Honam Mathematical J. **36** (2014), No. 4, pp. 739–753
http://dx.doi.org/10.5831/HMJ.2014.36.4.739

THE MANHEIM AND LIOVILLE FORMULAE BY THE BLASCHKE VECTORS IN R_1^3

Emin Ozyilmaz

Abstract. In this study, it is aimed to analyze how relationship among Blaschke vectors that the obtained formulae in [2, 3] change if parameter ruled surfaces of the spacelike line congruence are not choosed as principle ruled surfaces. Moreover, using the relation among Blaschke vectors, we obtain Manheim's and Liouville's formulae. This new method can be applied to congruences. Thus, we can obtain new formulae in lines space.

1. Introduction

A set of one-parameter of lines is called a ruled surface. Ruled surfaces, especially, developable ruled surfaces are used and applied several areas in mathematics and engineering, [1].

In kinematics, the dual number theory plays important role. Using these numbers, the lines in E^3 can be expressed by the unit dual vectors, which are defined as ID-Modul of these vectors set. An important analytical tool in the study of line trajectories is the introduction of dual number which was first introduced by Clifford [5], and rediscovered by Study [6, 7, 8] in 1903, E. Study gave a fundamental theorem. By this theorem, there exists one-to-one correspondence between the points of dual unit sphere S^2 and the lines of E^3. Thus we have a ruled surface in space of lines R^3, which corresponds a dual curve on S^2, by the set of unit dual vectors, [6], [8].

On the other hand, if we take two parameters in unit dual vectors, we have the line congruence. In practices, the line congruence has defined a family of ruled surfaces. The study of congruences of lines was started by E. Kummer [9], in which he gave a classificasion of those of order one.

Received August 12, 2014. Accepted September 15, 2014.
2010 Mathematics Subject Classification. 53A04.
Key words and phrases. dual space, Blaschke trihedron, dual curvature.

The applications of the line geometry and dual number representations of line trajectories have been developed by Blaschke [4] and Muller [10].

In [2] and [3], we see a formulae among the Blaschke vectors of any ruled surface $\vec{R_1}$ and the parameter ruled surfaces $\vec{R_{11}}$, $\vec{R_{21}}$. Here, he used the parameter ruled surface by choosing as principle ruled surfaces. And he gave some formulas of the Differential Geometry in the line of space. Furthermore, Onder has defined and studied dual timelike normal and dual timelike spherical curves in dual Minkowski space D_1^3 [16]. In [11], autours gave some fundamental formulae and facts about the geometry of spacelike congruences by means of dual hyperbolic and central angles. In [12], using dual elements, the Frenet and Blaschke frames, unit dual spherical curves are studied with ruled surfaces. Then, from some well-known approaches, new Blaschke approach of ruled surfaces is given. Moreover, kinematic interpretation of the moving Blaschke frame is presented with some theorems and their proofs.

2. Preliminary

We start with preliminaries on the geometry of Minkowski 3-space. Let R_1^3 be a Minkowski 3-space endowed with Lorentzian inner product $<,>$ of signature $(++-)$. A vector $\vec{a}$=(a_1, a_2, a_3) of R_1^3 is said to be timelike if $<\vec{a},\vec{a}>< 0$, spacelike if $<\vec{a},\vec{a}>> 0$ and lightlike (or null) if $<\vec{a},\vec{a}>= 0$. The set of all vectors $\vec{a}$ such that $\vec{a}^2 = 0$ is called the lightlike (or null) cone and is denoted by $\wedge$. The norm of a vector $\vec{a}$ is defined to be $|\vec{a}| = \sqrt{|<a,a>|}$. Time orientation is defined as follows: A timelike vector $\vec{a} = (a_1, a_2, a_3)$ is future pointing if and only if $a_3 > 0$ (resp. $a_3 < 0$) [16]. The hyperbolic and Lorentzian spheres of radius 1 in R_1^3 are given by

$$H_0^2 = \{\vec{a} = (a_1, a_2, a_3) \in R_1^3 \,|\, <\vec{a},\vec{a}>= -1\} \tag{2.1}$$

and

$$S_1^2 = \{\vec{a} = (a_1, a_2, a_3) \in R_1^3 \,|\, <\vec{a},\vec{a}>= 1\}, \tag{2.2}$$

respectively, [13].

Lemma 1. *Let $\vec{a}$ and $\vec{b}$ be two future pointing (resp. past pointing) timelike unit vectors in R_1^3. Then we have*

$$<\vec{a},\vec{b}>= -\cosh\theta, \tag{2.3}$$

[15].

As in the case of Euclidean 3-space, the Lorentzian cross product of $\vec{a}$ and $\vec{b}$ is defined by

$$(2.4) \qquad \vec{a} \wedge \vec{b} = (b_2a_3 - a_2b_3,\ a_1b_3 - a_3b_1,\ a_1b_2 - a_2b_1),$$

where $\vec{a}$=(a_1, a_2, a_3) and $\vec{b}$=(b_1, b_2, b_3) are the vectors of the space R_1^3, [13], [14].

Let $A = a + \varepsilon a_0$ be a dual number, $A \in D = \{(a, a_0)\,|\, a, a_0 \in IR\}$ and D be a commutative ring with a unit element. We call dual number $\varepsilon = (0, 1) \in D$ (dual unit) and $\varepsilon^2 = (0, 0)$. $(D^3, +)$ is a module on the dual number ring. We call it D-module, and dual vectors are the elements of this modul. We denote unit vector $\vec{A}$ as

$$(2.5) \qquad \vec{A} = (\vec{a}, \vec{a_0}) = \vec{a} + \varepsilon\vec{a_0},\ <\vec{a}, \vec{a}>= 1,\ <\vec{a}, \vec{a}_0>= 0,$$

where $\vec{a},\ \vec{a_0} \in R^3$. The dual vectors with unit length correspond to oriented lines of the Euclidean space E^3, [4].

