On approximation properties of sampling operators defined by dilated kernels

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Abstract:

In this paper we consider some generalized Shannon sampling operators, which are defined by band-limited kernels. In particular, we use dilated versions of some previously known kernels. We give also some examples of using sampling operators with dilated kernels in imaging applications.

1. Introduction

For the uniformly continuous and bounded functions $f \in C(\mathbb{R})$ the generalized sampling series with a kernel function $s \in L^1(\mathbb{R})$ are given by ($t \in \mathbb{R}$; $W > 0$)

$$(S_W f)(t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{W}) s(Wt - k) \tag{1}$$

where

$$\sum_{k=-\infty}^{\infty} s(u - k) = 1, \tag{2}$$

and their operator norms are

$$\|S_W\| = \sum_{k=-\infty}^{\infty} |s(u-k)| < \infty \quad (u \in \mathbb{R}).$$

If the kernel function is $s(t) = \mathrm{sinc}\,(t) := \frac{\sin \pi t}{\pi t}$, we get the classical (Whittaker-Kotel'nikov-)Shannon operator S_W^{sinc}. The idea to replace the sinc kernel ($\mathrm{sinc}\,(\cdot) \notin L^1(\mathbb{R})$) by another kernel function $s \in L^1(\mathbb{R})$ appeared first in [15], where the case $s(t) = (\mathrm{sinc}\,(t))^2$ was considered. A systematic study of sampling operators (1) for arbitrary kernel functions s was initiated at RWTH Aachen by P. L. Butzer and his students since 1977 (see [3], [4], [14] and references cited there).

In this paper we consider the generalized sampling series with even band-limited kernels s, defined as the Fourier transform of an even window function $\lambda \in C_{[-1,1]}$, $\lambda(0) = 1$, $\lambda(u) = 0$ ($|u| \geqslant 1$) by the equality

$$s(t) := s(\lambda; t) := \int_0^1 \lambda(u) \cos(\pi t u)\, du = \sqrt{\frac{\pi}{2}} \lambda^{\wedge}(\pi t). \tag{3}$$

These types of kernels arise in conjunction with window functions widely used in applications (e.g. [1], [2], [11], [16]), in Signal Analysis in particular. Many kernels can be defined by (3), e.g.

1) $\lambda(u) = 1$ defines the sinc function;

2) $\lambda(u) = 1 - u$ defines the Fejér kernel (cf. [15])

$$s_F(t) = \frac{1}{2} \mathrm{sinc}^2 \frac{t}{2} = O(|t|^{-2});$$

3) $\lambda_H(u) := \cos^2 \frac{\pi u}{2} = \frac{1}{2}(1 + \cos \pi u)$ defines the Hann kernel (see [7])

$$s_H(t) := \frac{1}{2} \frac{\mathrm{sinc}\, t}{1 - t^2} = O(|t|^{-3});$$

4) the general cosine window

$$\lambda_{C,\mathbf{b}}(u) := \sum_{j=0}^{m} b_j \cos j\pi u \tag{4}$$

defines the Blackman-Harris kernel (see [9])

$$s_{C,\mathbf{b}}(t) := \frac{1}{2} \sum_{j=0}^{m} b_j \Big(\mathrm{sinc}(t - j) + \mathrm{sinc}(t + j) \Big) \tag{5}$$

provided

$$\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} b_{2j} = \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} b_{2j-1} = \frac{1}{2}. \tag{6}$$

From approximation theory point of view at least two problems for the generalized sampling operators $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ have some interest:

1) to calculate the operator norms

$$\|S_W\| = \sup_{u \in \mathbb{R}} \sum_{k=-\infty}^{\infty} |s(u-k)|; \tag{7}$$

2) to estimate the order of approximation

$$\|f - S_W f\|_C \leqslant M \omega_k(f, \frac{1}{W}) \tag{8}$$

in terms of the k-th modulus of smoothness $\omega_k(f, \delta)$.

