ON EXISTENCE AND UNIQUENESS OF SOLUTION OF FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The purpose of this paper is to study the fuzzy fractional differential equations. We prove that fuzzy fractional differential equation is equivalent to the fuzzy integral equation and then using this equivalence existence and uniqueness result is establish. Fuzzy derivative is consider in the Goetschel-Voxman sense and fractional derivative is consider in the Riemann Liouville sense. At the end, we give the applications of the main result.

1. **Introduction**

The study of theory of the fuzzy differential equations has been growing rapidly. In many cases of the modeling of real world phenomena, fuzzy initial value problems appear naturally, because information about the behavior of a dynamical system is uncertain. In order to obtain a more adequate model, we have to take into account these uncertainties. Significant results from the theory of fuzzy differential equations and their applications can be found in [11], [13], [15], [18], [21] and [24].

Fractional calculus stems from the beginning of theory of differential and integral calculus [17],[19]. Fractional differential equations are a powerful tool for modeling many systems in various areas of sciences. There are many systems in nature with a complex behavior and fractional order model capture the properties of these kinds of systems but classical integer order model neglect such properties. Fractional differential equations have played an important role in many fields such as astrophysics, electronics, diffusion, material theory, chemistry, control theory, wave propagation, signal theory, electricity and thermodynamics (see [14],[22]).

The concept of solution of fuzzy fractional differential equations was first introduced in [2]. Generally, in fuzzy case, the fuzzy fractional differential equation

$$D^q y(t) = f(t, y(t)), \qquad \lim_{t\to 0^+} t^{1-q} y(t) = y_0, \tag{1}$$

is not equivalent to fuzzy integral equation

$$y(t) = y_0 t^{q-1} + I^q f(t, y(t)). \tag{2}$$

In [4], we established the existence and uniqueness of the solution of fuzzy fractional integral equation (2). Obviously, each solution of integral equation (2) is also a solution of (1). Also in [5], we have given the existence and uniqueness of fractional differential equation with fuzzy initial condition. Fuzzy fractional integral

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equations have been studied in [1]. Explicit solutions of fuzzy fractional differential equations have given in [3]. In [20] fuzzy laplace transform method have proposed to solve fuzzy fractional differential equations and modified Euler Method have presented in [16] to solve fuzzy fractional differential equations.

In this paper, we use the concept of Goetschel-Voxman derivative was defined in paper [10]. In the definition of this derivative non-standard fuzzy subtraction is used. The relation between Goetschel-Voxman derivative and other fuzzy derivatives can be found in the paper [7]. The advantage of this derivative is that we can use integration by parts formula and this formula does not work in case of other kind of fuzzy derivatives. We prove that fuzzy fractional differential equation (1) is equivalent to fuzzy integral equation (2). Using this equivalence the existence and uniqueness of solution of (1) is establish. For basic results related to differentiation and integration of fuzzy numbers we refer [8] and [10].

In case of Hukuhara derivative (H-derivative), the diameter of the solutions is nondecreasing as the time goes ahead. This property is major obstacle in application of H-differentiability in fuzzy mathematical modeling. In [23] authors gave comparative analysis of some alternative approaches, but this shortcoming was solved by Bede and Gal in [6] where they introduce the strongly generalized differentiability.

This paper is organized as follows: In section 2, we recall some basic and well known results related to fuzzy numbers and fractional differential equations. In section 3, we proved the existence and uniqueness of the solution of fuzzy fractional differential equations. At the end in section 4, we conclude the paper by providing the applications of the existence and uniqueness theorem.

2. Preliminaries

Let E denote the set of all fuzzy numbers. We recall that $y : \mathbb{R} \to [0,1]$ is a fuzzy number if it satisfies the following properties:
(i) There is a unique $\xi_0 \in \mathbb{R}$ such that $y(\xi_0) = 1$,
(ii) $[y]^0 = \mathrm{cl}\{\xi \in \mathbb{R} | y(\xi) > 0\}$ is bounded in $\mathbb{R}$,
(iii) y is strictly fuzzy convex on $[y]^0$, i.e.,

$$y(\lambda\xi_1 + (1-\lambda)\xi_2) > \min\{y(\xi_1), y(\xi_2)\} \quad \text{for all} \quad \xi_1, \xi_2 \in [y]^0, \xi_1 \neq \xi_2 \quad \text{for all } \lambda \in (0,1)$$

(iv) y is upper semi-continuous on $\mathbb{R}$.

Let $y \in E$. Then for each $\alpha \in (0,1]$, the set

$$[y]^\alpha = \{\xi \in \mathbb{R}; \quad y(\xi) \geq \alpha\},$$

is called the α-level set of y.