Definition 1. The scalar product of two dual vectors $\vec{A} = \vec{a} + \varepsilon\vec{a_0}$ and $\vec{B} = \vec{b} + \varepsilon\vec{b_0}$ is given by

$$(2.6) \qquad <\vec{A}, \vec{B}>=<\vec{a}, \vec{b}> +\varepsilon(<\vec{a}, \vec{b_0}> + <\vec{a_0}, \vec{b}>).$$

Definition 2. The vectoral product of two dual vectors $\vec{A} = \vec{a} + \varepsilon\vec{a_0}$ and $\vec{B} = \vec{b} + \varepsilon\vec{b_0}$ is given by

$$(2.7) \qquad \vec{A} \wedge \vec{B} = \vec{a} \wedge \vec{b} + \varepsilon(\vec{a} \wedge \vec{b_0} + \vec{a_0} \wedge \vec{b}),$$

[4].

The Lorentzian inner product of two dual vector $\vec{A} = \vec{a} + \varepsilon\vec{a_0}$ and $\vec{B} = \vec{b} + \varepsilon\vec{b_0}$ is defined as

$$(2.8) \qquad <\vec{A}, \vec{B}>=<\vec{a}, \vec{b}> +\varepsilon(<\vec{a}, \vec{b_0}> + <\vec{a_0}, \vec{b}>).$$

where $< A, B >$ is Lorentzian inner product of vectors $\vec{a}$ and $\vec{b}$ in Minkowski 3-space R_1^3 with signature $(+ + -)$, [15]. A dual vector $\vec{A}$ is said to be timelike if $<\vec{A}, \vec{A}>< 0$, spacelike if $<\vec{A}, \vec{A}>> 0$ and lightlile (or null) if $<\vec{A}, \vec{A}>= 0$. The set of all vectors A such that $<\vec{a}, \vec{a}>= 0$ is called the dual lightlike cone and is denoted by $\tilde{\wedge}$. Then, we call dual Lorentzian space the set of all dual timelike, spacelike and lightlike vectors and denote it by D_1^3. Then we define the dual hyperbolic and Lorentzian spheres of radius 1 in D_1^3 by

$$(2.9) \quad H_0^2 = \{\vec{A} = \vec{a} + \varepsilon\vec{a_0} \in D_1^3 \mid <\vec{A}, \vec{A}>= -1\,;\ \vec{a},\ \vec{a_0} \in R_1^3\}$$

and

$$(2.10)\quad S_1^2 = \{\vec{A} = \vec{a} + \varepsilon\vec{a_0} \in D_1^3 \mid <\vec{A}, \vec{A}>= 1\,;\ \vec{a},\ \vec{a_0} \in R_1^3\}$$

respectively, [15].

3. The space-like ruled surface and the space-like line congruence

A timelike ruled surface is defined as a surface generated by the motion of straight timelike line while a spacelike ruled surface is defined as a surface generated by the motion of a straight spacelike line in R_1^3. Using E. Study's mapping for the elements of Lorentzian space D_1^3 and R_1^3, spacelike ruled surfaces are represented by a dual spacelike unit vector of an arbitrary real parameter t: This means that the image of a spacelike (resp. timelike) ruled surface is a timelike curve on dual Lorentzian unit sphere S_1^2, [15]. The Blaschke trihedron $\{\vec{R_1}, \vec{R_2}, \vec{R_3}\}$ depends on the striction point of the ruled surface $\vec{R_1}(t)$ in dual space D^3, [4]. According to this, the first space-like axis $\vec{R_1}(t)$ of the trihedron is the generator which passes from the striction point of the ruled surface, the second time-like axis $\vec{R_2}(t)$ is the surface normal at this point and finally the third space-like axis $\vec{R_3}(t)$ is the tangent of the striction line at this point. The derivative formulas of the Blaschke trihedron $\{\vec{R_1}, \vec{R_2}, \vec{R_3}\}$ with respect to dual arc parameter S of the striction curve are written

$$(3.1)\qquad \vec{R_1'} = P\vec{R_2},\quad \vec{R_2'} = -P\vec{R_1} + Q\vec{R_3},\quad \vec{R_3'} = Q\vec{R_2},$$

where $P = \parallel \vec{R_1'} \parallel$, $Q = \det \frac{(R,R',R'')}{P^2}$.

A ruled surface is given as dual vector function by

$$(3.2)\qquad \vec{R_1}(t) = \vec{r}(t) + \varepsilon\vec{r_0}(t),\quad \vec{R_1}^{\,2} = 1.$$

On the other hand, a space-like congruence in D^3 can be represented by a unit dual vector which depend on two real parameters u and v as follows:

$$(3.3)\qquad \vec{R}(u,v) = \vec{r}(u,v) + \varepsilon\vec{r_0}(u,v),\quad \vec{R}^2 = 1.$$

The dual arc element of a ruled surface of the space-like line congruence can be given as

$$(3.4)\qquad dS^2 = d\vec{R}^2 = (\vec{R}_u du + \vec{R}_v dv)^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where $E = e + \varepsilon e_0 =< \vec{R}_u, \vec{R}_u >$, $F = f + \varepsilon f_0 =< \vec{R}_u, \vec{R}_v >$ and $G = g + \varepsilon g_0 =< \vec{R}_v, \vec{R}_v >$. Moreover, the differential form I and II of the space-like line congruence are

$$I = edu^2 + 2fdudv + gdv^2 \tag{3.5}$$

$$II = e_0du^2 + 2f_0dudv + g_0dv^2, \tag{3.6}$$

respectively. Thus, we have

$$dS^2 = I + \varepsilon II. \tag{3.7}$$

Definition 3. The drall of a ruled surface of a line congruence can be written as follows:

$$\frac{1}{d} = \frac{I}{2II}, \tag{3.8}$$

[4].

