# The Manifold of Planes that Intersect Four Straight Lines in Points of a Circle

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Our topic is the manifold of planes that intersect four straight lines in three-dimensional euclidean space in points of a circle. The solution manifold is of class seven and contains 24 single lines, four double lines, a triple plane and four dual conics. We compute the solution manifold's equation, visualize it and discuss the special case of the four base lines being contained in a regulus.

## 1 Introduction

In this text, we study the set $\mathcal{L}$ of planes in three-dimensional euclidean space $\mathbb{E}^3$ that intersect four straight lines in points of a circle. The manifold of solution planes $\mathcal{L}$ is, in general, of dimension two. Furthermore, there are numerous ways of seeing that it is algebraic. We will give two independent proofs for that. Both also yield the class of $\mathcal{L}$ (it is seven).

Our investigations heavily rely on a concept already used in Schröcker (2004) for studying the intersection conics of six straight lines. The present situation is, however, slightly more complicated and some important modifications are necessary.

In the projective extension $\mathbb{P}^3$ of euclidean three-space, circles are characterized as bisecant conics of the absolute circle. Therefore, our problem has an obvious projective generalization: Study the set of planes in $\mathbb{P}^3$ that intersect a conic section and four straight lines in six points of a conic section. Because of its generality and clarity, we will usually take this latter point of view. Only for computational issues (Section 4), the euclidean setting seems to be more appropriate. This is also the reason why all visualizations in this text refer to the original circle problem.

After introducing a few basic conventions and notions in Section 2, we discuss the use of Pascal's Theorem for identifying solution planes (Section 3). It leads to a method for counting the solution planes in a pencil and, consequently, to the mentioned result on the solution manifold's class (Theorem 1). The extension of this method to non-generic pencils of planes in Section 5 yields results on special planes, pencils of planes and dual conics (sets of tangent planes of quadratic cones) in $\mathcal{L}$.

In Section 6, we investigate the special case of the four base lines being contained in a regulus $\mathfrak{R}$. The solution manifold splits into the dual carrier quadric $\mathcal{R}$ of $\mathfrak{R}$ and an algebraic remainder $\mathcal{T}$ of class five. Our main result concerns the curious intersection of $\mathcal{R}$ and $\mathcal{T}$ which consists of eight pencils of planes and a dual conic.

Section 4 is dedicated to the computation of an algebraic equation for $\mathcal{L}$. It uses the geometry of circles in space and is the basis for visualization purposes throughout this text.

## 2 Definitions and conventions

In complex projective space $\mathbb{P}^3$ of dimension three, four straight lines $B_0, \ldots, B_3$ and a conic section $C$ are given. The straight lines $B_i$ will be referred to as *base lines* the conic $C$ as *base conic*. We give the following definition of fundamental concepts:

**Definition 1.** A conic section $D \subset \mathbb{P}^3$ is called a *solution conic*, if it intersects each base line $B_i$ in at least one and the base conic $C$ in at least two (possibly coinciding) points.

**Definition 2.** A plane $\varepsilon \subset \mathbb{P}^3$ is called *solution plane*, if it contains at least one solution conic. It is called *singular* if it contains at least one singular solution conic and *regular* otherwise.

The union $\mathcal{L}$ of all solution planes is called *solution manifold*. As already mentioned, it is algebraic. Note, however, that the singular solution planes are not necessarily singular in algebraic sense.

Unless stated otherwise, we assume a generic configuration of base lines and base conic. This ensures that $\mathcal{L}$ is of dimension two. Instances where all planes of $\mathbb{P}^3$ are solution planes (base lines and base conic have a common carrier quadric, four base lines or the base conic and two base lines lie in a common plane) are of little interest and will not be dealt with in this text.

## 3 Pascal's Theorem

Pascal's well-known theorem (Figure 1) provides a simple method for deciding whether a given plane $\varepsilon$ is a solution plane or not. We let $\mathbf{b}_i := \varepsilon \cap B_i$, denote the two intersection points of $\varepsilon$ and $C$ by $\mathbf{c}_0$ and $\mathbf{c}_1$ and define the