Theorem 2.1. [8] *Let $y \in E$ and for each $\alpha \in [0,1]$,*

$$y_1(\alpha) = \min[y]^\alpha \text{ and } y_2(\alpha) = \max[y]^\alpha,$$

then we have
(i) $y_1, y_2 \in C[0,1] = \{u : [0,1] \to \mathbb{R}; u \text{ is continuous on } [0,1]\}$,
(ii) *y_1 is monotone increasing and y_2 is monotone decreasing,*

(iii) $y_1(1) = y_2(1)$.
Conversely, if $x(\alpha), z(\alpha) : [0,1] \to \mathbb{R}$ *satisfy the above conditions* $(i) - (iii)$, *denote*

$$y(\xi) = \begin{cases} \sup\{\alpha \in [0,1] : x(\alpha) \le \xi \le z(\alpha)\}, & \xi \in [x(0), z(0)], \\ 0, & \xi \notin [x(0), z(0)]. \end{cases}$$

Then there exists $y \in E$ *such that* $[y]^\alpha = [x(\alpha), z(\alpha)]$, $y_1(\alpha) = x(\alpha)$, $y_2(\alpha) = z(\alpha)$, $\alpha \in [0,1]$.

In [1] parametric representation of fuzzy number was introduced, i.e. $y \in E$ can be written as $y = (y_1(\alpha), y_2(\alpha))$, $\alpha \in [0,1]$. (for sake of simplicity we write $y = (y_1, y_2)$). Therefore fuzzy number $y \in E$ can be considered as a continuous curve $\{(y_1(\alpha), y_2(\alpha)) : \alpha \in [0,1]\}$ in $\mathbb{R}^2$. For $t \in \mathbb{R}$, the membership function has the following form:

$$\mu_t(\xi) = \begin{cases} 1, & \xi = t, \\ 0, & \xi \ne t. \end{cases}$$

Then we have $t \in \mathbb{R} \subset E$, $[\mu_t]^\alpha = [t,t]$ for all $\alpha \in [0,1]$, therefore the parametric representation of $t \in \mathbb{R}$ is $t = (t,t)$, $\alpha \in [0,1]$.
For $w = (u,v) \in C[0,1] \times C[0,1]$, define the norm

$$\|w\| = \max_{0 \le \alpha \le 1} \max\{|u(\alpha)|, |v(\alpha)|\}.$$

It is obvious that $C[0,1] \times C[0,1]$ is a Banach space.

For $y = (y_1, y_2), z = (z_1, z_2) \in E, k \in \mathbb{R}$, we have the following operations based on Zadeh's extension principle,

i) $y \oplus z = (y_1 + z_1, y_2 + z_2)$,
ii) $y \ominus z = (y_1 - z_2, y_2 - z_1)$,
iii) $k \otimes y = \begin{cases} (ky_1, ky_2), & k \ge 0, \\ (ky_2, ky_1), & k < 0. \end{cases}$

It is easy to see that E is not a linear space under these operations. In [8] the following operations were introduced.
For all $y, z \in E$, define $y - z = (y_1(\alpha) - z_1(\alpha), y_2(\alpha) - z_2(\alpha))$, $\alpha \in [0,1]$.

$$E - E := \{w : w = y - z, \ y, z \in E\}.$$

If the H-difference (Hukuhara difference) of y and z exists, then $y - z$ is the H-difference of y and z.

Remark 2.2. [8] i) $E - E$ is a linear subspace of $C[0,1] \times C[0,1]$, and
a) $y \oplus z = y + z$ for all $y, z \in E$.
b) $k \otimes y = k \cdot y$ for all $k \in [0, \infty)$ for all $y \in E$.
where "$+$", "$\cdot$" are additive and product operations in linear space $C[0,1] \times C[0,1]$.
ii) E is a closed convex cone in Banach space $C[0,1] \times C[0,1]$.

In this paper, "$\oplus$", "$\ominus$", "$\otimes$" represent operations based on Zadeh's extension Principle, "$+$", "$-$", "$\cdot$" stand for operations based on linear space, if they agree, we use the later. We define a metric d on E by

$$d(y,z) = \sup_{0 \le \alpha \le 1} d_H([y]^\alpha, [z]^\alpha),$$

where d_H is the Hausdorff metric defined as

$$d_H([y]^\alpha,[z]^\alpha)=\max\{|y_1(\alpha)-z_1(\alpha)|,|y_2(\alpha)-z_2(\alpha)|\}.$$

It is well known that (E,d) is a complete metric space. We list some properties of the metric d:

$$d(y+w,z+w)=d(y,z),\ d(\lambda y,\lambda z)=|\lambda|d(y,z), \tag{3}$$

$$d(y,z)\leq d(y,w)+d(w,z) \tag{4}$$

$$d(\lambda y,\gamma y)\leq|\lambda-\gamma|d(y,\widehat{0}) \tag{5}$$

for all $y,z,w\in E$ and $\lambda,\gamma\in\mathbb{R}$.

Let $T\subset\mathbb{R}$ be an interval, $y:T\to E$ be a fuzzy function, and $t_0\in T$. If for each $\varepsilon>0$, there exists $\delta>0$, such that

$$d(y(t),y(t_0))<\varepsilon,$$

for all $t\in T$ with $|t-t_0|<\delta$, then y is said to be continuous at t_0. If y is continuous at each point of T, then y is said to be continuous on T. We denote by $C(T,E)$ the space of all continuous fuzzy functions on T.
Let $a>0$ and $r\geq 0$. We need the following notion before proceeding further.

$$C_r([0,a],E):=\{y\in C((0,a],E):\sup_{t\in[0,a]} d(t^r y(t),\widehat{0})<\infty\},$$

and on this set we define the metric d_r by

$$d_r(y,z):=\sup_{t\in[0,a]} t^r d(y(t),z(t)).$$

Also, $d_r(y,\widehat{0})$ will be denoted by $\|y\|_r$. Clearly, $C([0,a],E)=C_0([0,a],E)$.