4. The relation among the magnitudes of the space-like ruled surface $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{21}}$

Let we consider a space-like ruled surface $\vec{R} = \vec{R_1}(t)$ of the space-like line congruence $\vec{R} = \vec{R}(u, v)$ where u and v are the functions of t. Let we write the parameter ruled surfaces of the space-like line congruence as

$$\vec{R_{11}} = \vec{R_{11}}(u, v_0), \quad \vec{R_{11}}^2 = 1 \tag{4.1}$$

and

$$\vec{R_{21}} = \vec{R_{21}}(u_0, v), \quad \vec{R_{21}}^2 = 1, \tag{4.2}$$

where the ruled surfaces $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{21}}$ have common line defined as

$$\vec{R_0} = \vec{R}(u_0, v_0) = \vec{R_{11}}(u_0, v_0) = \vec{R_{21}}(u_0, v_0). \tag{4.3}$$

The Blaschke trihedrons of these ruled surfaces are given by

$$\{\vec{R_0}, \vec{R_2}, \vec{R_3}\}, \quad \{\vec{R_0}, \vec{R_{12}}, \vec{R_{13}}\}, \quad \{\vec{R_0}, \vec{R_{22}}, \vec{R_{23}}\} \tag{4.4}$$

and also the derivative formulae of these Blaschke trihedrons are

$$\vec{R_1}' = P\vec{R_2}, \quad \vec{R_2}' = P\vec{R_1} + Q\vec{R_3}, \quad \vec{R_3}' = Q\vec{R_2}, \tag{4.5}$$

$$\vec{R_{11}}' = P_1\vec{R_{12}}, \quad \vec{R_{12}}' = P_1\vec{R_{11}} + Q_1\vec{R_{13}}, \quad \vec{R_{13}}' = Q_1\vec{R_{12}}, \tag{4.6}$$

$$\vec{R_{21}}' = P_2\vec{R_{22}}, \quad \vec{R_{22}}' = -P_2\vec{R_{21}} + Q_2\vec{R_{23}}, \quad \vec{R_{23}}' = Q_1\vec{R_{22}}, \tag{4.7}$$

where $P_1 = \sqrt{|\vec{R_u}^2|} = \sqrt{E}$, $P_2 = \sqrt{\vec{R_v}^2} = \sqrt{G}$, and Q_1, Q_2 are dual curvatures of parameter ruled surfaces. The dual arc elements of the ruled surfaces are $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{21}}$ can be given respectively as

$$dS = Pdt, \quad dS_1 = P_1 du = \sqrt{E}du, \quad dS_2 = P_2 dv = \sqrt{G}dv. \tag{4.8}$$

Moreover, the Blaschke vectors of the Blaschke trihedrons are given by

$$\vec{B} = -Q\vec{R_0} + P\vec{R_3}, \ \vec{B_1} = -Q_1\vec{R_0} + P_1\vec{R_{13}}, \ \vec{B_2} = Q_2\vec{R_0} - P_2\vec{R_{23}}. \tag{4.9}$$

If we choose the parameter ruled surface as principle ruled surfaces, we may write $F = \vec{R_u}\vec{R_v} = 0$ [2]. But, in this study, we choose the F which is not equal to zero (i.e $F \neq 0$). Let we consider any parameter ruled surfaces of the space-like line congruence $\vec{R} = \vec{R}(u, v)$, $\vec{R}^2 = 1$.

Remark 1. The second edges of the parameter ruled surfaces can be written by

$$\vec{R_{12}} = \frac{\vec{R_{11}}'}{P_1} = \frac{\vec{R_u}}{\sqrt{E}}, \quad \vec{R_{22}} = \frac{\vec{R_{21}}'}{P_2} = \frac{\vec{R_v}}{\sqrt{G}}. \tag{4.10}$$

Thus, from (4.10), we have

$$\vec{R_0} = \frac{\vec{R_u} \wedge \vec{R_v}}{\|\vec{R_u} \wedge \vec{R_v}\|} = \frac{\vec{R_u} \wedge \vec{R_v}}{\cosh \Theta} \tag{4.11}$$

and

$$\vec{R_2} = \frac{\vec{R_1}'}{P} = \sqrt{E}\frac{du}{dS}\vec{R_{12}} + \sqrt{G}\frac{dv}{dS}\vec{R_{22}}. \tag{4.12}$$

On the other hand, let we consider the dual angle between the edges $\vec{R_{12}}$ and $\vec{R_{22}}$ as Θ, and the dual angle between the edges $\vec{R_2}$ and $\vec{R_{12}}$ as Φ. If we take dot product both sides (4.12) with $\vec{R_{12}}$ and $\vec{R_{22}}$, we have

$$\vec{R_2}.\vec{R_{12}} = -\cosh \Phi = -\sqrt{E}\frac{du}{dS} + \sinh \Phi \sqrt{G}\frac{dv}{dS}, \tag{4.13}$$

$$\vec{R_2}.\vec{R_{22}} = \sinh(\Theta - \Phi) = \sinh \Phi \sqrt{E}\frac{du}{dS} + \sqrt{G}\frac{dv}{dS}. \tag{4.14}$$

