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Convergence of numerical solutions to stochastic age-structured system of two species with diffusion

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Abstract: In general, most of stochastic age-structured system of two species with diffusion do not have explicit solutions, thus numerical approximation schemes are invaluable tools for exploring their properties. The main purpose of this paper is to develop a numerical scheme and show the convergence of the numerical approximation solution to the true solution for stochastic age-structured system of two species.

Introduction

Convergence of numerical solutions to stochastic age-structured population system is a very important problem and has been investigated by several mathematicians. For example, Qimin Zhang and Chongzhao Han [1] gave an introduction to numerical analysis for stochastic age-dependent population equations. Qimin Zhang [2] studied the Convergence of numerical solutions to stochastic age-structured population system with diffusion, and for a general nonlinear model has been investigated by El-Borai, etal.[3]. All of above,their work is mainly concerned with stochastic age-dependent single population dynamics with diffusion.

In this paper, we consider the convergence of stochastic age-structured system of two species with diffusion:

$$\begin{cases} \frac{\partial p_1}{\partial t}+\frac{\partial p_1}{\partial r}-k(r,t)\Delta p_1+\mu_1^{'}(r,t,x)p_1+\lambda_1^{'}(r,t,x)p_2(r,t,x)p_1=f_1(r,t,x,p_1)+g_1(r,t,x,p_1)\frac{\partial w}{\partial t}, & in \quad Q_A=(0,\ A)\times Q, \\ \frac{\partial p_2}{\partial t}+\frac{\partial p_2}{\partial r}-k(r,t)\Delta p_1+\mu_2^{'}(r,t,x)p_2-\lambda_2^{'}(r,t,x)p_1(r,t,x)p_2=f_2(r,t,x,p_2)+g_2(r,t,x,p_2)\frac{\partial w}{\partial t}, & in \quad Q_A=(0,\ A)\times Q, \\ p_i(0,t,x)=\int_0^A \beta_i^{'}(r,t,x)p_i(r,t,x)dr, & in \quad (0,\mathrm{T})\times\Gamma, \\ p_i(0,t,x)=p_{i0}(r,x), & in \quad Q_A=(0,\ A)\times\Gamma, \\ p_i(r,t,x)=0, & on \quad \sum_A=(0,\ A)\times(0,\mathrm{T})\times\partial\Gamma, \\ y_i(t,x)=\int_0^A p_i(r,t,x)dr, & in \quad Q, \end{cases} \tag{1}$$

where $t\in(0,T)$, $r\in(0,A)$, $x\in\Gamma$, $p_i(r,t,x)(i=1,2)$ denotes of two species density of age r at time t and in the location x, $\beta_i^{'}(r,t,x)$ denotes the fertility rate of females of age r at time t and in spatial position x, $\mu_i^{'}(r,t,x)$ denotes the mortality rate of age r at time t and in the location x, $\lambda_i^{'}(r,t,x)$ is the interaction coefficients $(i=1,2)$, Δ denotes the Laplace operator with respect to the space variable, $k(r,x)>0$ is the diffusion coefficient. $f_i(r,t,x,p_i)+g_i(r,t,x,p_i)\frac{dw_t}{dt}(i=1,2)$ denotes effects of external environment for population system, such as emigration and earthquake and so on, $A>0$ is the maximal age of the two species, the age can not reach the maximal with two of the species, so the two population'quantities are zero while the age is A, that is $p_i(A,t,x)=0$, $\Gamma\subset R^n(n\in N)$ is an open and bound habitat where the two species can be free to move, $\partial\Gamma$ is the boundary of Γ, $t\in(0,T)$, $r\in(0,A)$, $x\in\Gamma$. The effects of external environment has the deterministic and random parts which depend on r, t, x and $p_i(i=1,\ 2)$.

In general, stochastic age-dependent of two species equation (1) rarely has an explicit solution. Thus, numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we will develop an numerical approximation method for stochastic age-dependent of two species equation of the type described by Eq(1). We shall prove that the numerical solutions converge to the exact solutions and provide the order of convergence.

