal potential games has been done by Kukushkin (1999, 2002). Ui (2001) and Morris and Ui (2002) focus on robustness of equilibria and potentials.

Dubey, Haimanko and Zapechelnyuk (2002) do not provide a characterization of pseudo-potential games. Their important result should also deserve a clarification of why games with strategic substitutes or complements and aggregation are pseudo-potential games. It is also not clear how pseudo-potential games relate to other notions of potential games, and how pseudo-potential functions can be constructed. In this article we answer these questions. First, we provide a characterization of pseudo-potential games by the absence of strict best-response cycles and an order condition on the action space (Section 3). This order condition is automatically satisfied if either the action space is countable or a subset of the Euclidian space. Second, we show that any exact, weighted, ordinal, generalized ordinal and best-response potential game is a pseudo-potential game (Section 4). Third, we show that two-player games with strategic substitutes or complements are best-response potential games, and thus pseudo-potential games (Section 5). We show with a counter-example that this result fails for more than 2 players. Fourth, we show that n-player games with strategic substitutes or complements and with general aggregation of opponents' actions are pseudo-potential games since they do not contain any strict best-response cycles (Section 6). Finally, for countable pseudo-potential games we provide a simply method to find a pseudo-potential function by the incidence function and the Möbius inverse (Section 7).

2 Pseudo-Potential Games

We consider a strategic game $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ with $N = \{1, ..., n\}$ being the finite set of players, A_i is the set of actions of player $i \in N$, and for all $i \in N$ a preference relation $\succeq_i$ on $A = \times_{i \in N} A_i$. Let $A_{-i} := \times_{j \in N \setminus \{i\}} A_j$ and denote a typical element by $a_{-i} \in A_{-i}$. The (pure-action) best-response correspondence is defined by $b_i(a_{-i}) := \{a_i \in A_i : (a_i, a_{-i}) \succeq_i (a'_i, a_{-i}), \forall a'_i \in A_i\}$.

Definition 1 (Pseudo-Potential Game) *A strategic game $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ is a pseudo-potential game if there exists a (pseudo-potential) function $P : A \longrightarrow \mathbb{R}$ such that for all $i \in N$, for all $a_{-i} \in A_{-i}$,*

$$b_i(a_{-i}) \supseteq \arg\max_{a_i \in A_i} P(a_i, a_{-i}). \tag{1}$$

For each player, it suffices to maximize P in order to get some best-reply.

By the definition of Nash equilibrium we have immediately the following result:

Proposition 1 *Let $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ be a strategic game with pseudo-potential P.*

(i) If P has a maximum over A (e.g., if A is finite), then Γ has a Nash equilibrium.

(ii) The set of Nash equilibria of the game $\Gamma' = \langle N, (A_i), (P) \rangle$ is a subset of the set of Nash equilibria of the game $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$.

We define $\succ_i$ by $a \succ_i a'$ if and only if $a \succeq_i a'$ and not $a' \succeq_i a$. A sequential myopic best-response process is a sequence of combinations of actions in which at each step exactly one player is allowed to change his action, and this player moves to a best-response to his opponents' actions.

Proposition 2 *Let $\Gamma'' = \langle N, (A_i), (\succ_i) \rangle$ be a finite strategic game (with generic preferences) with pseudo-potential P. Then a sequential myopic best-response process reaches a Nash equilibrium after finite steps.*

Proof. P can not have infinitely many strict increases on a finite domain. □

3 Characterization

The characterization is analog to Voorneveld (2000) for best-response potential games, which itself is analog to Voorneveld and Norde (1997) for ordinal potential games.

A *path* in the strategy space A is a sequence $(a^1, a^2, ...)$ of elements $a^k \in A$ such that for all $k = 1, 2, ...$ the strategy profile a^k and a^{k+1} differ in exactly one component, e.g., the $i(k)$th component.

A path is *strict best-response compatible* if the deviating player moves to a best-response unless he plays already a best-response in which case he stays with it, i.e., for all $k = 1, 2, ...$,

$$a_{i(k)}^{k+1} \in \begin{cases} \{a_{i(k)}^k\} & \text{if} \quad a_{i(k)}^k \in b_{i(k)}(a_{-i(k)}^k) \\ b_{i(k)}(a_{-i(k)}^k) & \text{otherwise} \end{cases} \tag{2}$$

Note that the rule (2) is equivalent to the following two requirements

$$a_{i(k)}^{k+1} \in b_{i(k)}(a_{-i(k)}^k) \tag{3}$$

$$a_{i(k)}^{k+1} \neq a_{i(k)}^k \Longleftrightarrow (a_{i(k)}^{k+1}, a_{-i(k)}^k) \succ_{i(k)} (a_{i(k)}^k, a_{-i(k)}^k) \tag{4}$$

On a strict best-response compatible path, if players switch their action then it is a strict improvement over the current action, i.e., either $a_{i(k)}^{k+1} = a_{i(k)}^k$ (hence $a^{k+1} = a^k$) or $(a_{i(k)}^{k+1}, a_{-i(k)}^k) \succ_{i(k)} (a_{i(k)}^k, a_{-i(k)}^k)$ for $k = 1, 2, ...$ Note that the constant path $(a, ..., a)$ is strict best-response compatible.

A finite path $(a^1, ..., a^m)$ is called a *strict best-response cycle* if it is strict best-response compatible, $a^1 = a^m$, and $(a_{i(k)}^{k+1}, a_{-i(k)}^k) \succ_{i(k)} (a_{i(k)}^k, a_{-i(k)}^k)$ for some $k \in \{1, ..., m-1\}$.

We define a binary relation $\rightarrow$ on the action space A by $a' \rightarrow a''$ if there exists a strict best-response compatible path from a' to a'', i.e., there is a strict best-response compatible path $(a^1, ..., a^m)$ with $a^1 = a'$ and $a^m = a''$. Note that for all $a' \in A$, $a' \rightarrow a'$ (reflexivity) and for all $a', a'', a''' \in A$, if $a' \rightarrow a''$ and $a'' \rightarrow a'''$ then $a' \rightarrow a'''$ (transitivity).