Thus, from (4.13) and (4.14)

$$\frac{\cosh(\Theta - \Phi)}{\cosh \Theta} = \sqrt{E}\frac{du}{dS}, \tag{4.15}$$

$$-\frac{\sinh \Phi}{\cosh \Theta} = \sqrt{G}\frac{dv}{dS} \tag{4.16}$$

are written. Finally, if we put (4.15) and (4.16) into (4.12), we have following equation between the second edges of the Blaschke trihedrons of the ruled surfaces $\vec{R_2}$, $\vec{R_{12}}$ and $\vec{R_{22}}$ by

$$\vec{R_2} = \frac{\cosh(\Theta - \Phi)}{\cosh\Theta}\vec{R_{12}} - \frac{\sinh\Phi}{\cosh\Theta}\vec{R_{22}}. \tag{4.17}$$

Moreover, by the definition of the angle Θ between $\vec{R_{12}}$ and $\vec{R_{22}}$ we write

$$\vec{R_{12}}.\vec{R_{22}} = \mid \sinh\Theta \mid . \tag{4.18}$$

In the paper, the sign will be taken as positive, i.e, it will be assumed that $\vec{R_{12}}\vec{R_{22}} = \sinh\Theta$. Multipliying the equation (4.17) by $\vec{R_{12}}$ and $\vec{R_{22}}$ we have

$$\frac{\cosh(\Theta - \Phi)}{\cosh\Theta} = \frac{dS_1}{dS} = \sqrt{E}\frac{du}{dS}, \tag{4.19}$$

$$-\frac{\sinh\Phi}{\cosh\Theta} = \frac{dS_2}{dS} = \sqrt{G}\frac{dv}{dS}. \tag{4.20}$$

Corollary 1. *The third elements $\vec{R_3}$, $\vec{R_{13}}$ and $\vec{R_{23}}$ of the Blaschke trihedrons of the ruled surfaces $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{21}}$ are linear dependent as follows:*

$$\vec{R_3} = \frac{\cosh(\Theta - \Phi)}{\cosh\Theta}\vec{R_{13}} + \frac{\sinh\Phi}{\cosh\Theta}\vec{R_{23}}. \tag{4.21}$$

Proof. If we substitute the equation (4.17) in $\vec{R_3} = \vec{R_0} \wedge \vec{R_2}$ and consider the Blaschke trihedrons of $\vec{R_{11}}$ and $\vec{R_{21}}$ we have just obtain (4.21). □

Theorem 1. *The third elements $\vec{R_{13}}$ and $\vec{R_{23}}$ of the ruled surfaces $\vec{R_{11}}$ and $\vec{R_{21}}$ can be expressed by dual vectors $\vec{R_{12}}$ and $\vec{R_{21}}$ as follows:*

$$\vec{R_{13}} = \frac{1}{\cosh\Theta}(-\sinh\Theta\vec{R_{12}} - \vec{R_{22}}), \tag{4.22}$$

$$\vec{R_{23}} = \frac{1}{\cosh\Theta}(\vec{R_{12}} - \sinh\Theta\vec{R_{22}}). \tag{4.23}$$

Proof. From the equations (4.4) and (4.11) we write

$$\vec{R_{12}} \wedge \vec{R_{13}} = -\vec{R_0}, \quad \vec{R_{22}} \wedge \vec{R_{23}} = -\vec{R_0}, \quad \vec{R_{12}} \wedge \vec{R_{22}} = \cosh\Theta\vec{R_0}, \tag{4.24}$$

$$\vec{R_{12}} \wedge (\vec{R_{13}} + \frac{\vec{R_{22}}}{\cosh\Theta}) = \vec{0} \ \Rightarrow \ \cosh\Theta\vec{R_{13}} + \vec{R_{22}} = M\vec{R_{12}}, \tag{4.25}$$

$$(4.26)\qquad \vec{R_{22}} \wedge (\vec{R_{23}} - \frac{\vec{R_{12}}}{\cosh\Theta}) = \vec{0} \;\Rightarrow\; \cosh\Theta\vec{R_{23}} - \vec{R_{12}} = N\vec{R_{22}}.$$

Here, M and N dual scalars. And if we take dot product of (4.25) and (4.26) by the dual vectors $\vec{R_{12}}$ and $\vec{R_{22}}$ respectively, and also with the relation (4.4) and (4.21), we have $M = -\sinh\Theta$ and $N = -\sinh\Theta$. At the end substuting the dual vectors M and N in (4.25) and (4.26), we obtain (4.22) and (4.23). □

Corollary 2. *The Blaschke vectors of the ruled surfaces $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{21}}$, which are defined in (4.9), can be written among the unit dual vectors $\vec{R_{12}}$, $\vec{R_{22}}$ and $\vec{R_0}$ as follows:*

$$(4.27)\qquad \vec{B} = -Q\vec{R_0} + P\left(\frac{\cosh(\Theta-\Phi)}{\sinh\Theta}\vec{R_{12}} - \frac{\sinh\Theta}{\sinh\Theta}\vec{R_{22}}\right),$$

$$(4.28)\qquad \vec{B} = -Q_1\vec{R_0} + \frac{P_1}{\cosh\Theta}(-\sinh\Theta\vec{R_{12}} - \vec{R_{22}}),$$

$$(4.29)\qquad \vec{B} = Q_2\vec{R_0} + \frac{P_2}{\cosh\Theta}(\vec{R_{12}} - \sinh\Theta\vec{R_{22}}).$$

Theorem 2. *There are following relative between the non principle ruled surface $\vec{R_{11}}$ and $\vec{R_{21}}$ as*

$$(4.30)\qquad \vec{R_{12}}\frac{\partial\vec{R_{22}}}{\partial S_1} = -\vec{R_{22}}.\frac{\partial\vec{R_{12}}}{\partial S_1} = \frac{(\sqrt{E})_v - \sinh\Theta(\sqrt{G})_u}{\sqrt{EG}},$$

$$(4.31)\qquad \vec{R_{22}}\frac{\partial\vec{R_{12}}}{\partial S_2} = -\vec{R_{12}}.\frac{\partial\vec{R_{22}}}{\partial S_2} = \frac{(\sqrt{G})_u - \sinh\Theta(\sqrt{E})_v}{\sqrt{EG}}.$$