Theorem 2.3. *$(C_r([0,a],E),d_r)$ is a complete metric space.*

Proof. Let $\{y_n\}_{n=1}^\infty$ be a Cauchy sequence in $C_r([0,a],E)$. Then for each $\varepsilon>0$ there exists $M\in\mathbb{N}$ such that $d_r(y_n,y_m)<\varepsilon$ for all n, $m\geq M$. That is,

$$\sup_{t\in[0,a]} t^r d(y_n(t),y_m(t))<\varepsilon,\quad \text{for all } n,\ m\geq M,$$

Let $z_n(t):=t^r y_n(t),\ n\geq 1$. Then

$$\sup_{t\in[0,a]} d(z_n(t),z_m(t))<\varepsilon,\quad \text{for all } n,\ m\geq M.$$

This implies that $\{z_n\}_{n=1}^\infty$ is a Cauchy sequence in the complete metric space $C([0,a],E)$. Therefore z_n converges uniformly to $z\in C([0,a],E)$.
Let $y(t):=t^{-r}z(t),\ t\in(0,a]$. Clearly $y\in C((0,a],E)$. we have

$$\begin{aligned} d_r(y_n,y) &= \sup_{t\in[0,a]} t^r d(y_n(t),y(t)) \\ &= \sup_{t\in[0,a]} d(t^r y_n(t),t^r y(t)) \\ &= \sup_{t\in[0,a]} d(z_n(t),z(t))\to 0,\ \text{as } n\to\infty. \end{aligned}$$

Hence y_n converges to y. Since $\{y_n\} \subset C_r([0,a],E)$, we have

$$\sup_{t\in[0,a]} d(t^r y_n(t),\widehat{0}) < \infty, \quad \text{for all } n \geq 1. \tag{6}$$

Now using the inequality (4), we obtain

$$\begin{aligned}\sup_{t\in[0,a]} d(t^r y(t),\widehat{0}) &\leq \sup_{t\in[0,a]} [t^r d(y(t),y_M(t)) + d(t^r y_M(t),\widehat{0})] \\ &\leq \sup_{t\in[0,a]} t^r d(y(t),y_M(t)) + \sup_{t\in[0,a]} d(t^r y_M(t),\widehat{0}).\end{aligned}$$

Therefore by the convergence of y_n and from inequality (6), we get

$$\sup_{t\in[0,a]} d(t^r y(t),\widehat{0}) < \infty.$$

Hence $y \in C_r([0,a],E)$. Thus $C_r([0,a],E)$ is a complete metric space. □

Let $y : [a,b] \to E$ be a fuzzy function, $t_0 \in [a,b]$ and $\omega \in E$. If for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$d\left(\frac{y(t)-y(t_0)}{t-t_0},\omega\right) < \varepsilon,$$

for all $t \in [a,b]$ with $|t-t_0| < \delta$, then y is said to be derivable at t_0. We denote $y'(t_0) = \omega$ or $\frac{d}{dt}y(t_0) = \omega$. If y is derivable at each point of $[a,b]$, then y is said to be derivable on $[a,b]$. Obviously, if $y : [a,b] \to E$ is derivable at t_0, then y is continuous at t_0.

We denote by $C_r^1[0,a]$ the space of functions $y(t)$ which are continuously derivable on $(0,a]$ and have the derivative $y'(t)$ of order 1 on $(0,a]$ such that $y'(t) \in C_r[0,a]$.

Proposition 2.4. [8] *Let $y : [a,b] \to E$ be derivable on $[a,b]$ and $y(t) = (y_1(t,\alpha), y_2(t,\alpha))$, $t \in [a,b]$, $\alpha \in [0,1]$. Then*

$$y'(t) = \left(\frac{d}{dt}y_1(t,\alpha), \frac{d}{dt}y_2(t,\alpha)\right), \quad \alpha \in [0,1],$$

provided this equation defines a fuzzy number $y'(t) \in E$.

Remark 2.5. [8] i) If $y : [a,b] \to E$ is derivable on $[a,b]$, then y is H-derivable (Hukuhara derivable) on $[a,b]$, and the H-derivative is the same as the derivative. That is to say on an interval derivability is equivalent to H-derivability.

ii) If $y : [a,b] \to E$ is Riemann integrable on $[a,b]$, then the parametric representation of its integral is given by

$$\int_a^b y(t)dt = \left(\int_a^b y_1(t,\alpha)dt, \int_a^b y_2(t,\alpha)dt\right), \quad a,b \in T, \ \alpha \in [0,1].$$

Also, we know that the fuzzy integral is a fuzzy number.