The binary relation $\leftrightarrow$ on A is defined by $a' \leftrightarrow a''$ if $a' \rightarrow a''$ and $a'' \rightarrow a'$. Note that $\leftrightarrow$ is (i) reflexive, (ii) transitive, and (iii) symmetric (i.e., $a' \leftrightarrow a''$ if and only if $a'' \leftrightarrow a'$). Hence $\leftrightarrow$ is an equivalence relation. We denote the equivalence classes with respect to $\leftrightarrow$ by $[a]$, i.e., $[a] := \{a' \in A | a \leftrightarrow a'\}$. Let $A_{\leftrightarrow}$ be the set of equivalence classes. Define a binary relation $\lhd$ on this set of equivalence classes by $[a'] \lhd [a'']$ if $[a'] \neq [a'']$ and $a' \rightarrow a''$. This relation is well defined, the choice of the representative in the equivalence class is of no concern, i.e., for all $a', a'', a^{\bullet}, a^{\bullet\bullet} \in A$ with $a' \leftrightarrow a''$ and $a^{\bullet} \leftrightarrow a^{\bullet\bullet}$: $a' \rightarrow a^{\bullet}$ if and only if $a'' \rightarrow a^{\bullet\bullet}$. Note that $\lhd$ is irreflexive and transitive.

Since strict best-response cycles plays an important role in our characterization of pseudo-potential games, some characterizations of the absence of strict best-response cycles may be helpful. To this end, some further general definitions are required. $\langle S, \preccurlyeq \rangle$ is a partially ordered set if $\preccurlyeq$ is reflexive, transitive, and antisymmetric (i.e., $s \preccurlyeq s'$ and $s' \preccurlyeq s$ implies $s = s'$). $\langle S, \preccurlyeq \rangle$ is a chain if for all $s, s'' \in S$, either $s \preccurlyeq s'$ or $s' \preccurlyeq s$, i.e., there are no unordered pairs of elements in S.

Lemma 1 *Suppose a strategic game $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$. Then the following statements are equivalent:*

(i) A contains no strict best-response cycle.

(ii) $\langle A, \rightarrow \rangle$ is a partially ordered set.

(iii) Each $[a] \in A_{\leftrightarrow}$ is a singleton.

Proof. (i) implies (ii): By definition $\rightarrow$ is reflexive and transitive. To show antisymmetry, suppose by the way of contradiction that A contains no best-response cycles and $\rightarrow$ is symmetric. If $\rightarrow$ is symmetric then $a \rightarrow a'$ and $a' \rightarrow a$ for all $a, a' \in A$. If A contains no best-response cycles then there do not exists $a, a' \in A$ with $a \neq a'$ for which $a \rightarrow a'$ and $a \rightarrow a'$. Hence $a = a'$, contradicting that $\rightarrow$ is symmetric.

(ii) implies (iii): Let $a', a'' \in [a]$ with $a' \neq a''$. Since $[a] := \{a''' \in A | a \leftrightarrow a'''\}$ we have by definition of $\leftrightarrow$ that $a' \rightarrow a''$ and $a'' \rightarrow a'$. If $\langle A, \rightarrow \rangle$ is a partially ordered set, then $\rightarrow$ is antisymmetric. Hence $a' = a''$, a contradiction.

(iii) implies (i): Suppose there is a strict best-response cycle $(a^1, ..., a^m, ..., a^1)$ with $a^1 \in [a]$ and $a^m \in [a']$. Then $a^1 \rightarrow a^m$ and $a^m \rightarrow a^1$, thus $a^1 \leftrightarrow a^m$. Hence $a^1, a^m \in [a^1]$. Since $[a^1]$ is a singleton we must have $a^1 = a^m$, a contradiction to $(a^1, ..., a^m, ..., a^1)$ being a strict best-response cycle. □

Lemma 2 *The path $(a^1, a^2, ...)$ is a strict best-response compatible path of a strategic game $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$ if and only if it is a chain of the partially ordered set $\langle A, \rightarrow\rangle$.*

Proof. "$\Rightarrow$": Suppose there is a strict best-response path $(a^1, a^2, ...)$ that is not a chain. Being not a chain implies that there are $a^k, a^{k+1} \in (a^1, a^2, ...)$ that are unordered. Then there is no strict best-response path between a^k and a^{k+1}, a contradiction.

"$\Leftarrow$": Suppose $(a^1, a^2, ...)$ is a chain with $a^k \rightarrow a^{k+m}$ for any $a^k, a^{k+m} \in (a^1, a^2, ...)$. If both $a^k \rightarrow a^{k+m}$ and $a^{k+m} \rightarrow a^k$ then since $\langle A, \rightarrow\rangle$ is a partially ordered set $a^k = a^{k+m}$ (antisymmetry). Note, that any constant path is strict best-response compatible. Otherwise, if $a^k \rightarrow a^{k+m}$ and not $a^{k+m} \rightarrow a^k$, then by definition of $\rightarrow$ there is a strict best-response compatible path from a^k to a^{k+m}. □

The tuple $\langle S, \preccurlyeq\rangle$ is *properly ordered* if there exists a function $F : S \longrightarrow \mathbb{R}$ that preserves the order $\preccurlyeq$, i.e., for all $s', s'' \in S : s' \preccurlyeq s''$ implies $F(s') < F(s'')$.

Theorem 1 (Sufficiency) *Suppose a strategic game $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$. If*

(i) A contains no strict best-response cycle, and

(ii) $\langle A_{\leftrightarrow}, \lhd\rangle$ is properly ordered,

then Γ is a pseudo-potential game.

Proof. Assume that (i) and (ii) hold. Since $\langle A_{\leftrightarrow}, \lhd\rangle$ is properly ordered, there exists a function $F : A_{\leftrightarrow} \longrightarrow \mathbb{R}$ that preserves the order $\lhd$. Define $P : A \longrightarrow \mathbb{R}$ by $P(a) = F([a])$ for all $a \in A$.