Proof. Differentiating (4.10) according to the parameter v and u we have

$$(4.32)\qquad \frac{\partial\vec{R_{12}}}{\partial v} = \frac{\vec{R_{uv}}\sqrt{E} - (\sqrt{E})_v\vec{R_u}}{E}$$

and

$$(4.33)\qquad \frac{\partial\vec{R_{22}}}{\partial u} = \frac{\vec{R_{uv}}\sqrt{G} - (\sqrt{G})_u\vec{R_v}}{G}$$

Then,

$$(4.34)\quad \vec{R_{12}}\frac{\partial\vec{R_{22}}}{\partial u} = \frac{\vec{R_u}}{\sqrt{E}}\left(\frac{\vec{R_{uv}}\sqrt{G} - (\sqrt{G})_u\vec{R_v}}{G}\right) = \frac{(\sqrt{E})_v - \sinh\Theta(\sqrt{G})_u}{\sqrt{G}},$$

$$(4.35)\quad \vec{R_{22}}\frac{\partial \vec{R_{12}}}{\partial v}=\frac{\vec{R_v}}{\sqrt{G}}\left(\frac{\vec{R_{uv}}\sqrt{E}-(\sqrt{E})_v\vec{R_u}}{E}\right)=\frac{(\sqrt{G})_u-\sinh\Theta(\sqrt{E})_v}{\sqrt{E}}.$$

are written. On the other hand, considering (4.8) in (4.34) and (4.35) we obtain

$$(4.36)\qquad \vec{R_{12}}.\frac{\partial \vec{R_{22}}}{\partial S_1}=\frac{1}{\sqrt{E}}\vec{R_{12}}\frac{\partial \vec{R_{22}}}{\partial u}=\frac{(\sqrt{E})_v-\sinh\Theta(\sqrt{G})_u}{\sqrt{EG}},$$

$$(4.37)\qquad \vec{R_{22}}.\frac{\partial \vec{R_{12}}}{\partial S_2}=\frac{1}{\sqrt{G}}\vec{R_{22}}\frac{\partial \vec{R_{12}}}{\partial v}=\frac{(\sqrt{G})_u-\sinh\Theta(\sqrt{E})_v}{\sqrt{EG}}.$$

Finally, differentiating dual arcs S_1 and S_2

$$(4.38)\qquad \vec{R_{12}}.\frac{\partial \vec{R_{22}}}{\partial S_1}=-\vec{R_{22}}\frac{\vec{R_{12}}}{\partial S_1},$$

$$(4.39)\qquad \vec{R_{22}}.\frac{\partial \vec{R_{12}}}{\partial S_2}=-\vec{R_{12}}\frac{\partial \vec{R_{22}}}{\partial S_2}.$$

are written. Thus, from (4.36) and (4.38), (4.30) is obtained. Similarly, (4.31) is obtained. □

Proposition 1. *The Blaschke trihedrons $\{\vec{R_0},\vec{R_{12}},\vec{R_{13}}\}$ and $\{\vec{R_0},\vec{R_{22}},\vec{R_{23}}\}$ of the space-like parameter ruled surfaces of the spacelike congruence $R(\vec{u},v)$ are always coincide such that the dual unit vectors between $\vec{R_{12}}$ and $\vec{R_{22}}$ are orthogonal. Moreover, the Blaschke derivative formulae are given by*

$$(4.40)\qquad \frac{d\vec{R_{12}}}{dS_2}=\vec{B_2}\wedge\vec{R_{12}},\quad \frac{d\vec{R_{22}}}{dS_1}=\vec{B_1}\wedge\vec{R_{22}},\quad \frac{d\vec{R_0}}{dS}=\vec{B}\wedge\vec{R_0},$$

$$(4.41)\qquad \frac{d\vec{R_{12}}}{dS_1}=\vec{B_1}\wedge\vec{R_{12}},\quad \frac{d\vec{R_{22}}}{dS_2}=\vec{B_2}\wedge\vec{R_{22}}.$$

Proof. If we take $\Theta=\frac{\pi}{2}$ in (4.22) and (4.23), we have first statement of proposition 1.

$$(4.42)\qquad \vec{R_{13}}=-\vec{R_{22}},\quad \vec{R_{23}}=\vec{R_{12}}.$$

Differentiating (4.22) according to the dual arc parameter S_1

$$\begin{aligned} \frac{\partial \vec{R_{13}}}{\partial S_1} &= \frac{1}{\cos\Theta}\left(-\sinh\Theta\frac{\vec{R_{12}}}{\partial S_1} - \frac{\vec{R_{22}}}{\partial S_1}\right), \\ &= \frac{1}{\cosh\Theta}\left(-\sinh\Theta(\vec{B_1}\wedge\vec{R_{12}}) - \frac{\vec{R_{22}}}{\partial S_1}\right). \end{aligned} \tag{4.43}$$

Moreover, we know

$$\begin{aligned} \frac{\partial \vec{R_{13}}}{\partial S_1} &= \vec{B_1}\wedge\vec{R_{13}} \\ &= \vec{B_1}\wedge\left(\frac{1}{\cos\Theta}(-\sinh\Theta(\vec{B_1}\wedge\vec{R_{12}}) - (\vec{B_1}\wedge\vec{R_{22}}))\right). \end{aligned} \tag{4.44}$$