Lemma 2.6. *Let $y : [a,b] \to E$ and $z : [a,b] \to E$ be integrable on $[a,b]$. If the function $g : [a,b] \to \mathbb{R}$ defined by*

$$g(t) := d(y(t),z(t))$$

is Riemann integrable on $[a,b]$. Then

$$d\left(\int_a^b y(t)dt, \int_a^b z(t)dt\right) \le \int_a^b d(y(t), z(t))dt.$$

Proof. It can be proved easily using the Riemann sum. □

The following results are given in [8].

Theorem 2.7. *Let $f : [a, b] \to E$ be continuous on $[a, b]$, then the fuzzy function $F : [a, b] \to E$ given by*

$$F(t) = \int_a^t f(s)ds,\ t \in [a, b],$$

is derivable on $[a, b]$ and

$$F'(t) = f(t),\ t \in [a, b].$$

Corollary 2.8. *Assume that $f : [a, b] \to E$ is continuously derivable on $[a, b]$, then*

$$\int_a^b f'(t)dt = f(b) - f(a),$$

where $f(b) - f(a)$ is the H-difference of $f(b)$ and $f(a)$.

Theorem 2.9. *Let $\varphi : [a, b] \to \mathbb{R}$ be continuously derivable and $y : [a, b] \to E$ be continuously derivable. Then*

$$\int_a^b y(t)\varphi'(t)dt = [\varphi(t) \cdot y(t)]_a^b - \int_a^b \varphi(t) \cdot y'(t)dt.$$

Let y be a real valued function on $[0, a]$. The Riemann-Liouville fractional integral $I^q y$ of order $q > 0$ is defined by

$$I^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y(s)ds, \qquad 0 < t < a,$$

provided that the expression on the right hand side is defined.

The Riemann-Liouville fractional derivative $D^q y$ of y of order $0 < q < 1$ is defined by

$$D^q y(t) = \frac{d}{dt} I^{1-q} y(t), \qquad 0 < t < a,$$

provided the expression on right hand side is defined.

Lemma 2.10. *Let $q > 0$ and $y : [0, a] \to E$ be such that $y(t) = (y_1(t, \alpha), y_2(t, \alpha))$ for all $t \in [0, a]$ and $\alpha \in [0, 1]$. Then the family of pairs*

$$F_\alpha := \left(\frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y_1(s, \alpha)ds, \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y_2(s, \alpha)ds\right),\ \alpha \in [0, 1]$$

define a fuzzy number $u \in E$ such that $(u_1(\alpha), u_2(\alpha)) = F_\alpha$.

Proof. Fix $t \in [0, a]$, then by Theorem 2.1(i), $y_1(., \alpha)$, $y_2(., \alpha) \in C([0, 1], \mathbb{R})$, for all $\alpha \in [0, 1]$. It is easy to see that $I^q y_1(t, \alpha)$ and $I^q y_2(t, \alpha)$ are continuous with respect to α. From Theorem 2.1(ii), we have $y_1(\alpha) \le y_1(\beta)$ for $\alpha \le \beta$, then

$$\frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y_1(s, \alpha)ds \le \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} y_1(s, \beta)ds,$$

therefore $I^q y_1$ is monotone increasing with respect to α, similarly, y_2 is monotone decreasing. $y \in E$ implies $y_1(1) = y_2(1)$ which gives

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s,1)ds = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_2(s,1)ds.$$

Hence by Theorem 2.1 there exists a fuzzy number $u \in E$ such that $(u_1(\alpha), u_2(\alpha)) = F_\alpha$. □

Let $y \in C([0,a], E)$, where $y = (y_1, y_2)$, we define the fractional integral of order $q > 0$ of y by

$$I^q y(t) = \left(\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_1(s,\alpha)ds, \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y_2(s,\alpha)ds \right), \ \alpha \in [0,1].$$

Proposition 2.11. *Let $p, q > 0$ and $y \in C([0,a], E)$. Then*

$$I^p I^q y = I^{p+q} y.$$

Proof. Similar to the proof of Lemma 4.1 in [4]. □

Let $y \in C([0,a], E)$, where $y = (y_1, y_2)$. We define the Riemann-Liouville fractional derivative of order $0 < q < 1$ of y by

$$D^q y(t) = \frac{d}{dt} I^{1-q} y(t).$$

The Riemann-Liouville derivative $D^q y(t)$ can be represented parametrically as follows

$$D^q y(t) = \left(\frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} y_1(s,\alpha)ds, \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} y_2(s,\alpha)ds \right),$$

where $\alpha \in [0,1]$.