Let $a_i \in \arg\max_{a_i \in A_i} P(a_i, a_{-i})$ and suppose there exists $a'_i \in A_i \backslash \{a_i\}$ with $(a_i, a_{-i}) \rightarrow (a'_i, a_{-i})$. By the absence of strict best-response cycles, not $(a'_i, a_{-i}) \rightarrow (a_i, a_{-i})$. Hence $[(a_i, a_{-i})] \lhd [(a'_i, a_{-i})]$, which implies $P(a_i, a_{-i}) = F([a_i, a_{-i}]) < F([a'_i, a_{-i}]) = P(a'_i, a_{-i})$, contradicting that $a_i \in \arg\max_{a_i \in A_i} P(a_i, a_{-i})$. Hence for all $i \in N$, for all $a_{-i} \in A_{-i}$: $b_i(a_{-i}) \supseteq \arg\max_{a_i \in A_i} P(a_i, a_{-i})$. □

We say that A contains no strict best-response cycle for $\Gamma' = \langle N, (A_i), (P)\rangle$ if it does not contain any strict best-response cycle based on the pseudo-potential best-responses $\arg\max_{a_i \in A_i} P(a_i, a_{-i})$ for all $i \in N$. Following remark follows immediately from the definition of Pseudo-potential.

Remark 1 *If for $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$, A contains no strict best-response cycle and $\langle A_{\leftrightarrow}, \lhd\rangle$ is properly ordered, then A contains no strict best-response cycle for $\Gamma' = \langle N, (A_i), (P)\rangle$.*

Theorem 2 (Necessity) *Suppose a strategic game* $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$*. If* Γ *is a pseudo-potential game with pseudo-potential* P*, then for* $\Gamma' = \langle N, (A_i), (P) \rangle$

(i) A *contains no strict best-response cycle, and*

(ii) $\langle A_{\leftrightarrow}, \lhd \rangle$ *is properly ordered.*

Proof. Assume that P is a pseudo-potential for Γ. (i) Suppose by the way of contradiction that $(a^1, ..., a^m)$ is a strict best-response cycle with respect to the game $\langle N, (A_i), (P) \rangle$. Then $(a^1, ..., a^m)$ is a strict best-response path. Suppose $P(a^{k+1}) < P(a^k)$ for some $k \in \{1, ..., m-1\}$. Then $a^{k+1}_{i(k)} \notin \arg\max_{a_i \in A_{i(k)}} P(a_i, a^k_{-i(k)})$, a contradiction. Hence $P(a^{k+1}) \geq P(a^k)$ for all $k \in \{1, ..., m-1\}$. Since $(a^1, ..., a^m)$ is a strict best-response cycle with respect to the game $\langle N, (A_i), (P) \rangle$ we have $P(a^{k+1}) > P(a^k)$ for some $k \in \{1, ..., m-1\}$. Conclude $P(a^1) < P(a^m) = P(a^1)$, a contradiction to the existence of a strict best-response cycle.

(ii) Define $F : A_{\leftrightarrow} \longrightarrow \mathbb{R}$ by taking for all $[a] \in A_{\leftrightarrow} : F([a]) = P(a)$. F is well-defined, since by (i) the game $\langle N, (A_i), (P) \rangle$ has no strict best-response cycles, which implies by Lemma 1 that each $[a] \in A_{\leftrightarrow}$ is a singleton. Hence, if $a', a'' \in [a]$ then $a' = a''$ and thus $P(a') = P(a'')$.

Take $[a'], [a''] \in A_{\leftrightarrow}$ with $[a'] \lhd [a'']$. Since $a' \to a''$, there is a strict best-response compatible path from a' to a''. Hence $P(a') \leq P(a'')$. Moreover, since a' and a'' are in difference equivalence classes, some player must have strictly gained from deviating along this path: $P(a') < P(a'')$. Hence $F([a']) < F([a''])$. □

The proper order condition on $\langle A_{\leftrightarrow}, \lhd \rangle$ may be hard to check in specific applications. Thus it is desirable to find sufficient conditions for it.

Theorem 3 *Suppose a strategic game* $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ *with* A *countable.*

(i) Sufficiency: If A *contains no strict best-response cycle, then* Γ *is a pseudo-potential game.*

(ii) Necessity: If Γ *is a pseudo-potential game with pseudo-potential* P*, then* A *contains no strict best-response cycle for* $\Gamma' = \langle N, (A_i), (P) \rangle$*.*

Proof. If A is countable then $A_{\leftrightarrow}$ is countable. It follows from Fishburn (1970), Theorem 2.2, that $\langle A_{\leftrightarrow}, \lhd \rangle$ is properly ordered. Hence we can apply Theorem 1 and 2. □

The countability of A may be quite restrictive in some games, and we may want to consider alternative sufficient conditions. By Lemma 1 in Ok (2002) (alternatively, see Fishburn, 1970, Theorem 3.1), a sufficient condition for $\langle A_{\leftrightarrow}, \lhd \rangle$ being properly ordered is the weak separability of $\lhd$. Let B be a subset of $A_{\leftrightarrow}$. Suppose $[a'], [a''] \in A_{\leftrightarrow} \setminus B$ with $[a'] \lhd [a'']$. If there exists an $[a] \in B$ such that $[a'] \lhd [a]$ and $[a] \lhd [a'']$, then B is called *order dense* in $\langle A_{\leftrightarrow}, \lhd \rangle$. We say that $\langle A_{\leftrightarrow}, \lhd \rangle$ is *weakly separable* if there exists a countable subset $B \subseteq A_{\leftrightarrow}$ that is order dense in $\langle A_{\leftrightarrow}, \lhd \rangle$. Unfortunately, $\lhd$ is not weakly separable. It is easy to see that if for example $[a']$ *covers* $[a]$, i.e., there is no $[a'']$ such that $[a] \lhd [a''] \lhd [a']$, then there is no countable subset $B \subseteq A_{\leftrightarrow}$ that is order dense in $\langle A_{\leftrightarrow}, \lhd \rangle$. However, we can find a weakly separable strict partial order that includes $\lhd$ if action sets are subsets of the real line.

Theorem 4 *Suppose a strategic game* $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ *and let* $A_i \subseteq \mathbb{R}$, *for all* $i \in N$.

(i) Sufficiency: If A *contains no strict best-response cycles, then* Γ *is a pseudo-potential game.*

(ii) Necessity: If Γ *is a pseudo-potential game with pseudo-potential* P, *then* A *contains no strict best-response cycles for* $\Gamma' = \langle N, (A_i), (P) \rangle$.