Thus, from the second equality of (4.43) and (4.44), we get the second equality of (4.40). Similarly, it can be shown that other equalities are valid. □

Theorem 3. *The dual curvatures Q_1, Q_2 and Q of the parameter ruled surfaces $\vec{R_{11}}$, $\vec{R_{21}}$ and any ruled surface $\vec{R_1}$ are found as follows:*

$$Q_1 = \frac{1}{\cosh\Theta\sqrt{G}}((\sqrt{E})_v - \sinh\Theta(\sqrt{G})_u), \tag{4.45}$$

$$Q_2 = -\frac{1}{\cosh\Theta\sqrt{E}}((\sqrt{G})_u - \sinh\Theta(\sqrt{E})_v). \tag{4.46}$$

Proof. From (4.36) and (4.8) we have

$$\vec{R_{12}}.\frac{\partial\vec{R_{22}}}{\partial S_1} = \frac{(\sqrt{E})_v - \sinh\Theta(\sqrt{G})_u}{\sqrt{EG}}, \tag{4.47}$$

$$\frac{\partial\vec{R_{22}}}{\partial S_1} = \frac{1}{\sqrt{E}}\frac{\partial\vec{R_{22}}}{\partial u} = \frac{1}{\sqrt{E}}\vec{B_1}\wedge\vec{R_{22}}. \tag{4.48}$$

Then, we obtain

$$\begin{aligned} \frac{(\sqrt{E})_v - \sinh\Theta(\sqrt{G})_u}{\sqrt{EG}} &= \frac{1}{\sqrt{E}}\vec{R_{12}}\frac{\partial\vec{R_{22}}}{\partial u} \\ &= \frac{1}{\sqrt{E}}\vec{R_{12}}(\vec{B_1}\wedge\vec{R_{22}}) = \frac{-\cosh\Theta\vec{R_0}\vec{B_1}}{\sqrt{E}}. \end{aligned} \tag{4.49}$$

Other than, from (4.9), taking dot product $\vec{B_1}$ with $\vec{R_0}$ we have

$$\frac{-\cosh\Theta\vec{R_0}\vec{B_1}}{\sqrt{E}} = \frac{Q_1\cosh\Theta}{\sqrt{E}}. \tag{4.50}$$

Thus, from (4.49) and (4.50) we get Q_1. We can proof the other cases easily. □

Theorem 4. *Let we consider any ruled surface $\vec{R_1}$ of a space-like line congruence $R(\vec{u}, v)$ and the dual arc elements of the ruled surfaces $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{22}}$ as S, S_1 and S_2, respectively. Let the Blaschke vectors of $\vec{R_{11}}$ and $\vec{R_{21}}$ be $\vec{B_1}$, $\vec{B_2}$ and if the dual angle between the line $\vec{R_2}$ and $\vec{R_{12}}$ is Φ, then*

$$\vec{C} = \frac{\cosh(\Theta - \Phi)}{\cosh \Phi}\vec{B_1} - \frac{\sinh \Phi}{\cosh \Theta}\vec{B_2} \tag{4.51}$$

and

$$\frac{d\vec{R_{12}}}{dS} = \vec{C} \wedge \vec{R_{12}}, \quad \frac{d\vec{R_{22}}}{dS} = \vec{C} \wedge \vec{R_{22}}, \quad \frac{d\vec{R_0}}{dS} = \vec{C} \wedge \vec{R_0} \tag{4.52}$$

are obtained.

Proof. We know that the dual vector $\vec{R_{12}}$ is the function of S_1 and S_2. Hence we have

$$\frac{d\vec{R_{12}}}{dS} = \frac{\partial \vec{R_{12}}}{\partial S_1}\frac{dS_1}{dS} + \frac{\partial \vec{R_{12}}}{dS_2}\frac{\partial S_2}{dS}. \tag{4.53}$$

From the equalities (4.19), (4.20) and (4.41) we obtain

$$\begin{aligned} \frac{dR_{12}}{dS} &= \frac{\cosh(\Theta - \Phi)}{\cosh \Theta}(\vec{B_1} \wedge \vec{R_{12}}) - \frac{\sinh \Phi}{\cosh \Theta}(\vec{B_2} \wedge \vec{R_{12}}) \\ &= \left(\frac{\cosh(\Theta - \Phi)}{\cosh \Theta}\vec{B_1} - \frac{\sinh \Phi}{\cosh \Theta}\vec{B_2}\right) \wedge \vec{R_{12}} \\ &= \vec{C} \wedge \vec{R_{12}}. \end{aligned}$$

Thus, we have the first equalities of (4.52). Other equals of (4.52) can be seen by using same method. □

Theorem 5. *Let we consider the ruled surfaces $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{22}}$ which have common line $\vec{R_0}$ on the space-like line congruence $R(\vec{u}, v)$. Let the Blaschke vectors of these ruled surfaces be $\vec{B}$, $\vec{B_1}$ and $\vec{B_2}$, respectively. We obtain following equality among the Blaschke vectors*

$$\vec{B} = \frac{\cosh(\Theta - \Phi)}{\cosh \Theta}\vec{B_1} - \frac{\sinh \Phi}{\cosh \Theta}\vec{B_2} + \frac{d\Phi}{dS}\vec{R_0}. \tag{4.54}$$