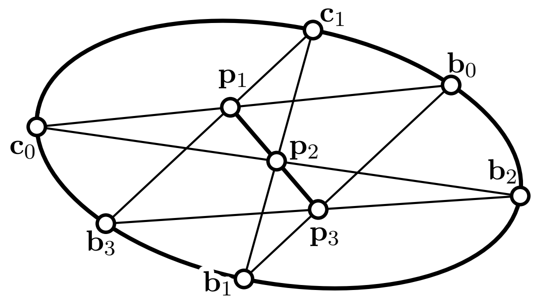


**Figure 1:** Pascal's Theorem.

*Pascal points* of $\varepsilon$ as

$$\begin{aligned} \mathbf{p}_1 &:= [\,\mathbf{b}_0, \mathbf{c}_0\,] \cap [\,\mathbf{b}_3, \mathbf{c}_1\,], \\ \mathbf{p}_2 &:= [\,\mathbf{b}_2, \mathbf{c}_0\,] \cap [\,\mathbf{b}_1, \mathbf{c}_1\,], \\ \mathbf{p}_3 &:= [\,\mathbf{b}_0, \mathbf{b}_1\,] \cap [\,\mathbf{b}_2, \mathbf{b}_3\,]. \end{aligned} \tag{1}$$

Now it would be tempting to say that $\varepsilon$ is a solution plane iff its Pascal points are collinear. Unfortunately, this is not true – not even when all Pascal points are well-defined. The following problems might occur:

- One of the Pascal points $\mathbf{p}_i$ is not well-defined for whatever reason. Planes with that property will be called *special* (as opposed to *ordinary* planes).
- The plane $\varepsilon$ is tangent to the base conic $C$. In this case, the points $\mathbf{c}_0$ and $\mathbf{c}_1$ coincide and Pascal's Theorem in the version of Equation 1 will always (and usually wrongly) identify $\varepsilon$ as a solution plane.

Despite these exceptional cases, Pascal's Theorem is still an excellent tool for identifying solution planes and we can at least state

**Proposition 1.** *Special planes and ordinary planes with collinear Pascal points that are not tangent to $C$ are always solution planes.*

Tangent planes of $C$ cannot be judged by means of Equation 1. Of course it is possible to use different Pascal points for the solution plane test. This will, however, not be necessary in this text.

We want to use Pascal's Theorem for counting the ordinary solution planes in a pencil of planes. For that purpose we choose the pencil axis $E$ and a rational quadratic parametrization $\mathbf{c}_0(t)$ of $C$. It defines a quadratic parametrization $\varepsilon(t) := [E, \mathbf{c}_0(t)]$ of the pencil of planes through $E$ and induces rational parameterized equations for the points $\mathbf{b}_i(t) \in B_i$, $\mathbf{c}_1(t) \in C$ and $\mathbf{p}_i(t)$ from Equation 1.

As $t$ varies in the parameter space $\mathbb{C} \cup \{\infty\}$, the trajectory of the point $\mathbf{p}_i(t)$ is a rational curve $P_i$. Generic position of $E$ provided, every plane $\varepsilon$ through $E$ and hence every point of $P_3$ belongs to two parameter values, while the points of $P_1$ and $P_2$ belong to one one parameter only. The curves $P_i$ will be called the *Pascal curves* to the straight line $E$. The degree of $\mathbf{b}_i(t)$ and $\mathbf{c}_1(t)$ is two, the degree $p_i$ of $\mathbf{p}_i(t)$ will be determined soon (in Lemma 1).

A second rational parametrization of the pencil of planes through $E$ is obtained as

$$\varphi(t)\colon \mathbf{u}(t) = \mathbf{p}_0(t) \wedge \mathbf{p}_1(t) \wedge \mathbf{p}_2(t).$$

The plane coordinate vector $\mathbf{u}(t)$ is proportional to the induced rational parameterization $\mathbf{v}(t)$ of $\varepsilon(t)$ and of degree $p = p_1 + p_2 + p_3$. Since $\mathbf{v}(t)$ is quadratic, there exist $p - 2$ zeros $t_0, \ldots, t_{p+1}$ of $\mathbf{u}(t)$. Two of them describe tangent planes of $C$. The remaining values correspond to the ordinary solution planes through $E$. Because two parameter values $t_i$ and $t_j$ describe the same plane, the total number of solution planes in the generic case is given as

$$\varrho(E) := (p - 4)/2. \tag{2}$$

If $E$ is in a general position, all its solution planes are ordinary and $\varrho(E)$ equals the class of $\mathcal{L}$. From these considerations, it follows that $\mathcal{L}$ is algebraic. The class will be known as soon as we can compute $\varrho(E)$ for generic arguments:

**Lemma 1.** *The parameterizations $\mathbf{p}_i(t)$ of the Pascal curves $P_i$ associated to a generic pencil axis $E$ are of degree six.*

*Proof.* We begin with the Pascal curve $P_3$. It lies in the intersection of the quadrics $[E, B_0, B_1]$ and $[E, B_2, B_3]$ (i.e., the carrier quadric of the regulus defined by $E$, $B_i$ and $B_j$) and is therefore a twisted cubic. Since every point of $P_3$ belongs to two parameter values, the degree of $\mathbf{p}_3(t)$ is really six.