Proof. Let $C_i \subseteq \mathbb{R}$ be a convex hull of A_i and $C = \times_{i \in N} C_i$. We show that the absence of strict best-response cycles in A implies that that there exists a strict partial order $\lessdot$ on C that includes $\lhd$ and for which $\langle C, \lessdot \rangle$ is properly ordered.

Assume that A contains no strict best-response cycles. Suppose $[a''']$ covers $[a]$ with respect to $\lhd$, and define a binary relation $\lessdot_a^{a'''}$ on C as follows: $a' \lessdot_a^{a'''} a''$ if and only of there exists $\lambda', \lambda'' \in [0,1]$ with $a'_i = \lambda' a_i + (1-\lambda') a'''_i$, $a''_i = \lambda'' a_i + (1-\lambda'') a'''_i$, and $[\lambda' < \lambda''$ if and only if $a < a''']$, whereby i denotes the unique player who switches her action on the single step path $a \to a'''$. Since by Lemma 1 (iii) each $[a]$ is a singleton, such binary relation is well defined.

We claim that for any $a, a''' \in A$ with $[a] \lhd [a''']$ such that $[a''']$ covers $[a]$, the binary relation $\lessdot_a^{a'''}$ on C is irreflexive and transitive. Irreflexivity follows immediately from $\lambda' \neq \lambda''$. For transitivity note that if $a' \lessdot_a^{a'''} a''$ and $a'' \lessdot_a^{a'''} a^{IV}$ then $a' \lessdot_a^{a'''} a^{IV}$ since $\lambda' < \lambda''$ and $\lambda'' < \lambda^{IV}$ implies $\lambda' < \lambda^{IV}$.

Define a binary relation $\lessdot$ on C by $a' \lessdot a''$ if (i) $[a'] \lhd [a'']$ or (ii) $a' \lessdot_a^{a'''} a''$ for some $a, a''' \in A$ with $[a] \lhd [a''']$ such that $[a''']$ covers $[a]$, or (iii) $[a' \lessdot_a^{a'''} a'''$ and $a'' \lessdot_{a^{IV}}^{a^V} a^V$ for some $a, a''', a^{IV}, a^V \in A$ with $[a] \lhd [a''']$, $[a'''] \lhd [a^{IV}]$, $[a^{IV}] \lhd [a^V]$ such that $[a''']$ covers $[a]$ and $[a^V]$ covers $[a^{IV}]]$. Again, since by Lemma 1 (iii) each $[a]$ is a singleton, such binary relation is well defined.

We claim that $\lessdot$ on C is irreflexive and transitive. Note that both $\lhd$ and $\lessdot_a^{a'''}$ for any $a, a''' \in A$ with $[a] \lhd [a''']$ with $[a''']$ covering $[a]$ is irreflexive and transitive, hence $\lessdot$ is irreflexive and transitive.

We claim that $\lessdot$ is weakly separable. To check this let $a \lessdot a'$. If $[a] \lhd [a']$ and $[a']$ covers $[a]$ then consider any nontrivial convex combination of a and a' (as in the definition of $\lessdot_a^{a'}$ above). Denote such convex combination by a''. Clearly, $a \lessdot_a^{a'} a''$ or $a'' \lessdot_a^{a'} a'$, and thus $a \lessdot a'' \lessdot a'$. Thus there is a set $B \subseteq C \setminus \{a, a'\}$ that is order dense in $\langle C, \lessdot \rangle$. To see that there is a countable set $B \subseteq C \setminus \{a, a'\}$ that is order dense in $\langle C, \lessdot \rangle$, consider only nontrivial convex combinations of a and a' (as in the definition of $\lessdot_a^{a'}$ above) that are rational numbers. Since the set of rational numbers is countable, such B must be countable. Hence $\lessdot$ is weakly separable. By Lemma 1 in Ok (2002) (alternatively, see Fishburn, 1970, Theorem 3.1), $\langle C, \lessdot \rangle$ is properly ordered. I.e., there is an order preserving real valued function of $\lessdot$ on C. Since by definition $\lessdot$ includes $\lhd$, such functions preserves also the order of $\lhd$. Hence we can apply Theorem 1 and 2. □

Note that we do not require that A_i is a convex subset of $\mathbb{R}$ nor that A_i is a subset of $\mathbb{R}_+$.

The theorem is related to results on the existence of a (semi-)continuous numerical representation of a partial order under topological conditions that unfortunately do not apply in our case (Peleg, 1970, Jaffray, 1975, and Sondermann, 1980).

Let $\langle S, \preccurlyeq \rangle$ be a partially ordered set and $B \subseteq S$. An *upper bound* for B is an element $s \in S$ such that $s' \preccurlyeq s$ for all $s' \in B$.

Proposition 3 *Suppose the strategic game $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ is a pseudo-potential game. If each strict best-response compatible path has an upper bound, then Γ has a Nash equilibrium.*

Proof. Let $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ be a pseudo-potential game. The result is equivalent to if each strict best-response compatible path has an upper bound, then $\langle A, \rightarrow \rangle$ has a maximal element. By Lemma 2 each strict best-response compatible path is a chain. Thus Zorn's Lemma (see Davey and Priestley, 2002, p. 230) implies the result. □

4 Relations to other Potential Games

Let $u_i : A \longrightarrow \mathbb{R}$ player i's payoff or utility function.