2. Interpolating generalized sampling operators with dilated kernels

Let us consider the dilated kernel $s_\alpha(t) = \alpha s(\alpha t)$. The Shannon operators with sinc kernel satisfy the interpolatory conditions

$$(S_W^{\mathrm{sinc}})(\frac{k}{W}) = f(\frac{k}{W}) \quad (k \in \mathbb{Z}). \tag{9}$$

When we replace the sinc kernel with a band-limited one (3), we may lose the interpolatory property (9), but using the dilated kernel $\tilde{s}(t) = 2s(2t)$, we can recover the interpolatory property. If $s \in B^1_\pi$, then $s_\alpha \in B^1_{\alpha\,\pi}$, and the condition (2) is valid for $0 < \alpha \leqslant 2$, therefore we get the sampling operator $S_{W,\alpha} : C(\mathbb{R}) \to B^\infty_{\alpha\pi W} \subset C(\mathbb{R})$. Here B^p_σ stands for the Bernstein class consisting of those bounded functions $f \in L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ which can be extended to an entire function $f(z)$ $(z \in \mathbb{C})$ of exponential type σ.
Using the Nikolskii inequality [13], we get the bounds for the operator norm.

Theorem 1. *Let the operators $S_W : C(\mathbb{R}) \to B^\infty_{W\pi} \subset C(\mathbb{R})$, $S_{W,\alpha} : C(\mathbb{R}) \to B^\infty_{\alpha W\pi} \subset C(\mathbb{R})$ are defined by* (1) *with kernels s and s_α, respectively. Then*

$$\|s\|_1 \leqslant \|S_{W,\alpha}\| \leqslant (1+\alpha\pi)\|S_W\| \quad (0 < \alpha \leqslant 2).$$

The order of approximation by operators $S_{W,\alpha}$ we can estimate via modulus of smoothness $\omega_k(f,\sigma)$. Next theorem generalizes slightly the result in [10] (Th. 1.3).

Theorem 2. *Let $S_W : C(\mathbb{R}) \to C(\mathbb{R}),\ S_{W,\alpha} : C(\mathbb{R}) \to B^\infty_{\alpha W\pi} \subset C(\mathbb{R})$ be sampling operators defined by* (1) *with kernel functions $s \in B^1_\pi,\ s_\alpha \in B^1_{\alpha\,\pi}$, respectively.*
1) If $0 < \alpha \leqslant 1$, then there exist positive constants $C_{1,\alpha}$ and $C_{2,\alpha}$ such that

$$C_{1,\alpha}\|S_{\alpha W}f - f\|_C \leqslant \|S_{W,\alpha}f - f\|_C \leqslant C_{2,\alpha}\|S_{\alpha W}f - f\|_C.$$

2) Moreover, if $0 < \alpha < 2$, then

$$\|S_W f - f\|_C \leqslant M_k\omega_k(f, \frac{1}{W}), \qquad (10)$$

implies

$$\|S_{W,\alpha}f - f\|_C \leqslant M_{k,\alpha}\omega_k(f, \frac{1}{W})$$

for some constant $M_{k,\alpha} > 0$.

Example. The Blackman-Harris sampling operator $C_{W,\mathbf{b}}$ is defined by the window function

$$\lambda_{C,\mathbf{b}} := \sum_{j=0}^{m} b_j \cos(\pi j u).$$

In [9] we proved that for some values of the parameters $\mathbf{b} = (b_0, b_1, \ldots, b_m) \in \mathbb{R}^{m+1}$ we can estimate the order of approximation by operators $C_{W,\mathbf{b}} : C(\mathbb{R}) \to B^\infty_{W\pi} \subset C(\mathbb{R})$ via the modulus of continuity $\omega_{2\ell}(f, \frac{1}{W})$ $(\ell \leqslant m)$. More precisely (see [9], Th. 3), let ℓ, $1 \leqslant \ell \leqslant m$, be fixed. If for every $k = 0, \ldots, \ell - 1$

$$\sum_{j=0}^{m} j^{2k} b_j = 0 \ \ (0^0 = 1), \qquad (11)$$

then

$$\|f - C_{W,\mathbf{b}}f\|_C \leqslant M_{\mathbf{b},\ell}\,\omega_{2\ell}(f, \frac{1}{W}). \qquad (12)$$