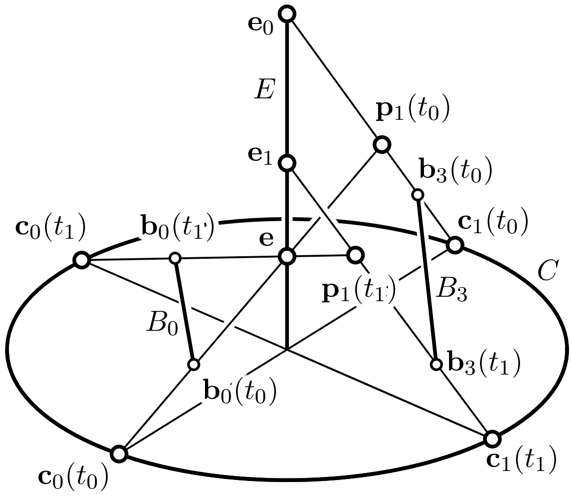


**Figure 2:** The algebraic correspondence on $E$.

The first and second Pascal curve are generated in identical ways. Therefore, it is enough to study only one of them, say $P_1$. In contrast to $P_3$, it has to be counted with multiplicity one. Consequently, $P_1$ and $\mathbf{p}_1(t)$ are of equal degree $p_1$. For a generic parameter value $t$, the plane $\varepsilon(t)$ intersects $P_1$ in $p_1$ points, two of which are *not* contained in $E$. The intersection points of $E$ and $P_1$ are found as fixed points of an algebraic $(2,2)$-correspondency $\alpha$ on $E$, relating a point

$$\begin{aligned} \mathbf{e} &= E \cap [\mathbf{c}_0(t_0), \mathbf{b}_0(t_0)] \\ &= E \cap [\mathbf{c}_0(t_1), \mathbf{b}_0(t_1)] \end{aligned}$$

to the intersection points $\mathbf{e}_i$ of $E$ and the span of $\mathbf{c}_1(t_i)$ and $\mathbf{b}_3(t_i)$ (Figure 2). By Chasles' principle of correspondence (see for example Müller and Krames, 1931, p. 259) there exist four fixed points of $\alpha$. Hence, the degree of $P_1$ is $4 + 2 = 6$. $\square$

As a consequence of Lemma 1 we have computed the degree of $\varphi(t)$ as $\deg(\varphi) = p_1 + p_2 + p_3 = 18$. By Equation 2 we find $\varrho(E) = 7$ and the proof of this section's central theorem is finished:

**Theorem 1.** *The solution manifold $\mathcal{L}$ is algebraic and of class seven.*

*Remark.* Schröcker (2004) shows that the manifold of planes that intersect six straight lines in points of a conic section is of class eight. If two of the straight lines intersect, a bundle of planes splits off and the remaining manifold is of class seven. In this sense, Theorem 1 holds for a singular base conic as well.

An alternative proof for Theorem 1 will be given in Section 4. Instead of synthetic reasoning, it uses straightforward computation. Note further that the presented concept goes far beyond the result of Theorem 1. In Section 5 we will use it extensively for investigating special plane sets in $\mathcal{L}$.

## 4 The algebraic equation

As we are approaching more "depictable" properties of $\mathcal{L}$, it is time for a few remarks on the computation of its algebraic equation and its visualization. We propose an approach via the classic geometry of circles in space (see for example Coolidge, 1971). It is not elementary but lucid and straightforward. In contrast to the preceding sections, we will use the euclidean setting.