Definition 2 *A strategic game $\Gamma''' = \langle N, (A_i), (u_i) \rangle$ is*

(E) an exact potential game if there exists an (exact potential) function $P : A \longrightarrow \mathbb{R}$ such that for all $i \in N$, for all $a_{-i} \in A_{-i}$, and all $a_i', a_i'' \in A_i$,

$$u_i(a_i', a_{-i}) - u_i(a_i'', a_{-i}) = P(a_i', a_{-i}) - P(a_i'', a_{-i}), \tag{5}$$

(W) a weighted potential game if there exists a (weighted potential) function $P : A \longrightarrow \mathbb{R}$ and a vector $(w_i)_{i \in N} \in \mathbb{R}^n_{++}$ such that for all $i \in N$, for all $a_{-i} \in A_{-i}$, and all $a_i', a_i'' \in A_i$,

$$u_i(a_i', a_{-i}) - u_i(a_i'', a_{-i}) = w_i[P(a_i', a_{-i}) - P(a_i'', a_{-i})], \tag{6}$$

(O) *an ordinal potential game if there exists an (ordinal potential) function* $P : A \longrightarrow \mathbb{R}$ *such that for all* $i \in N$*, for all* $a_{-i} \in A_{-i}$*, and all* $a'_i, a''_i \in A_i$*,*

$$u_i(a'_i, a_{-i}) - u_i(a''_i, a_{-i}) > 0 \Longleftrightarrow P(a'_i, a_{-i}) - P(a''_i, a_{-i}) > 0, \tag{7}$$

(G) *a generalized ordinal potential game if there exists a (generalized ordinal potential) function* $P : A \longrightarrow \mathbb{R}$ *such that for all* $i \in N$*, for all* $a_{-i} \in A_{-i}$*, and all* $a'_i, a''_i \in A_i$*,*

$$u_i(a'_i, a_{-i}) - u_i(a''_i, a_{-i}) > 0 \Longrightarrow P(a'_i, a_{-i}) - P(a''_i, a_{-i}) > 0, \tag{8}$$

(B) *a best-reponse potential game if there exists a (best-response potential) function* $P : A \longrightarrow \mathbb{R}$ *such that for all* $i \in N$*, for all* $a_{-i} \in A_{-i}$*,*

$$b_i(a_{-i}) = \arg\max_{a_i \in A_i} P(a_i, a_{-i}). \tag{9}$$

Monderer and Shapley (1996) characterized exact potential games by the property that changes in payoff to deviating players along a cycle sum to zero. Voorneveld and Norde (1997) characterized ordinal potential games by the absence of weak improvement cycles and an order condition on the action space analog to the one in our Theorems 1 and 2. A weak improvement cycle is a finite path $(a^1, ..., a^m)$ for which $a^{k+1} \succeq_{i(k)} a^k$, $k \in \{1, ..., m-1\}$, $a^1 = a^m$, and $a^{k+1} \succ_{i(k)} a^k$ for some $k \in \{1, ..., m-1\}$. Monderer and Shapley (1996) characterized finite generalized ordinal potential games by the absence of strict improvement cycles. A strict improvement cycle is a finite path $(a^1, ..., a^m)$ for which $a^{k+1} \succ_{i(k)} a^k$, $k \in \{1, ..., m-1\}$ and $a^1 = a^m$. Thus a generalized ordinal potential of a finite game is a necessary and sufficient condition for the existence of a finite strict improvement path that terminates in Nash equilibrium. Voorneveld (2000) characterized best-response potential games by the absence of best-response cycles and an order condition on the action space analog to the one in our Theorems 1 and 2. A best-response cycle is a finite path $(a^1, ..., a^m)$ for which $a^{k+1}_{i(k)} \in b_{i(k)}(a^k_{-i(k)})$, $k \in \{1, ..., m-1\}$, $a^1 = a^m$, and $a^{k+1} \succ_{i(k)} a^k$ for some $k \in \{1, ..., m-1\}$.

Yet, we can think of a more general notion of potential.

Definition 3 (Quasi-Potential) *Let* $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ *be a strategic game with* $b(a) = \times_{i \in N} b_i(a_{-i})$ *the joint best-response correspondence. A function* $P : A \longrightarrow \mathbb{R}$ *is called a quasi-potential of the game* Γ *if*

$$a \in b(a) \Longleftrightarrow a \in \arg\max_{a \in A} P(a). \tag{10}$$

Clearly, any strategic game with a pure strategy Nash equilibrium has a quasi-potential. While a maximizer of a quasi-potential implies existence, it does not imply any convergence properties unlike other notions of potentials above.

Denote the classes of finite exact, weighted, ordinal, generalized ordinal, best-response, pseudo-potential, and quasi-potential games by E, W, O, G, B, P, and Q respectively. If X and Y are classes, denote by $X \subset Y$ that X is a proper subclass of Y.

Proposition 4 *Consider the class of finite strategic games of type* $\Gamma''' = \langle N, (A_i), (u_i) \rangle$. *Then*

(i) $\emptyset \neq E \subset W \subset O \subset G \subset P \subset Q$,

(ii) $\emptyset \neq E \subset W \subset O \subset B \subset P \subset Q$,

(iii) $G \cap B \neq \emptyset$, $G \setminus B \neq \emptyset$, $B \setminus G \neq \emptyset$.

Proof. (i) $E \subset W \subset O \subset G$ follows by definitions of those classes (see Monderer and Shapley, 1996). To see $G \subset P$, note that G is characterized by the absence of strict improvement cycles whereas P is characterized by the absence of strict improvements cycles restricted to best-responses. That is, any strategic game that has no strict improvement cycles has also no strict best-response cycles (but not vice versa). Hence $G \subset P$. To see that $B \nsubseteq G$ (and hence $P \nsubseteq G$), refer to Voorneveld (2000), Example 4.2.

(ii) follows by definitions of those classes (see Voorneveld, 2000). To see that $P \not\subset B$, consider following example

	L	R
T	$0, 0$	$0, 1$
D	$0, 1$	$1, 0$

	L	R
T	0	1
D	3	2

Left is the payoff matrix of the 2x2 game. The game has no best-response potential since a best-response potential would need to satisfy $P(TL) < P(TR) < P(DR) < P(DL) < P(TL)$, which is impossible. However, a pseudo-potential does not need to satisfy $P(DL) < P(TL)$. The matrix to the right is a pseudo-potential for the game.

(iii) follows from examples by Voorneveld (2000). $\square$

Proposition 5 *Consider the class of strategic games of type* $\Gamma'' = \langle N, (A_i), (\succ_i) \rangle$ *(with generic strict preferences). Then*

(i) $O = G$,

(ii) $B = P$.