Lemma 2.12. *Let $x : [0,a] \to E$ be a continuous function and $g : \mathbb{R}^2 \to \mathbb{R}$ be such that $g(t,s)$ is non negative and non decreasing with respect to t and continuous with respect to s and $\frac{\partial}{\partial t} g(t,s)$ is continuous with respect to t. Then the function $G : [0,a] \to E$ given by*

$$G(t) = \int_0^t g(t,s)x(s)ds, \quad t \in [0,a],$$

is derivable and

$$G'(t) = g(t,t)x(t) + \int_0^t \frac{\partial}{\partial t} g(t,s)x(s)ds, \quad t \in [0,a]. \tag{7}$$

Proof. For any $h > 0$ and by Proposition 3.5 in [8], we have

$$\begin{aligned} G(t+h) &= \int_0^{t+h} g(t+h,s)x(s)ds \\ &= \int_t^{t+h} g(t+h,s)x(s)ds + \int_0^t (g(t+h,s) - g(t,s))x(s)ds \\ &\quad + \int_0^t g(t,s)x(s)ds \end{aligned}$$

Now using Lemma 2.6, equation (3) and inequality (5), we have

$$
\begin{aligned}
& d\left(\frac{1}{h}\otimes(G(t+h)-G(t)), g(t,t)x(t)+\int_0^t \frac{\partial g(t,s)}{\partial t}x(s)ds\right) \\
\leq\ & d\left(\frac{1}{h}\int_0^t (g(t+h,s)-g(t,s))x(s)ds, \int_0^t \frac{\partial g(t,s)}{\partial t}x(s)ds\right) \\
& +d\left(\frac{1}{h}\int_t^{t+h} g(t+h,s)x(s)ds, g(t,t)x(t)\right) \\
\leq\ & \int_0^t d\left(\frac{g(t+h,s)-g(t,s)}{h}x(s), \frac{\partial g(t,s)}{\partial t}x(s)\right)ds \\
& +\frac{1}{h}\int_t^{t+h} d(g(t+h,s)x(s), g(t,t)x(t))ds \\
\leq\ & \int_0^t \left|\frac{g(t+h,s)-g(t,s)}{h} - \frac{\partial g(t,s)}{\partial t}\right| d(x(s),\widehat{0})ds \\
& +\frac{1}{h}\int_t^{t+h} [d(g(t+h,s)x(s), g(t,s)x(s)) + d(g(t,s)x(s), g(t,s)x(t)) \\
& +d(g(t,s)x(t), g(t,t)x(t))]ds \\
\leq\ & \int_0^t \left|\frac{g(t+h,s)-g(t,s)}{h} - \frac{\partial g(t,s)}{\partial t}\right| d(x(s),\widehat{0})ds \\
& +\frac{1}{h}\int_t^{t+h} (g(t+h,s)-g(t,s))d(x(s),\widehat{0})ds \\
& \quad +\frac{1}{h}\int_t^{t+h} g(t,s)d(x(s),x(t))ds \\
& \quad +\frac{1}{h}\int_t^{t+h} |g(t,s)-g(t,t)|d(x(t),\widehat{0})ds \to 0 \quad \text{as } h\downarrow 0.
\end{aligned}
$$

by the continuity of $x(t)$ and the results in analysis. Similarly, we have

$$d\left(\frac{1}{h}\otimes(G(t)-G(t-h), g(t,t)x(t)+\int_0^t \frac{\partial g(t,s)}{\partial t}x(s)ds\right) \to 0 \quad \text{as } h\downarrow 0.$$

Therefore $\lim_{h\to 0}\frac{1}{h}\otimes(G(t+h)-G(t))$ and $\lim_{h\to 0}\frac{1}{h}\otimes(G(t)-G(t-h))$ exist. It follows that $G'(t)$ exists and (7) holds. □

Lemma 2.13. *Let $0<q<1$. Then the following assertions are true:*
a) If $y(t)\in C_{1-q}([0,a],E)$, then

$$D^q I^q y(t) = y(t) \text{ for all } t\in(0,a].$$

b) If $I^{1-q}y(t)\in C^1_{1-q}([0,a],E)$, then

$$I^q D^q y(t) = y(t) - \frac{t^{q-1}}{\Gamma(q)} I^{1-q}y(0) \text{ for all } t\in(0,a].$$

Proof. a) Using the definition of fractional derivative, Proposition 2.11 and Theorem 2.7, we obtain

$$\begin{aligned} D^q I^q y(t) &= \frac{d}{dt} I^{1-q} I^q y(t) = \frac{d}{dt} I^1 y(t) \\ &= \frac{d}{dt} \int_0^t y(s) ds = y(t). \end{aligned}$$

b)By the definition of fractional integral and Lemma 2.12, we get

$$\begin{aligned} I^q D^q y(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} D^q y(s) ds \\ &= \frac{1}{\Gamma(q+1)} \frac{d}{dt} \int_0^t (t-s)^q D^q y(s) ds. \end{aligned}$$

Now by Theorem 2.9, we have

$$\begin{aligned} \frac{1}{\Gamma(q+1)} \int_0^t (t-s)^q D^q y(s) ds &= \frac{1}{\Gamma(q+1)} \int_0^t (t-s)^q \frac{d}{ds} I^{1-q} y(s) ds \\ &= \frac{1}{\Gamma(q+1)} [(t-s)^q I^{1-q} y(s)|_0^t \\ &\quad + q \int_0^t (t-s)^{q-1} I^{1-q} y(s) ds] \\ &= \frac{1}{\Gamma(q+1)} [-t^q I^{1-q} y(0) + q\Gamma(q) I^q (I^{1-q} y(t))] \\ &= \frac{-t^q}{\Gamma(q+1)} I^{1-q} y(0) + I^1 y(t). \end{aligned}$$