### 4.1 Circle geometry

We consider the mapping $\gamma$ from the set of euclidean planes and spheres into $\mathbb{P}^4$, defined as follows: The sphere $\Sigma$ with center $\mathbf{m} = (1:a:b:c)^{\mathrm{T}}$ and radius $r$ is mapped to the point

$$\gamma(\Sigma) = (\mu + 1 : 2a : 2b : 2c : \mu - 1)^{\mathrm{T}}$$

with $\mu = a^2 + b^2 + c^2 - r^2$. The $\gamma$-image of the plane $\varepsilon\colon ax + by + cz + d = 0$ is the point

$$\gamma(\varepsilon) = (-d : a : b : c : -d)^{\mathrm{T}}.$$

Embedding $\mathbb{E}^3$ in $\mathbb{E}^4 \subset \mathbb{P}^4$ via the natural identification

$$(1 : x : y : z)^{\mathrm{T}} \mapsto (1 : x : y : z : 0)^{\mathrm{T}}$$

reveals the geometric meaning of $\gamma$: We denote the unit hypersphere in $\mathbb{E}^4 \subset \mathbb{P}^4$ by M and the stereographic projection of $\mathbb{E}^3 \subset \mathbb{E}^4$ from its north pole $\mathbf{n}$ by $\sigma$. Now, it is not difficult to verify (see Paluszny and Bühler, 1998):

**Proposition 2.** *The $\gamma$-image of a sphere (or plane) $\Sigma$ is the pole of the span of $\Sigma$'s stereographic projection onto the unit hypersphere $\mathrm{M} \subset \mathbb{E}^4$.*

A circle $D \subset \mathbb{E}^3$ is incident with a one-parametric set of spheres (a *pencil of spheres*). Via $\gamma$, this pencil is transformed into a straight line $\gamma(D)$ that can be addressed as the $\gamma$-image of $D$. Similarly, we call the straight line $\gamma(L)$, whose points correspond to the planes of the pencil of planes through a straight line $L$, the $\gamma$-image of $L$. It lies in the tangent hyperplane $\mathcal{N}$ of M in the north-pole $\mathbf{n}$. In this way, the geometry of circles in space is transformed to the line geometry of $\mathbb{P}^4$.

Since all spheres through a given point have stereographic images in hyperplanes through a point of M, their $\gamma$-images are located in the corresponding tangent hyperplane and we find

**Proposition 3.** *Two circles $D, E \subset \mathbb{E}^3$ have a common point, if their $\gamma$-images span a tangent hyperplane of* M. *They have two common points, if the span of $\gamma(D)$ and $\gamma(E)$ is a two-dimensional plane.*

Note that Proposition 3 is still valid, if $D$ is a straight line. If we agree on a common point $\infty$ of all planes in $\mathbb{E}^3$ (conformal closure of $\mathbb{E}^3$), it also holds for two straight lines.

### 4.2 Computing the equation

Proposition 3 can be used for a straightforward computation of the algebraic equation of $\mathcal{L}$: Via $\gamma$, the four base line $B_i$ correspond to straight lines $H_i \subset \mathcal{N}$. Together with a further point $\mathbf{u} = (-u_3 : u_0 : u_1 : u_2 : -u_3)^{\mathrm{T}}$ of $\mathcal{N}$ (the $\gamma$-image of an undetermined plane $\varepsilon$ in $\mathbb{P}^3$), the straight line $H_i$ spans a plane $\eta_i$ that is incident with two tangent hyperplanes of M: The north-pole hyperplane $\mathcal{N}$ and a further hyperplane $\mathcal{T}_i$. The computation of $\mathcal{T}_i$ is a linear problem, its coefficients are rational of degree two in the coordinates $u_i$ of $\mathbf{u}$.

By Proposition 3, the point $\mathbf{u}$ is the $\gamma$-image of a solution plane if and only if the hyperplanes $\mathcal{T}_i$ are in special position. With the help of an arbitrary hyperplane $\mathcal{V} = (v_0 : \ldots : v_4)^{\mathrm{T}}$, this can be tested by evaluating the equation

$$\det(\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{V}) = 0. \tag{3}$$

It is algebraic of degree eight and describes the union of the solution manifold $\mathcal{L}$ with a bundle of planes (the $\gamma$-pre-images of $\mathcal{N} \cap \mathcal{V}$). The equation of $\mathcal{L}$ is found by splitting off the linear component

$$-(v_0 + v_4)u_0 + v_1 u_1 + v_2 u_2 + v_3 u_3 = 0.$$

Thus, the algebraic equation of $\mathcal{L}$ is of degree seven. This provides a second, computational, proof for Theorem 1.