Proof. Immediate by the definitions of those classes restricted to the class of strategic games of type $\Gamma'' = \langle N, (A_i), (\succ_i) \rangle$. $\square$

5 Strategic Complements or Substitutes

Let A_i be a chain and denote by $\geq$ the (total) order relation. A strategic game $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$ is called a game with ordered actions if A_i is a chain for all $i \in N$. The extension of the order $\geq$ to $A_{-i} = \times_{j\in N\setminus\{i\}} A_j$ is defined by $a''_{-i} \geq a'_{-i}$ if and only if $a''_j \geq a'_j$ for all $j \in N\setminus\{i\}$. The best-response correspondence b_i is increasing (decreasing) if $a''_{-i} \geq a'_{-i}$ implies that there is $a''_i \in b_i(a''_{-i})$ and $a'_i \in b_i(a'_{-i})$ with $a''_i \geq (\leq) a'_i$.

Definition 4 (Strategic Complements/Substitutes) *A strategic game with ordered actions $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$ is a game with strategic complements/substitutes if for any $i \in N$, the best-response correspondence b_i is increasing/decreasing on A_{-i}.*

A path $(a^1, a^2, ...)$ in the action space A is *best-response compatible* if the deviating player $i(k)$ plays $a^{k+1}_{i(k)} \in b_{i(k)}(a^k_{-i(k)})$, $k = 1, ...$ A finite path $(a^1, ..., a^m)$ is a *best-response cycle* if it is best-response compatible, $a^1 = a^m$, and $(a^{k+1}_{i(k)}, a^k_{-i(k)}) \succ_{i(k)} (a^k_{i(k)}, a^k_{-i(k)})$ for some $k \in \{1, ..., m\}$.

Define a binary relation $\rightharpoondown$ on the action space A as follows: $a \rightharpoondown a'$ if there exists a best-response compatible path from a to a'. Note that $a \rightharpoondown a$, for all $a \in A$. Define $\leftrightharpoons$ on A by $a \leftrightharpoons a'$ if $a \rightharpoondown a'$ and $a' \rightharpoondown a$. Since $\leftrightharpoons$ satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation. We denote the equivalence class of $a \in A$ with respect to $\leftrightharpoons$ by $[\tilde{a}]$, i.e., $[\tilde{a}] = \{a' \in A | a \leftrightharpoons a'\}$. We define a binary relation $\trianglelefteq$ on the set $A_{\leftrightharpoons}$ of equivalence classes as follows: $[\tilde{a}] \trianglelefteq [\tilde{a}']$ if $[\tilde{a}] \neq [\tilde{a}']$ and $a \rightharpoondown a'$. This relation is indeed well defined (see Voorneveld, 2000). Moreover, $\trianglelefteq$ is irreflexive and transitive on $A_{\leftrightharpoons}$.

Proposition 6 *Let the strategic game with ordered action space $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$ be a game with strategic complements or substitutes, $N = \{1, 2\}$, and $\langle A_{\leftrightharpoons}, \trianglelefteq\rangle$ be properly ordered. Then Γ is a best-response potential game.*

Proof. By Voorneveld (2000), Theorem 3.1, a strategic game Γ is a best-response potential game if and only if it contains no best-response cycle and $\langle A_{\leftrightharpoons}, \trianglelefteq\rangle$ is properly ordered.

Suppose to the contrary that Γ is a game with strategic complements but has a best-response cycle. Let $(a^1, ...)$ be a best-response compatible path containing a best-response cycle.

We claim that for $i = 1, 2$, if $a^{k+1}_i \geq (\leq) a^k_i$, $a^k, ..., a^{k+3}$ in a best-response compatible path $(a^1, ...)$ then $a^{k+3}_i \geq (\leq) a^{k+2}_i$ for $n = 1, ...$ To see this, assume $i = i(k)$, and note that if $a^{k+1}_{i(k)} \geq (\leq) a^k_{i(k)}$, then since best-responses are increasing (strategic complements) $a^{k+2}_{-i(k)} \geq (\leq) a^{k+1}_{-i(k)} (= a^k_{-i(k)})$. Again, since best-responses are increasing $a^{k+3}_{i(k)} \geq (\leq) a^{k+2}_{i(k)} (= a^{k+1}_{i(k)})$.

Since $(a^1, ...)$ contains a best-response cycle $(a^1, ..., a^m)$ we have $(a^{k+1}_{i(k)}, a^k_{-i(k)}) \succ_{i(k)} (a^k_{i(k)}, a^k_{-i(k)})$ for some $k \in \{1, ..., m-1\}$. Thus $a^{k+1}_{i(k)} \neq a^k_{i(k)}$. Hence by previous claim we can not have $a^1 = a^m$, which is a contradiction to the existence of a best-response cycle.

The proof extends to two-player games with strategic substitutes by reversing the order of one player's action set which turns it into a game with strategic complements (alternatively, one can show directly that if one player's actions go up the other players actions must go down (and vice versa)). $\square$

The intuition of the proof is straight-forward: If for any of the two player's best-responses are increasing then the best-response compatible path must be increasing too. This rules out any best-response cycles.

(Quasi)supermodularity is a sufficient condition of strategic complementarities. It is a property of preferences/utility functions that is easy to check.

Definition 5 *A strategic game $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$ is a quasisupermodular game if it has the single crossing property, i.e., if for $a''_i \geq a'_i$ and $a''_{-i} \geq a'_{-i}$,*

$$(a''_i, a'_{-i}) \succeq_i (a'_i, a'_{-i}) \quad \Rightarrow \quad (a''_i, a''_{-i}) \succeq_i (a'_i, a''_{-i}), \tag{11}$$
$$(a''_i, a'_{-i}) \succ_i (a'_i, a'_{-i}) \quad \Rightarrow \quad (a''_i, a''_{-i}) \succ_i (a'_i, a''_{-i}). \tag{12}$$

Corollary 1 *Let the strategic game $\Gamma = \langle N, (A_i), (\succeq_i)\rangle$ be quasisupermodular, $N = \{1, 2\}$, and let $\langle A_{\leftrightharpoons}, \trianglelefteq\rangle$ be properly ordered. Then Γ is a best-response potential game.*

Proof. The corollary follows from a known result by Milgrom and Shannon (1994) (see for example Topkis, 1998) and Proposition 6. $\square$