Proof. From (4.17)

$$\vec{R_2} = \frac{\cosh(\Theta - \Phi)}{\cosh \Theta}\vec{R_{12}} - \frac{\sinh \Phi}{\cosh \Theta}\vec{R_{22}} \tag{4.55}$$

can be written. Then, by taking derivative with respect to dual arc S from equation (4.17), we obtain

$$\begin{aligned}\frac{d\vec{R_2}}{dS} &= \frac{d\vec{R_{12}}}{dS}\frac{\cosh(\Theta-\Phi)}{\cosh\Theta} - \frac{d\vec{R_{22}}}{dS}\frac{\sinh\Phi}{\cosh\Theta} \\ &\quad -\frac{\sinh(\Theta-\Phi)}{\cosh\Theta}\frac{d\Phi}{dS}\vec{R_{12}} - \frac{\cosh\Phi}{\cosh\Theta}\frac{d\Phi}{dS}\vec{R_{22}}.\end{aligned} \tag{4.56}$$

On the other hand, considering the Blaschke trihedrons $\{\vec{R_0}, \vec{R_{12}}, \vec{R_{22}}\}$ and $\{\vec{R_0}, \vec{R_{22}}, \vec{R_{23}}\}$, we write

$$\begin{aligned}\vec{R_{13}} &= \vec{R_{12}} \wedge \vec{R_0} = \vec{R_{12}} \wedge \left(\frac{\vec{R_{12}} \wedge \vec{R_{22}}}{\cosh\Theta}\right) \\ &= \frac{-<\vec{R_{12}}, \vec{R_{22}}>\vec{R_{12}} + <\vec{R_{12}}, \vec{R_{12}}>\vec{R_{22}}}{\cosh\Theta} \\ &= -\frac{1}{\cosh\Theta}(\sinh\Theta\vec{R_{12}} - \vec{R_{22}})\end{aligned} \tag{4.57}$$

and

$$\begin{aligned}-\vec{R_{23}} &= \vec{R_{22}} \wedge \vec{R_0} = \vec{R_{22}} \wedge \left(\frac{\vec{R_{12}} \wedge \vec{R_{22}}}{\cosh\Theta}\right) \\ &= \frac{-<\vec{R_{22}}, \vec{R_{22}}>\vec{R_{12}} + <\vec{R_{22}}, \vec{R_{12}}>\vec{R_{22}}}{\cosh\Theta} \\ &= \frac{1}{\cosh\Theta}(-\vec{R_{12}} + \cosh\Theta\vec{R_{22}}).\end{aligned} \tag{4.58}$$

And then, substituting the equations (4.56) and (4.57) in $-\vec{R_{12}} = \vec{R_0} \wedge \vec{R_{13}}$ and $\vec{R_{22}} = \vec{R_0} \wedge \vec{R_{23}}$ we have (4.55) as follows

$$\begin{aligned}\frac{\vec{R_2}}{dS} &= \frac{d\vec{R_{12}}}{dS}\frac{\cosh(\Theta-\Phi)}{\cosh\Theta} - \frac{d\vec{R_{22}}}{dS}\frac{\sinh\Theta}{\cosh\Theta} \\ &\quad + \left[-\vec{R_0} \wedge \frac{(-\sinh\Theta\vec{R_{12}} - \vec{R_{22}})}{\cosh\Theta}\frac{\sinh(\Theta-\Phi)}{\cosh\Theta}\right] \\ &\quad + \vec{R_0} \wedge \frac{(\vec{R_{12}} - \sinh\Theta\vec{R_{22}})}{\cosh\Theta}\frac{\cosh\Phi}{\cosh\Theta}\frac{\partial\Phi}{\partial S}.\end{aligned} \tag{4.59}$$

According to the theorem 4

$$\frac{d\vec{R_{12}}}{dS} = \vec{C} \wedge \vec{R_{12}}, \quad \frac{d\vec{R_{22}}}{dS} = \vec{C} \wedge \vec{R_{22}}, \tag{4.60}$$

$$\vec{C} = \frac{\cosh(\Theta-\Phi)}{\cosh\Phi}\vec{B}_1 - \frac{\sinh\Phi}{\cosh\Theta}\vec{B}_2 \tag{4.61}$$

are known. And, by using the dual trigonometric expression, we find

$$\begin{aligned}\frac{d\vec{R_2}}{dS} &= \frac{\cosh(\Theta-\Phi)}{\cosh\Theta}\vec{C}\wedge\vec{R_{12}} - \frac{\sinh\Phi}{\cosh\Theta}\vec{C}\wedge\vec{R_{22}} \\ &\quad +\vec{R_0}\wedge\left[\frac{\cosh(\Theta-\Phi)}{\cosh\Theta}\vec{R_{12}} - \frac{\sinh\Phi}{\cosh\Theta}\vec{R_{22}}\right]\frac{d\Phi}{dS} \\ &= \left(\vec{C}+\vec{R_0}\frac{d\Phi}{dS}\right)\wedge\vec{R_2}.\end{aligned} \tag{4.62}$$

Thus, if we consider

$$\vec{M} = \vec{C} + \vec{R_0}\frac{d\Phi}{dS} \tag{4.63}$$

we write

$$\vec{C} = \vec{M} - \vec{R_0}\frac{d\Phi}{dS}. \tag{4.64}$$

Thus

$$\frac{d\vec{R_2}}{dS} = \vec{M}\wedge\vec{R_2}. \tag{4.65}$$

By the last equation of (4.52) we have

$$\begin{aligned}\frac{d\vec{R_0}}{dS} &= \vec{C}\wedge\vec{R_0} \\ &= \left(\vec{M} - \vec{R_0}\frac{d\Theta}{dS_1}\right)\wedge\vec{R_0} \\ &= (\vec{M}\wedge\vec{R_0})\end{aligned} \tag{4.66}$$

$$\frac{d\vec{R_0}}{dS} = \vec{M}\wedge\vec{R_0}.$$

The derivative formula with Blaschke vector of $\frac{d\vec{R_2}}{dS}$ and (4.65) we obtain