Hence

$$I^q D^q y(t) = y(t) - \frac{t^{q-1}}{\Gamma(q)} I^{1-q} y(0).$$

□

3. Existence and Uniqueness

Consider the following fuzzy fractional differential equation

$$D^q y(t) = f(t, y(t)), \tag{8}$$

where $0 < q < 1$, and $f : [0, a] \times E \to E$ is continuous on $(0, a] \times E$.
A fuzzy function $y : (0, a] \to E$ is a solution of fuzzy fractional differential equation (8) if it is continuous on $(0, a]$ and

$$D^q y(t) = f(t, y(t)),$$

for all $t \in (0, a]$. We can associate with the fuzzy fractional differential equation with the following initial condition

$$\lim_{t \to 0^+} t^{1-q} y(t) = y_0 \in E. \tag{9}$$

Remark 3.1. Let $0 < q < 1$ and $y(t) \in C_{1-q}((0,a],E)$. Using a similar proof as in Lemma 4.1 in [4], we have
a) If

$$\lim_{t\to 0^+} t^{1-q}y(t) = b \in E,$$

then

$$I^{1-q}y(0^+) := \lim_{t\to 0^+} I^{1-q}y(t) = b\Gamma(q).$$

b) If

$$\lim_{t\to 0^+} I^{1-q}y(t) = c \in E,$$

and if there exists the limit $\lim\limits_{t\to 0^+} t^{1-q}y(t)$, then

$$\lim_{t\to 0^+} t^{1-q}y(t) = \frac{c}{\Gamma(q)}.$$

Lemma 3.2. *Let $0 < q < 1,\ K > 0,$ and $a > 0$. Define*

$$G = \{(t,y) \in [0,a] \times E : y \in E \text{ for } t = 0 \text{ and } d(t^{1-q}y, y_0) < K \text{ else } \},$$

and assume that the function $f : G \to E$ is a continuous and bounded in G and there exists a constant $A > 0$ such that,

$$d(f(t,u), f(t,v)) \leq Ad(u,v),$$

for all (t,u), $(t,v) \in G$. If $y(t) \in C((0,a],E)$, then $y(t)$ satisfies the relations (8) and (9) if and only if $y(t)$ satisfies the integral equation

$$y(t) = y_0 t^{q-1} + I^q f(t,y(t)). \tag{10}$$

Proof. Suppose $y(t) \in C((0,a],E)$ satisfy (8),(9), we define $u(t) := f(t,y(t))$. By assumption, u is a continuous function and

$$u(t) = f(t,y(t)) = D^q y(t) = \frac{d}{dt}(I^{1-q}y)(t).$$

Thus $\frac{d}{dt}(I^{1-q}y)(t) \in C_{1-q}([0,a],E)$. Therefore $I^{1-q}y(t) \in C^1_{1-q}([0,a],E)$. Applying I^q to both sides of (8) and using Lemma 2.13 (b), we have

$$y(t) - \frac{t^{q-1}}{\Gamma(q)} I^{1-q}y(0) = I^q f(t,y(t)).$$

Therefore by Lemma 3.1,

$$y(t) = y_0 t^{q-1} + I^q f(t,y(t)).$$

Suppose that $y \in C((0,a],E)$ satisfy (10). Applying D^q to both sides of (10) and then using Lemma 2.13(a), we obtain

$$\begin{aligned} D^q y(t) &= D^q(y_0 t^{q-1}) + D^q I^q f(t,y(t)) \\ &= f(t,y(t)). \end{aligned}$$

□

The following theorem help us to prove the next result.

Theorem 3.3. [9] *Let (U,d) be non empty complete metric space, and let $\beta_n \geq 0$ for all $n \in \{0,1,2,...\}$ be such that $\sum_{n=0}^{\infty} \beta_n$ converges. Moreover, let the mapping $T : U \to U$ satisfy the inequality*

$$d(T^n u, T^n v) \leq \beta_n d(u,v)$$

for all $n \in \mathbb{N}$ and for all $u, v \in U$. Then the operator T has a unique fixed point $u^* \in U$. Furthermore, for any $u_0 \in U$, the sequence $\{T^n u_0\}_{n=1}^{\infty}$ converges to the above fixed point u^*.

Theorem 3.4. *Let $0 < q < 1$, $K > 0$ and $a^* > 0$. Define*

$$G = \{(t,y) \in [0,a^*] \times E : y \in E \text{ for } t = 0 \text{ and } d(t^{1-q}y, y_0) < K\},$$

and assume that the function $f : G \to E$ is a continuous and bounded in G and there exists a constant $A > 0$ such that,

$$d(f(t,u), f(t,v)) \leq A d(u,v), \tag{11}$$

for all (t,u), $(t,v) \in G$. Then there exists a unique solution $y(t) \in C((0,a],E)$ to the Cauchy problem (8) and (9), where

$$a := \min\left\{a^*, \widetilde{a}, \left(\frac{\Gamma(q+1)K}{M}\right)\right\},$$

with $M := \sup_{(t,y)\in G} d(f(t,y), \widehat{0})$ and $\widetilde{a}$ being a positive number such that