Now by Theorem 2 we obtain for the corresponding dilated sampling operator $C_{W,\mathbf{b};\alpha} : C(\mathbb{R}) \to B^\infty_{\alpha W\pi} \subset C(\mathbb{R})$ with $0 < \alpha < 2$ the estimate

$$\|f - C_{W,\mathbf{b};\alpha}f\|_C \leqslant M_{\mathbf{b},\ell,\alpha}\,\omega_{2\ell}(f, \frac{1}{W}). \qquad (13)$$

The case $m = \ell = 1$ gives the Hann sampling operator $H_W : C(\mathbb{R}) \to C(\mathbb{R})$, which often has been used in practise. For the corresponding dilated operator $H_{W,\alpha} : C(\mathbb{R}) \to B^\infty_{\alpha W\pi} \subset C(\mathbb{R})$ for $0 < \alpha < 2$ we obtain

$$\|f - H_{W,\alpha}f\|_C \leqslant M_\alpha\omega_2(f, \frac{1}{W}). \qquad (14)$$

See Figure 2 for corresponding kernels.
The next theorem gives hints how to construct the interpolating sampling series.

Theorem 3. *Let the sampling operator $\tilde{S}_W$ be defined by (1) using the kernel $\tilde{s}(t) := 2s(2t)$, where the kernel $s \in B^1_\pi \subset L^1(\mathbb{R})$ is generated by (3) with a window function λ. If*

$$\lambda(u) + \lambda(1-u) = 1 \ \ (u \in [0,1]) \qquad (15)$$

then $\tilde{S}_W : C(\mathbb{R}) \to B^\infty_{2W\pi} \subset C(\mathbb{R})$ is an interpolating sampling operator.

Examples. For the Hann window function $\lambda_H(u)$ the condition (15) holds and we get the interpolating Hann sampling operator $\tilde{H}_W : C(\mathbb{R}) \to B^\infty_{2W\pi} \subset C(\mathbb{R})$. Taking $b_0 = 1/2,\ b_{2j} = 0 (j \in \mathbb{N})$ in (11) gives us the Blackman-Harris window function for which the condition (15) is fullfilled (see [10]).
In the case when $s \in B^1_{\beta\pi}$, $0 < \beta < 1$ and (15) holds for the corresponding window function we can prove the following theorem.

Theorem 4. *Let the sampling operator $\tilde{S}_W$ be defined by (1) using the kernel $\tilde{s}(t) := 2s(2t)$, where the kernel $s \in B^1_{\beta\pi} \subset L^1(\mathbb{R})$, $0 < \beta < 1$, is generated by (3) with a window function λ. If (15) is valid, then for every $k \in \mathbb{N}$ there exist a constant M_k such that*

$$\|\tilde{S}_W f - f\|_C \leqslant M_k\omega_k(f, \frac{1}{W}).$$

Example. So-called Lanczos n-kernels

$$\tilde{s}_{L,n}(t) := \operatorname{sinc}\frac{t}{n}\operatorname{sinc} t,$$

which has been often used in image processing. The Lanczos 3-kernel is especially popular in imaging ((see [16] and references cited there). They are defined by De la Vallée Poussin window function

$$\lambda_{L,n}(u) := \begin{cases} 1, & 0 \leqslant u \leqslant \frac{n-1}{2n}, \\ \frac{1}{2}(1 + n(1-2u)), & \frac{n-1}{2n} < u < \frac{n+1}{2n}, \\ 0, & u \geqslant \frac{n+1}{2n}. \end{cases}$$

If $n > 1$, then the De la Vallée Poussin window function $\lambda_{L,n}$ satisfies the conditions (15) and $\tilde{s}_{L,n} \in B^1_{(\frac{n+1}{2n})\pi}$, hence Theorem 4 is applicable. If $n = 1$, then we get the Fejér sampling operator (cf. [15]), for which we do not have even an estimate via the modulus of continuity ω_1.