$$\vec{M}\wedge\vec{R_2} - \vec{B}\wedge\vec{R_2} = 0$$

$$(\vec{M}-\vec{B})\wedge\vec{R_2} = 0$$

$$\vec{M}-\vec{B} = \Lambda\ \vec{R_2};\quad \Lambda\in D \tag{4.67}$$

and

$$\vec{M}\wedge\vec{R_0} - \vec{B}\wedge\vec{R_0} = 0$$

$$(\vec{M}-\vec{B})\wedge\vec{R_0} = 0$$

$$\vec{M} - \vec{B} = \Omega \ \vec{R_0}, \quad \Omega \in D. \tag{4.68}$$

By (4.67) and (4.68) we have

$$\Lambda \vec{R_2} = \Omega \vec{R_0}, \tag{4.69}$$

$$\Lambda = \Omega = 0. \tag{4.70}$$

Consequently it is seen that

$$\vec{M} = \vec{C}. \tag{4.71}$$

That is, we have

$$\vec{B} = \vec{C} + \vec{R_0} \frac{d\Theta}{dS}. \tag{4.72}$$

□

Theorem 6. (Manheim's Formula) *There are following relation*

$$\begin{aligned} P \quad = \quad & P_1 \frac{\cosh(\Theta - \Phi)}{\cosh \Theta} \left[\frac{\cosh(\Theta - \Phi)}{\cosh \Theta} + \frac{\sinh \Theta \sinh \Phi}{\cosh \Theta} \right] \\ & - P_2 \frac{\sinh \Phi}{\cosh \Theta} \left[\frac{\cosh(\Theta - \Phi) \sinh \Theta - \sinh \Phi}{\cosh \Theta} \right] \end{aligned} \tag{4.73}$$

between the dual curvatures of ruled surfaces $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{21}}$ of the spacelike line congruence $R(\vec{u}, v)$.

Proof. If we substitute the Blaschke vectors $\vec{B}$, $\vec{B_1}$ and $\vec{B_2}$ in (4.54) and then taking dot product both of sides by $\vec{R_3}$, desired equation is obtained. □

Theorem 7. (Liouville's Formula) *There are following relation among the dual torsions of the ruled surfaces $\vec{R_1}$, $\vec{R_{11}}$ and $\vec{R_{21}}$ of the spacelike line congruence $R(\vec{u}, v)$ as*

$$Q = Q_1 \frac{\cosh(\Theta - \Phi)}{\cosh \Theta} - Q_2 \frac{\sinh \Phi}{\cosh \Theta} + \frac{d\Phi}{dS}. \tag{4.74}$$

Proof. If we substitute the Blaschke vectors $\vec{B}$, $\vec{B_1}$ and $\vec{B_2}$ in (4.54) and then taking dot product both of sides by $\vec{R_0}$, desired equation is obtained. □

References

[1] Ravani, B., Ku Ts., *Bertrand offsets of ruled and developable surface*, Computer Aided Geometric Design, Elsevier **23(2)** (1991), 145-152.

[2] Caliskan, A., *On the Studying of a line Congruence by Choosing Parameter Ruled Surfaces as Principal Ruled Surface*, Journal of Faculty Science of Ege University Series A **(1)** (1987).

[3] Caliskan, A., *The Relation Among Blaschke Vectors of Ruled Surfaces on a Line Congruence And Its Consequance*, Commun. Fac. Sci. Univ. Ank. Series An **(1-2)** (1989), 77-86.

[4] W. Blaschke, *Vorlesungen uber Differentialgeometrie und geometrische Grundlagen von Einsteins Relativit atstheorie*, Dover Publications, New York, 1945.

[5] Clifford, W. K., *Preliminary sketch of bi-quaternions*, Proc. London Math. Soc. **4(64, 65)** (1873), 361-395.

[6] Gugenheimer. H. W., *Differential Geometry*, Graw-Hill, New York, 1956.

[7] Karger, A., Novak, J., *Space Kinematics and Line Groups*, Gordon and Breach Science Publishers, New York, 1985.

[8] Veldkamp, G. R., *On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics*, Mech. Mach. Theory **11** (1976), 141-156.

[9] Kummer, E., *ber die algebraischen Strahlensysteme, insbesondere uber die der ersten undzweiten Ordnung*, Abh. K. Preuss. Akad. Wiss. Berlin (1866), 1-120, also in E. E. Kummer. Collected Papers, Springer Verlag, 1975.

[10] Muller, H. R., *Kinematik Dersleri*, Ankara University Press, 1963.

[11] Ugurlu, H. H., Caliskan, A., *On The Geometry of Space-Like Congruence*, Commun. Fac. Sci. Univ. Ank. series A1 **50** (2001), 9-24.

[12] Yayl, Y., Saracoglu S., *Ruled Surfaces with Different Blaschke Approach*, Applied Mathematical Sciences **6(59)** (2012), 2945-2955.

[13] O'Neil, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, London, 1983.

[14] Birman, G. S., Nomizo, K., *Trigonometry in Lorentzian Geometry*, Ann. Math. Mont. **91(9)** (1984), 543-549.

[15] Ugurlu, H. H., Caliskan, A., *The Study Mapping for Directed Spacelike and Timelike Lines in Minkowski 3-space* R_{13}, Mathematical and Computational Applications **1(2)** (1996), 142-148.

[16] Onder, M., *Dual timelike normal and dual timelike spherical curves in dual Minkowski space* D_1^3, SDU Fen-Edebiyat Fakultesi Fen Dergisi **1(1-2)** (2006), 77-86.

Emin Ozyilmaz
Department of Mathematics, Faculty of Science, University of Ege,
Bornova, Izmir, TURKEY.
E-mail: ozyilmaz@ege.edu.tr