Definition 6 *A strategic game $\Gamma''' = \langle N, (A_i), (u_i)\rangle$ is a supermodular game if for all $i \in N$, u_i has increasing differences in (a_i, a_{-i}) on $A_i \times A_{-i}$, i.e., if for $a''_i \geq a'_i$ and $a''_{-i} \geq a'_{-i}$,*

$$u_i(a''_i, a'_{-i}) - u_i(a'_i, a'_{-i}) \leq u_i(a''_i, a''_{-i}) - u_i(a'_i, a''_{-i}). \tag{13}$$

Corollary 2 *Let the strategic game $\Gamma'' = \langle N, (A_i), (u_i)\rangle$ be supermodular, $N = \{1, 2\}$, and let $\langle A_{\leftrightharpoons}, \trianglelefteq\rangle$ be properly ordered. Then Γ is a best-response potential game.*

Proof. Note that supermodularity implies quasisupermodularity (but not vice versa). Hence the corollary follows from the previous result. $\square$

Remark 2 *Proposition 6 and above corollaries hold if we assume countability of A instead the proper order condition.*

Proposition 6 and above corollaries do not extend to strategic game with more than two players as following counter-example shows.

Example 1 Consider following three-player game:

	a	b		a	b
a	$1,1,1$	$0,0,1$	a	$1,0,0$	$0,1,0$
b	$0,1,0$	$1,0,0$	b	$0,0,1$	$1,1,1$
	a			b	

Take $a \leq b$ for all players. Clearly, each player's best-response function increases in opponents' actions. Thus it is a game with strategic complements. However, there is a strict best-response cycle $(b,a,b) \rightarrow (a,a,b) \rightarrow (a,b,b) \rightarrow (a,b,a) \rightarrow (b,b,a) \rightarrow (b,a,a) \rightarrow (b,a,b)$. Thus it can neither have a best-response potential nor a pseudo-potential.

6 Aggregation of Actions

Let for all $i \in N$, $A_i \subseteq X$ with X being chain, and denote by $\geq$ the (total) order relation. Analog to the previous section, this order relation is extended to any k-times Cartesian product of X, $1 \leq k \leq n$.

The following definition introduces a concept of general symmetric order preserving aggregation of actions:

Definition 7 (Aggregator) *An aggregator is defined recursively by a function* $\alpha : X \times X \longrightarrow X$ *such that*

(i) for all $i \in N$*, for all* $a_i \in A_i$*,* $\alpha^1(a_i) = a_i$*,*

(ii) for all $a_1, ..., a_{k+1} \in X$*, and for all* $1 \leq k \leq n-1$*,*

$$\alpha^{k+1}(a_1, ..., a_k, a_{k+1}) = \alpha(\alpha^k(a_1, ..., a_k), a_{k+1}), \tag{14}$$

(iii) for all $1 \leq k \leq n$*,* α^k *is symmetric, i.e.,* $\alpha^k(a_1, ..., a_k) = \alpha^k(a_{f(1)}, ..., a_{f(k)})$ *for all bijections* $f : \{1, ..., k\} \longrightarrow \{1, ..., k\}$*.*

(iv) for all $1 \leq k \leq n$*,* α^k *is order-preserving, i.e., if* $(a_1, ..., a_k) \leq (a'_1, ..., a'_k)$ *implies* $\alpha^k(a_1, ..., a_k) \leq \alpha^k(a'_1, ..., a'_k)$*.*

A strategic game $\tilde{\Gamma} = \langle N, (A_i), (\succeq_i), \alpha \rangle$ is called a game with aggregation if α is an aggregator according to Definition 7 and for all $i \in N$ the preference relation $\succeq_i$ is defined on $A_i \times X$. A similar concept of games with aggregation was used by Corchón (1994), Ania and Alós-Ferrer (2004), and Schipper (2004a). A specific prominent example of a game with aggregation is the Cournot oligopoly, where the aggregator is simply the sum of quantities but our concept of aggregation is much more general (see Schipper, 2004a, for further examples). It entails for example aggregators considered by Dubey, Haimanko and Zapechelnyuk (2002), i.e., products of actions, sums of such products of actions, and linear combinations of such sums of products of actions, that play a role in many applications such as in tournaments or team production.

Proposition 7 below provides a generalization and "intuification" of the result by Dubey, Haimanko, and Zapechelnyuk (2002). The proof of the proposition is formally independent of Dubey, Haimanko and Zapechelnyuk (2002).

Proposition 7 *Let the strategic game* $\tilde{\Gamma} = \langle N, (A_i), (\succeq_i), \alpha \rangle$ *be a game with strategic complements or substitutes, and with aggregation, and let* $\langle A_{\leftrightarrow}, \triangleleft \rangle$ *be properly ordered. Then* $\tilde{\Gamma}$ *is a pseudo-potential game.*

Proof. By Theorem 1, it suffices to show that A has not strict best-response cycles. Suppose by the way of contradiction that $(a^1, ..., a^m)$ is a strict best-response cycle. Assume that $\tilde{\Gamma}$ has strategic complements and aggregation. Select a $k \in \{1, ..., m-1\}$ s.t. that $\alpha^n(a^k) \geq \alpha^n(a^l)$ for all $l \in \{1, ..., m-1\}$. Since $(a^1, ..., a^m)$ is strict best-response compatible, there exists exactly one $i(k) \in N$ s.t. $a^{k+1}_{i(k)} \in b_{i(k)}(\alpha^{n-1}(a^k_{-i(k)}))$ and $a^{k+1}_{i(k)} \neq a^k_{i(k)}$ (unless the path is constant in which case there is nothing to prove). Since α^n is order-preserving, and k was selected such that $\alpha^n(a^k) \geq \alpha^n(a^l)$ for all $l \in \{1, ..., m-1\}$, we must have that $a^{k+1}_{i(k)} \leq a^k_{i(k)}$. Since $(a^1, ..., a^m)$ is a strict best-response cycle there must be a $h \in \{1, ..., m-1\}$ with $i(h) = i(k)$ and $a^k_{i(k)} = a^{h+1}_{i(k)} \in b_{i(k)}(\alpha^{n-1}(a^h_{-i(k)}))$. If $\alpha^{n-1}(a^h_{-i(k)}) > \alpha^{n-1}(a^k_{-i(k)})$ then $\alpha^n(a^k_{i(k)}, a^h_{-i(k)}) > \alpha^n(a^k)$, a contradiction to $\alpha^n(a^k) \geq \alpha^n(a^l)$ for all $l \in \{1, ..., m-1\}$. If $\alpha^{n-1}(a^h_{-i(k)}) = \alpha^{n-1}(a^k_{-i(k)})$ then $a^k_{i(k)} \in b_{i(k)}(\alpha^{n-1}(a^h_{-i(k)}))$ implies $a^k_{i(k)} \in b_{i(k)}(\alpha^{n-1}(a^k_{-i(k)}))$. Hence by strict best-response compatibility, $a^k_{i(k)} = a^{k+1}_{i(k)}$, a contradiction to the choice of $i(k)$ above. Finally, if $\alpha^{n-1}(a^h_{-i(k)}) < \alpha^{n-1}(a^k_{-i(k)})$ then by strategic complements $a^k_{i(k)} \leq a^{k+1}_{i(k)}$. Since we showed already that $a^k_{i(k)} \geq a^{k+1}_{i(k)}$ we must have $a^k_{i(k)} = a^{k+1}_{i(k)}$, a contradiction to strict best-response compatibility. Hence we proved that a game with strategic complements and aggregation does not have a strict best-response cycle.