$$\widetilde{a} < \left(\frac{\Gamma(2q)}{\Gamma(q)A}\right)^{\frac{1}{q}}.$$

Proof. Define the set

$$U := \{y \in C((0,a],E) : \sup_{t\in[0,a]} d(t^{1-q}y, y_0) \leq K\}.$$

U is a closed subset of the complete metric space $C_{1-q}([0,a],E)$. Therefore U is a complete metric space. We define the operator $T : U \to C_{1-q}([0,a],E)$ by

$$Ty(t) := y_0 t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds.$$

In order to prove the desired result, it is sufficient to prove that the operator T has a unique fixed point. Note that for $y \in U$, Ty is also a continuous function on $(0,a]$. Moreover,

$$\begin{aligned} d(t^{1-q}Ty(t), y_0) &\leq d\left(\frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,y(s))ds, \widehat{0}\right) \\ &\leq \frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s,y(s)), \widehat{0}) ds \\ &\leq \frac{t^{1-q}M}{\Gamma(q)} \int_0^t (t-s)^{q-1} ds \\ &\leq \frac{aM}{\Gamma(q+1)} \leq K \end{aligned}$$

for $t \in (0, a]$. This shows that the operator T maps the set U into itself. Now we show by induction that for $y,\, z \in U$,

$$\|T^n y - T^n z\|_{1-q} \leq \left(\frac{Aa^q\Gamma(q)}{\Gamma(2q)}\right)^n \|y - z\|_{1-q}. \tag{12}$$

For $n = 0$, this statement is trivially true. Suppose that (12) is true for $n \geq 1$. Then from inequality (11), we have

$$\begin{aligned}
& \|T^{n+1}y - T^{n+1}z\|_{1-q} \\
= & \sup_{t\in[0,a]} t^{1-q} d((T^{n+1}y(t) - T^{n+1}z(t)), \widehat{0}) \\
= & \sup_{t\in[0,a]} t^{1-q} d((TT^n y(t) - TT^n z(t)), \widehat{0}) \\
= & \sup_{t\in[0,a]} \frac{t^{1-q}}{\Gamma(q)} d\left(\int_0^t (t-s)^{q-1}(f(s, T^n y(s)) - f(s, T^n z(s)))ds, \widehat{0}\right) \\
\leq & \sup_{t\in[0,a]} \frac{t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} d(f(s, T^n y(s)) - f(s, T^n z(s))), \widehat{0}) ds \\
\leq & \frac{A}{\Gamma(q)} \sup_{t\in[0,a]} t^{1-q} \int_0^t (t-s)^{q-1} d(T^n y(s) - T^n z(s)), \widehat{0}) ds \\
\leq & \frac{A}{\Gamma(q)} \sup_{t\in[0,a]} t^{1-q} \int_0^t (t-s)^{q-1} s^{q-1} s^{1-q} d(T^n y(s) - T^n z(s)), \widehat{0}) ds \\
\leq & \frac{A}{\Gamma(q)} \|T^n y - T^n z\|_{1-q} \sup_{t\in[0,a]} t^{1-q} \int_0^t (t-s)^{q-1} s^{q-1} ds \\
= & \left(\frac{Aa^q\Gamma(q)}{\Gamma(2q)}\right) \|T^n y - T^n z\|_{1-q}.
\end{aligned}$$

Now using induction we get (12). Therefore we can apply Theorem 3.3 with $\beta_n = \left(\frac{Aa^q\Gamma(q)}{\Gamma(2q)}\right)^n$. It remains to show that the series $\sum_{n=0}^{\infty} \beta_n$ is convergent. Since $a \leq \widetilde{a}$ and the definition of $\widetilde{a}$ implies that $\left(\frac{Aa^q\Gamma(q)}{\Gamma(2q)}\right) < 1$. Thus by Theorem 3.3, there exists a unique solution of the integral equation (10). Then using Lemma 3.2 yields the existence and uniqueness of the Cauchy problem (8) and (9). □

4. Examples

Example 4.1.

$$\begin{cases} D^q y(t) = \lambda y(t) + b(t) \\ \lim\limits_{t\to 0^+} t^{1-q} y(t) = (1|3|4), \end{cases}$$

where $t \in [0, a]$, $0 < q \leq 1$, $\lambda \geq 0$, and $\lim\limits_{t\to 0^+} t^{1-q} y(t) = (1|3|4) \in E$ is a fuzzy triangular number, that is, $\lim\limits_{t\to 0^+} t^{1-q} y(t) = (2\alpha + 1, 4 - \alpha)$ for $\alpha \in (0, 1]$. If we put