3. Applications in 2D imaging

A natural application of sampling operators with dilated kernels is imaging. We can represent an discrete 2D image f as a continuous function using sampling series

$$(Sf)(x,y) := \sum_{j,k} f(j,k)s_1(x-j)s_2(y-k). \qquad (16)$$

Figure 1: Original image, derivatives with Hann kernel $\tilde{s}_H(t) = 2s_H(2t)$ and $s_{H,1/4}(t) = \frac{1}{2}s_H(\frac{1}{4}t)$ $(\varphi = \frac{2\pi}{3})$.

Many image resizing (resampling) algorithms use such type of representation (see [16], [12], [6]). If the image data is exact, then we can take interpolating kernels s_1 and s_2, like interpolating Hann, Blackman-Harris or Lanczos, and enlarge (up-sample) image, having $(Sf)(j,k) = f(j,k)$. If we want to reduce the image size (down-sample) (magnification $\gamma < 1$) then, for eliminating artifacts, we can choose a dilated kernel s_α with in some sense optimal value of $\alpha = 2\gamma$ (see Figure 2). The artifacts in down-sampled images appear, because details that are resized to smaller than one pixel will be misrepresented by larger aliases (see [5], [6]). Depending on the choice of the parameter value α we have $S_{W,\alpha} : C(\mathbb{R}) \to B^\infty_{\alpha W\pi}$ i.e. a function belonging to a class for bandlimited functions, for which the Fourier' transform vanishes outside of the interval $[-\alpha W\pi, \alpha W\pi]$. This approach eliminates higher spatial frequencies, being equivalent to the use of low-pass filter. Also in the case, when the resolution of the optical system is less than the resolution of the sensor, we can choose the value of the dilation parameter α accordingly.

Using the representation (16) we can apply different imaging technics. For image enhancement we can use the unsharp masking (see [5], [6]), i.e. to subtract a blurred version of an image from the image itself. For the representation of original image $f(x,y)$ we can choose in (16) the interpolating kernels (dilation by $\alpha = 2$), but to get blurred version $f_b(x,y)$, we choose in (16) the dilated kernels with small parameter α, like $s_{H,1/2}$ in Figure 2. We can control the amount of unsharp masking choosing the parameter $a < 0$:

$$f_{usm}(x,y) = (1-a)f(x,y) + af_b(x,y).$$

Another well-known image enhancement method uses the derivatives of image. First derivatives in image processing are implemented using the magnitude of the gradient. The representation (16) gives us a natural way to implement derivatives. Indeed

$$f_x(x,y) := \sum_{j,k} f(j,k)s_1'(x-j)s_2(y-k),$$

$$f_y(x,y) := \sum_{j,k} f(j,k)s_1(x-j)s_2'(y-k).$$

Surprisingly, if we choose Hann kernel $s_1 = s_2 = s_H$ and $x, y \in \mathbb{Z}$, then the discrete convolution

$$f_x(p,q) \approx \sum_{j=p-1}^{p+1} \sum_{k=q-1}^{q+1} f(j,k)s_H'(p-j)s_H(q-k) \quad (17)$$

gives us the well-known Sobel filter (see [5], [6])

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}.$$

Indeed, $s_H(k) = 0$ $(k \in \mathbb{Z})$ if $|k| > 1$ (see Figure 1) and we get $\frac{1}{4}(1,2,1)$. For s_H' we use the first 3 values only, i.e. $\frac{3}{8}(1,0,-1)$.