We extend the proof to games with strategic substitutes by considering an order reversing map $\rho : X \longrightarrow X$ such that if $x'' \geq x'$ implies $\rho(x'') \leq \rho(x')$. Clearly, the best-response correspondence $b_i(\rho(\alpha^{n-1}(a_{-i}))) = \{a_i \in A_i | (a_i, \rho(\alpha^{n-1}(a_{-i}))) \succeq_i (a'_i, \rho(\alpha^{n-1}(a_{-i}))),$ $\forall a'_i \in A_i\}$ is increasing in ρ, implying strategic complements.

Theorem 1 implies that a game with strategic complements and aggregation is a pseudo-potential game, which completes the proof of the proposition. □

Proposition 8 *Let the strategic game $\tilde{\Gamma} = \langle N, (A_i), (\succeq_i), \alpha \rangle$ be a game with strategic complements or substitutes, and with aggregation, and let $A_i \subseteq \mathbb{R}$ for all $i \in N$. Then $\tilde{\Gamma}$ is a pseudo-potential game.*

Proof. In the proof of Proposition 7 we showed that if a strategic game $\tilde{\Gamma} = \langle N, (A_i), (\succeq_i), \alpha \rangle$ is a game with strategic complements or substitutes, and with aggregation, then it has no strict best-response cycles. By Theorem 4 it implies that $\tilde{\Gamma}$ is a pseudo-potential game if $A_i \subseteq \mathbb{R}$ for all $i \in N$. □

Remark 3 *Proposition 7 holds if we assume countablility of A instead the proper order condition.*

Remark 4 *Proposition 7 and Proposition 8 hold if we assume (quasi)supermodularity/ (quasi)submodularity instead strategic compelements/substitutes.*

Remark 5 *Proposition 7, Proposition 8, Remark 3, and Remark 4 hold if we assume $N = \{1, 2\}$ instead aggregation.*

In a recent article, Echenique (2004) shows that (i) a game with a unique Nash equilibrium has generalized strategic complementarities if and only if the Cournot best-response dynamics has no cycles and (ii) all games with multiple Nash equilibria have generalized strategic complementarities. However, he defines games with generalized strategic complementarities by an increasing joint best-response correspondence whereas we require that individual best-responses are increasing. Moreover, he considers also multilateral deviations in the Cournot best-response dynamics whereas we restrict best-response compatible paths to unilateral deviations. Motivated by Echenique's (2004) result we may ask whether any pseudo-potential game is a game with strategic complements. Following example shows that this is not the case. I.e., it is not true that if Γ is a pseudo-potential game (or best-response potential game) then there is an order on the actions set of each player such that Γ is a game with strategic complements.

Example 2. Consider following three-player game:

	α	β
α	$-1, -2, -1$	$-1, -1, -1$
β	$1, 1, 0$	$0, 0, 1$
γ	$0, 0, 1$	$1, 1, 0$

α

	α	β
α	$1, 1, 1$	$1, 0, 1$
β	$0, 1, 1$	$0, 0, 0$
γ	$-1, 0, 0$	$-1, 1, 1$

β

There is no order on the action set such that the game has strategic complements or generalized strategic complements (see Echenique, 2004). However, it has a best-response potential as follows:

$$\begin{array}{ccc} & \alpha & \beta \\ \alpha & 1 & 2 \\ \beta & 4 & 1 \\ \gamma & 2 & 3 \\ & \alpha & \end{array} \qquad \begin{array}{ccc} & \alpha & \beta \\ \alpha & 6 & 5 \\ \beta & 5 & 0 \\ \gamma & 1 & 4 \\ & \beta & \end{array} .$$

7 Construction of Potential Functions

In many applications of potential functions to games (e.g., Dubey, Haimanko and Zapechelnyuk, 2002) a (pseudo-)potential function "falls from heaven". For finite pseudo-potential games, there are simple ways of constructing and computing pseudo-potential functions. Consider the *incidence function* $m : A \times A \longrightarrow \mathbb{N}$ of the finite partially ordered set $\langle A, \rightarrow \rangle$ defined by

$$m(a, a') := \begin{cases} 1 & \text{if } a \rightarrow a' \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

This is the characteristic function of the relation $\rightarrow$.

Proposition 9 *Suppose the countable strategic game* $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ *is a pseudo-potential game. Then*

$$P(a) := \sum_{a' \in A} m(a', a) \tag{16}$$

is a pseudo-potential.

Proof. Suppose the countable strategic game $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ is a pseudo-potential game and $P(a) := \sum_{a' \in A} m(a', a)$ is not a pseudo-potential. Then there are $a', a'' \in A$ with $a' \rightarrow a''$ but $P(a') > P(a'')$. $P(a') > P(a'')$ implies that there exists $a''' \in A$ with $a''' \rightarrow a'$ but not $a''' \rightarrow a''$. Since $a' \rightarrow a''$ we must have by transitivity also $a''' \rightarrow a''$, a contradiction. □

Intuitively, for a pseudo-potential function we may just count for each combinations of actions the number of strict best-response compatible paths that lead to this combination of action.

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