$y(t) = (y_1(t,\alpha), y_2(t,\alpha))$, then $D^q y(t) = (D^q y_1(t,\alpha), D^q y_2(t,\alpha))$. Thus we have

$$\begin{cases} D^q y_1(t,\alpha) = \lambda y_1(t,\alpha) + b_1(t,\alpha), \\ \lim\limits_{t\to 0^+} t^{1-q} y_1(t,\alpha) = 2\alpha + 1, \end{cases} \tag{13}$$

and

$$\begin{cases} D^q y_2(t,\alpha) = \lambda y_2(t,\alpha) + b_2(t,\alpha), \\ \lim\limits_{t\to 0^+} t^{1-q} y_2(t,\alpha) = 4 - \alpha, \end{cases} \tag{14}$$

solution of (13) and (14) are given by (see [14])

$$y_1(t,\alpha) = \frac{2\alpha+1}{\Gamma(q)} t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) b_1(s,\alpha) ds,$$

and

$$y_2(t,\alpha) = \frac{4-\alpha}{\Gamma(q)} t^{q-1} E_{q,q}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) b_2(s,\alpha) ds,$$

where

$$E_{q,q}(\lambda t^q) = \sum_{n=0}^{\infty} \frac{(\lambda t^q)^n}{\Gamma(q(n+1))}.$$

Example 4.2.

$$\begin{cases} D^q y(t) = -\lambda y(t) + b(t), \ \lambda \geq 0 \\ \lim\limits_{t\to 0^+} t^{1-q} y(t) = (1|3|4). \end{cases} \tag{15}$$

We obtain the following system

$$\begin{aligned} D^q y_1(t,\alpha) &= -\lambda y_2(t,\alpha) + b_1(t,\alpha), \quad \lim_{t\to 0^+} t^{1-q} y_1(t,\alpha) = 2\alpha+1, \\ D^q y_2(t,\alpha) &= -\lambda y_1(t,\alpha) + b_2(t,\alpha), \quad \lim_{t\to 0^+} t^{1-q} y_2(t,\alpha) = 4-\alpha, \end{aligned}$$

or

$$\begin{aligned} D^q z(t) = A z(t) + B(t), \\ \lim_{t\to 0^+} t^{1-q} z(t) = c, \end{aligned} \tag{16}$$

where

$$z(t) = \begin{bmatrix} y_1(t,\alpha) \\ y_2(t,\alpha) \end{bmatrix}, \ A = \begin{bmatrix} 0 & -\lambda \\ -\lambda & 0 \end{bmatrix}, \ B(t) = \begin{bmatrix} b_1(t,\alpha) \\ b_2(t,\alpha) \end{bmatrix} \ c = \begin{bmatrix} 2\alpha+1 \\ 4-\alpha \end{bmatrix}.$$

Using the same method as in [12], we obtain the solution of (16). It is given by

$$z(t) = t^{q-1} E_{q,q}(At^q) c + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) B(s) ds,$$

where

$$E_{q,q}(At^q) = \sum_{i=0}^{\infty} \frac{(At^q)^i}{\Gamma(q(i+1))} = \begin{bmatrix} W_1(t) & 0 \\ 0 & W_1(t) \end{bmatrix} + \begin{bmatrix} 0 & -W_2(t) \\ -W_2(t) & 0 \end{bmatrix},$$

and

$$W_1(t) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^{2k}}{\Gamma(q(2k+1))}, \ \ W_2(t) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^{2k+1}}{\Gamma(2q(k+1))}.$$

Let

$$S_1(t) = \sum_{k=0}^{\infty} \frac{\lambda^{2k} t^{(2k+1)q-1}}{\Gamma(q(2k+1))}, \ \ S_2(t) = \sum_{k=0}^{\infty} \frac{\lambda^{2k+1} t^{2(k+1)q-1}}{\Gamma(2q(k+1))}.$$

Then

$$t^{q-1} E_{q,q}(At^q)c = \begin{bmatrix} S_1(t) & -S_2(t) \\ -S_2(t) & S_1(t) \end{bmatrix} \begin{bmatrix} 2\alpha+1 \\ 4-\alpha \end{bmatrix} = \begin{bmatrix} U_1(t,\alpha) \\ U_2(t,\alpha) \end{bmatrix}$$

where

$$U_1(t,\alpha) = S_1(t)(2\alpha+1) - S_2(t)(4-\alpha), \ \ U_2(t,\alpha) = S_1(t)(4-\alpha) - S_2(t)(2\alpha+1)$$

if we take

$$\begin{aligned} V_1(t,s,\alpha) &= S_1(t-s)b_1(s,\alpha) - S_2(t-s)b_2(s,\alpha), \\ V_2(t,s,\alpha) &= S_1(t-s)b_2(s,\alpha) - S_2(t-s)b_1(s,\alpha)), \end{aligned}$$

then we get

$$\int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)B(s)ds = \begin{bmatrix} \int_0^t V_1(t,s,\alpha)ds \\ \int_0^t V_2(t,s,\alpha)ds \end{bmatrix}.$$

Then we obtain

$$\begin{aligned} y_1(t,\alpha) &= U_1(t,\alpha) + \textstyle\int_0^t V_1(t,s,\alpha)ds, \\ y_2(t,\alpha) &= U_2(t,\alpha) + \textstyle\int_0^t V_2(t,s,\alpha)ds. \end{aligned}$$

It easy to see that $(y_1(t,\alpha), y_2(t,\alpha))$ define a fuzzy number, and therefore it is the solution of the fuzzy fractional differential equation (14).

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