*Remark.* It would be possible to compute the algebraic equation of $\mathcal{L}$ in a completely *elementary* way, using only concepts of basic euclidean geometry (intersection of straight planes and lines, planes of symmetry...). From a pragmatic point of view, this is entirely satisfactory. We presented a different way because of some theoretical deficiencies of the elementary approach. In particular, a computational proof for Theorem 1 is not that easy: The elementary computation results in an algebraic equation of high degree, i.e., it describes unwanted components. Of course, they can easily be eliminated but – in contrast to the approach via circle geometry – it is not easy to justify this in an accurate manner. Furthermore, the circle geometry seems to have some potential for deriving further results in related questions (compare also Section 8).

### 4.3 A few words on visualization

In Figure 3, an example of the solution manifold for a generic base line and base conic configuration is displayed. In order to produce this picture we

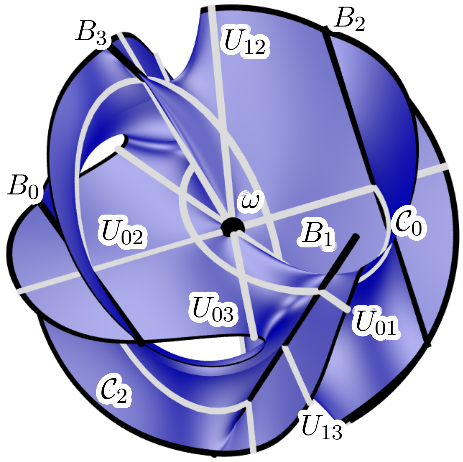


**Figure 3:** Visualization of the solution manifold in the general case.

1. computed the algebraic equation of $\mathcal{L}$ as has been described,
2. chose a suitable affine sheet (by substituting, for example, $u_0 = 1$, $u_1 = x$, $u_2 = y$, $u_2 = z$ and
3. fed the resulting equation into a ray-tracing program.

This implies the interpretation of plane coordinates as point coordinates. In other words, the surface depicted in Figure 3 is obtained from $\mathcal{L}$ by polarization at the complex sphere with equation

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

In real euclidean space, this transformation is the composition of the polarization at the unit sphere and the reflection at the origin.

Pencils of planes in $\mathcal{L}$ are transformed into straight lines (different from the pencil axes), the tangent planes of quadratic cones into conic sections. Examples of that can be seen in Figure 3. Their meaning will be revealed in the coming section.

## 5 Special planes, lines and cones

We continue with the investigation of pencils of planes through a straight line and other special plane sets in $\mathcal{L}$. The setting is switched back to projective space.

If a pencil of planes through a straight line $L$ is contained in the solution manifold $\mathcal{L}$, we say for short that $L$ *is contained* in $\mathcal{L}$. Examples of straight lines in $\mathcal{L}$ are easily spotted. We find

- the four base lines $B_i$,
- their two transversals $T_0$ and $T_1$ (not necessarily real),
- the six straight lines $U_{ij}$ in the plane of $C$ that intersect $B_i$ and $B_j$ and
- the sixteen straight lines $V_{0i}, \ldots, V_{3i}$, that meet $C$ and three base lines $B_j$, $B_k$, $B_l$ (where $\{i, j, k, l\} = \{0, 1, 2, 3\}$).

The solution planes through these lines are exclusively singular. In Figure 3, the four base lines $B_i$ and the lines $U_{ij}$ in the plane of $C$ are depicted. Since the latter lie in a common plane, their representatives are concurrent. The base line transversals $T_0$ and $T_1$ as well as the straight lines $V_{ij}$ are complex.

### 5.1 Double lines

A glance at Figure 3 suggests that the base lines $B_i$ are actually double lines of $\mathcal{L}$. In order to prove this, we use the concept of Section 3. We choose a generic transversal $E$ of $B_0$ and $B_3$ and study the pencil of planes through $E$. Its only special solution planes are $\varepsilon_0 = [\,E, B_0\,]$ and $\varepsilon_1 = [\,E, B_3\,]$. For reasons of symmetry, their multiplicities as elements of $\mathcal{L}$ are identical. As in Section 3, a quadratic parametrization $\mathbf{c}(t)$ of $C$ leads to Pascal curves $P_1$, $P_2$ and $P_3$. This time, $P_1$ is a conic (part of the intersection curve of the two quadratic cones through $C$), $P_2$ is rational of degree six and $P_3$ is a quadratically parameterized straight line. Thus, we have $\varrho(E) = 3$ and the multiplicity of $\varepsilon_0$ and $\varepsilon_1$ is

$$\frac{\deg(\mathcal{L}) - \varrho(E)}{2} = \frac{7-3}{2} = 2.$$

Hence, $B_0$ and $B_3$ are double lines. By similar means one can show that the other straight lines on $\mathcal{L}$ are usually of multiplicity one. Thus, we can state:

**Theorem 2.** *In general, the solution manifold $\mathcal{L}$ contains 28 straight lines: The four base lines, their two transversals, the six straight lines in the plane of $C$ that intersect two base lines and the sixteen transversals of $C$ and three base lines. The base lines are double lines of $\mathcal{L}$.*

### 5.2 The base conic plane

Similar (and even simpler) reasoning shows that the plane $\omega$ of $C$ is a triple plane of $\mathcal{L}$: We assume this time that the straight line $E$ lies in $\omega$ and that the pencil of planes through $E$ is parameterized *linearly* by $\varepsilon(t)$. Because the intersection points of $\varepsilon(t)$ with $C$ remain fixed, the corresponding pascal curves $P_1$, $P_2$ and $P_3$ are, in that order, of degree one, one and three. Thus, there exist four solution planes besides $\omega$ through $E$ and we find

**Theorem 3.** *The base conic plane $\omega$ is a triple plane of the solution manifold $\mathcal{L}$.*

The triple plane of $\mathcal{L}$ is visualized in Figure 3. It corresponds to the intersection point of the straight lines $U_{ij}$ and is a triple point of the displayed surface. As far as the euclidean circle problem is concerned, there exist two noteworthy corollaries to Theorem 1 and Theorem 3:

**Corollary 1 (euclidean setting).** *In general, a pencil of parallel planes contains four euclidean solution planes while a pencil with finite axis contains seven.*

**Corollary 2 (euclidean setting).** *The manifold of planes that intersect the four sides of a spatial quadrilateral in points of a circle consists of four bundles of planes and the tangent planes of a cubic curve in the plane at infinity.*

*Proof.* The bundles of planes through the quadrilateral's vertices are singular components of the solution manifold. The remaining component is of class three and has $\omega$ as a plane of multiplicity three or higher. Hence, it is a cubic dual cone with "vertex" $\omega$. This plane set in a euclidean interpretation is described in the theorem. □

### 5.3 Dual conics

Besides the straight lines enumerated in Theorem 2, we also find dual conics (set of tangent planes of a quadratic cone) in $\mathcal{L}$. Consider the bundle of planes through $\mathbf{g}_0 := \omega \cap B_0$. It intersects the solution manifold in the three straight lines $U_{01}$, $U_{02}$, $U_{03}$, the double line $B_0$ and a remainder $\mathcal{C}_0$ of class two, i.e., a dual conic.

The geometric meaning of $\mathcal{C}_0$ is related to the quadric $\mathcal{Q}_0$, spanned by the base lines $B_1$, $B_2$ and $B_3$. Since every tangent plane of $\mathcal{Q}_0$ through $\mathbf{g}_0$ is a (singular) solution plane, the dual conic $\mathcal{C}_0$ is the *tangent cone of* $\mathcal{Q}_0$ *with vertex* $\mathbf{g}_0$. All in all, four quadratic cones $\mathcal{C}_0, \ldots, \mathcal{C}_3$ of that type are contained in $\mathcal{L}$. In Figure 3, two of them are visualized as conic sections.

## 6 Four base lines in a regulus

So far we have assumed a generic configuration of the base lines $B_0, \ldots, B_3$ and the base conic $C$ (with exception of Corollary 2). In this section, we will study a special configuration of particular interest. We assume that the four base lines are contained in a *regulus* $\mathfrak{R}$. The carrier quadric of $\mathfrak{R}$ be denoted by $\mathcal{Q}$, the set of its tangent planes (a dual quadric) by $\mathcal{R}$.

The solution manifold $\mathcal{L}$ splits into $\mathcal{R}$ and a remainder $\mathcal{T}$ of class five – a curious pair of dual manifolds, as we shall see. In Figure 4, an example is depicted. Apparently, the intersection of $\mathcal{T}$ and $\mathcal{R}$ contains a quadratic dual conic $\mathcal{C}$ – visualized as a conic section – and the four base lines. The latter property is no surprise: $B_i$ is a double line of $\mathcal{L}$, a single line of $\mathcal{R}$ and hence a single line of $\mathcal{T}$ as well. The first property is a consequence of the following lemma and the fact that there exist four base line transversals $W_0, \ldots, W_3$ that also intersect $C$. It can be proved by following the concept of Section 3.