We can easily compute a directional derivative

$$f_\varphi(x,y) := \sum_{j,k} f(j,k)s_1'\Big((x-j)\cos\varphi - (y-k)\sin\varphi\Big)\times$$
$$\times s_2\Big((y-k)\cos\varphi + (x-j)\sin\varphi\Big),$$

To get the edges with different spatial frequency, we choose the dilation parameter (see Figure 1).

Second derivatives in image processing are implemented using the Laplacian. Using the representation (16) we get

$$\triangle f(x,y) := f_{xx}(x,y) + f_{yy}(x,y) =$$
$$\sum_{j,k} f(j,k)\big(s''(x-j)s(y-k) + s(x-j)s''(y-k)\big).$$

In image processing we use the derivatives for edge detection. Changing the dilation parameter α for the kernel $s_\alpha(t) = \alpha s(\alpha t)$ we can detect edges with different spatial frequencies.

In calculations we must use the truncated sampling series $(p, q \in \mathbb{Z})$

$$(Sf)_{mn}(p,q) := \sum_{j=p-m}^{p+m} \sum_{k=q-n}^{q+n} f(j,k)s_1(p-j)s_2(q-k)$$

and have the truncation error. We can use kernels with finite support like the combinations of B-splines, considered in [4], to get rid of the truncation error, but in some

Figure 2: Unsharp mask with Hann kernel $s_{H,1/2} = \frac{1}{2}s_H(\frac{1}{2}t)$, $a = -1.7$.

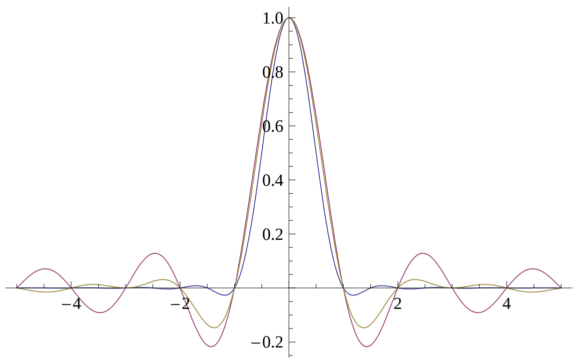

Figure 3: Hann kernel $\tilde{s}_H(t) = O(|t|^{-3})$, Lanczos kernel $s_{L,3}(t) = O(|t|^{-2})$ and $\mathrm{sinc}(t) = O(|t|^{-1})$.

cases other types of kernels are more suitable. For minimizing the truncation error the kernel $s(t)$ must decrease rapidly when $|t| \to \infty$. The sinc function does not belong even to L^1. Therefore using the kernels in form $s(t) = \theta(t)\,\mathrm{sinc}\,t$, where $\theta(t)$ is some window function (see [11]), is well-known. In most cases of we lose the important property (2) and do not get a generalized sampling series anymore. The kernels in our approach, i.e. kernels defined via Fourier transform of window functions, allow us to get good approximation properties and are rapidly decreasing. In Figure 3 we take the Hann kernel $\tilde{s}_H(t) = O(|t|^{-3})$ and compare it with the Lanczos kernel $s_{L,3}(t) = O(|t|^{-2})$, which is one of the most used kernels in imaging (see [16]). In the case of Blackman-Harris kernels (5), considered more precisely in [9], we have $s_{C,\mathbf{b}} = (|t|^{-2\ell-1})$ if for every $k = 0, \ldots, \ell - 1$

$$\sum_{j=0}^{m} j^{2k} b_j = 0.$$

We defined many rapidly decreasing kernels also in [8], [7], [10].

4. Acknowledgments

This research was supported by European Social Fund Funds Doctoral Studies and Internationalisation Programme DoRa, by the Estonian Science Foundation, grants 6943, 7033, and by the Estonian Min. of Educ. and Research, projects SF0132723s06, SF0140011s09. The second author wants to thank Archimedes Foundation for support.

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