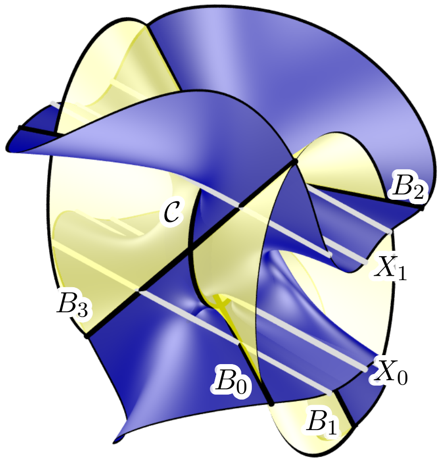


**Figure 4:** The four base lines lie in a regulus.

**Lemma 2.** *A transversal of the four base lines and the base conic is a double line of* $\mathcal{L}$.

Note that Lemma 2 is also valid if they base lines are not contained in a regulus. We summarize its consequences in

**Theorem 4.** *If the four base lines lie in a regulus, the solution manifold splits into a dual quadric* $\mathcal{R}$ *and a remaining dual manifold* $\mathcal{T}$ *of degree five. The intersection of* $\mathcal{R}$ *and* $\mathcal{T}$ *consists of eight straight lines (the four base lines and four of their transversals) and a dual conic.*

It is well known that there exist six pencils of parallel planes (four of them complex) whose planes intersect a given euclidean quadric $\mathcal{Q}$ in circles. By projective generalization we find that $\mathcal{T}$ contains six pencils of planes that have not been mentioned in Theorem 2. Their axes $X_i$ are the diagonals of the quadrangle formed by the intersection points of $\mathcal{Q}$ and $C$ (see also Figure 4, where the two real straight lines $X_0$ and $X_1$ are displayed). All in all, $\mathcal{T}$ contains a total of at least 16 straight lines.

The dual conic $\mathcal{C}$ in the intersection of $\mathcal{R}$ and $\mathcal{T}$ (Theorem 4) has a very simple geometric meaning: Its planes are tangent to $\mathcal{Q}$ along the intersection of $\mathcal{Q}$ and $\omega$. In order to see this, we have to understand how the concept of Pascal curves works in our special case:

In general, the first and second Pascal curves $P_1$ and $P_2$ are still rational of degree six. The third Pascal curve is a straight line because the intersection of the two quadrics $[E, B_0, B_1]$ and $[E, B_2, B_3]$ is highly reducible: It consists of $E$, two transversals of $E$ and the base lines and a fourth straight line $P_3$. Thus, we have $\varrho(E) = 5$ for a generic pencil axis $E$. If $E$ is a tangent of the intersection conic of $\omega$ and $\mathcal{Q}$, the Pascal curves $P_1$ and $P_2$ are straight lines (compare the proof of Theorem 3). The third Pascal curve $P_3$ is even more degenerate: It is a single point because the quadrics $[E, B_0, B_1]$ and $[E, B_2, B_3]$ touch along $E$ and a *transversal* of $E$. Since the multiplicity of $\omega$ as element of $\mathcal{L}$ is still three (Theorem 3), there exists a second special solution plane of $\mathcal{T}$ through $E$ – the tangent plane of $\mathcal{Q}$.

**Theorem 5.** *The dual conic in the intersection of $\mathcal{T}$ and $\mathcal{R}$ (compare Theorem 4) consists of the tangent planes of $\mathcal{Q}$ along its intersection with the base conic plane $\omega$.*

If the quadric $\mathcal{Q}$ has $\omega$ as tangent plane we make two further observations:

1. The intersection of $\mathcal{Q}$ and $\omega$ consists of two straight lines $E$ and $F$. One of them, say $E$, intersects the four base lines.
2. Two of the four base line transversals that also meet the base conic $C$ coincide with $E$.

Hence, $E$ is a triple line of $\mathcal{T}$ and the intersection of $\mathcal{T}$ with $\mathcal{R}$ consists only of straight lines. We state this as a corollary to Theorem 4 and use the euclidean setting for its formulation:

**Corollary 3 (euclidean setting).** *If the four base lines are skew generators of a hyperbolic paraboloid, the intersection of $\mathcal{R}$ and $\mathcal{T}$ consists of eight straight lines, one of them of multiplicity three.*

An example is depicted in Figure 5. Six straight lines of the intersection are clearly visible, among them the triple line of $\mathcal{T}$. The two further lines are complex.

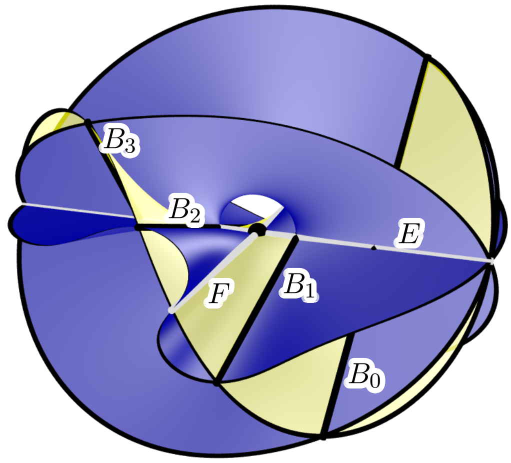


**Figure 5:** The four base lines lie on a hyperbolic paraboloid.

## 7 Concyclic base lines on a cylindroid

A curious base line configuration with reducible solution manifold $\mathcal{L}$ has been described in Stachel (1995), though under a slightly different viewpoint. The author characterizes those quadrupels of pairwise skew straight lines that are tangent to a one-parameter family of spheres. Besides the obvious solution of four generators of a hyperboloid of revolution there also exist special quadrupels of *concyclic generators* on a cylindroid (Plücker conoid) $\Psi$. Every tangent planes of $\Psi$ intersects the concyclic generators in four points of a circle and is hence a solution plane. The solution manifold $\mathcal{L}$ splits into the tangent plane manifold $\mathcal{C}$ of $\Psi$ and a remainder $\mathcal{D}$ of class four. A dual view of both plane manifolds is depicted in Figure 6

The line at infinity $G$ of $\Psi$ intersects all four base lines and the base conic (circle at infinity) twice. Its multiplicity as element of $\mathcal{L}$ is four. Since it is a double line of $\mathcal{C}$, it is also a double line of $\mathcal{D}$ – a fact that can be observed in Figure 6.

The intersection of $\mathcal{C}$ and $\mathcal{D}$ consists of the double line $G$, counted with multiplicity four, the four base lines and a remainder of class four.

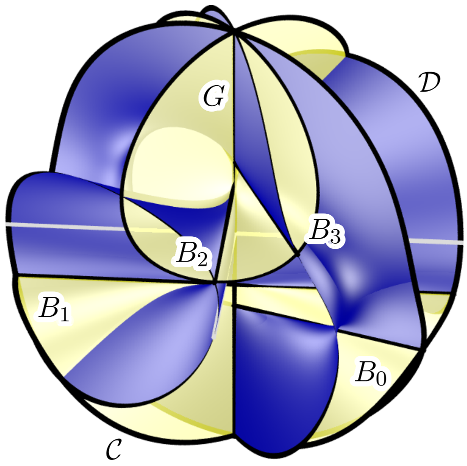


**Figure 6:** Concyclic generators on a cylindroid.

## 8 Future research

We have studied the manifold of planes that intersect four straight lines in points of a circle, we computed its algebraic equation, showed that it is of class seven and presented a general concept for the investigation of pencils of planes, quadratic cones etc. on the solution manifold. However, some questions in this context remain open.

In particular, we only considered solution planes. At least in the euclidean setting, also the solution circles themselves or geometric objects associated to them (congruence of axes, surface of midpoints) might be of interest. Possibly, the geometry of circles in space as used in Subsection 4.1 is an appropriate tool for investigations of this kind.

In the projective setting, we suggest a more general viewpoint that comprises the investigations of Schröcker (2004) and the present text: For all integers $n \in \{0, \ldots, 3\}$ determine the planes that intersect $n$ conic sections and $6 - 2n$ straight lines in six points of a conic section. The known results for $n = 0$ and $n = 1$ (compare Theorem 1 and its subsequent remark) indicate that the class of the solution manifold is $8 - n$. A study of the missing cases $n = 2$ and $n = 3$ shall be left to a separate paper.

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