Preliminaries and approximation

Consider stochastic age-structured of two species system with diffusion (1). A is the maximal age of two species, so $p_i(r,t,x)=0,\ \forall r\geq A,(i=1,2)$.

By Eq.(1), integrating on $[0,A]$ to Eqs.(1) with respect to r, then we obtain the following system :

$$\begin{cases}\frac{\partial y_1}{\partial t}-k(x)\Delta y_1+\mu_1^{'}(t,x)y_1-\beta_1^{'}(t,x)y_1+\lambda_1^{'}(t,x)y_2(t,x)y_1 & \\ =f_1(t,x,y_1)+g_1(t,x,y_1)\frac{dW}{dt}, & in\ \ Q=(0,T)\times\Gamma, \\ \frac{\partial y_2}{\partial t}-k(x)\Delta y_2+\mu_2^{'}(t,x)y_2-\beta_2^{'}(t,x)y_2-\lambda_2^{'}(t,x)y_1(t,x)y_2 & \\ =f_2(t,x,y_2)+g_2(t,x,y_2)\frac{dw}{dt}, & in\ \ Q=(0,T)\times\Gamma, \\ y_i(0,x)=y_{i0}(x), & in\ \ \Gamma, \\ y_i(t,x)=0, & on\ \sum=(0,T)\times\partial\Gamma,\end{cases} \tag{2}$$

where $\beta_i(t,x)\equiv(\int_0^A\beta_i^{'}(r,t,x)p_i(r,t,x)dr)(\int_0^A p_i(r,t,x)dr)^{-1},(i=1,2)$, $\int_0^A p_i(r,t,x)dr$ is the total number, and the birth process is described by the nonlocal boundary conditions $\int_0^A\beta_i^{'}(r,t,x)p_i(r,t,x)dr$, clearly, $\beta_i(t,x)$ denotes the fertility rate of total number at time t and in the location x. $\mu_i(t,x)\equiv(\int_0^A\mu_i^{'}(r,t,x)p_i(r,t,x)dr)(\int_0^A p_i(r,t,x)dr)^{-1}$, $\mu_i(t,x)$ denotes the mortality rate at time t and in the location x.

$$\lambda_1(t,x) \equiv (\int_0^A \lambda_1^{'}(r,t,x) p_2 p_1 dr)(\int_0^A p_2(r,t,x)dr)^{-1}(\int_0^A p_1(r,t,x)dr)^{-1},$$

$$\lambda_2(t,x) \equiv (\int_0^A \lambda_2^{'}(r,t,x) p_1 p_2 dr)(\int_0^A p_2(r,t,x)dr)^{-1}(\int_0^A p_1(r,t,x)dr)^{-1},$$

$$f(t,x,y_1) \equiv \int_0^A f_1(r,t,x,p_1)dr \quad , \quad g(t,x,y_1) \equiv \int_0^A g_1(r,t,x,p_1)dr \, ,$$

$$f^{'}(t,x,y_2) \equiv \int_0^A f_2(r,t,x,p_2)dr \quad , \quad g^{'}(t,x,y_2) \equiv \int_0^A g_2(r,t,x,p_2)dr \, .$$

First, we state the assumptions about the stochastic age-dependent population system with diffusion that will be considered.

(i) $f(t,x,0)=0,\ g(t,x,0)=0, f^{'}(t,x,0)=0,\ g^{'}(t,x,0)=0;$

(ii) (Lipschitz condition) there exists a positive constant K such that $p_1, p_2 \in H$

$$\left|f(t,x,p_1)-f(t,x,p_2)\right| \vee \left\|g(t,x,p_1)-g(t,x,p_2)\right\|_2 \le K\left|p_1-p_2\right|, a.e.t\,;$$

$$\left|f^{'}(t,x,p_1)-f^{'}(t,x,p_2)\right| \vee \left\|g^{'}(t,x,p_1)-g^{'}(t,x,p_2)\right\|_2 \le K\left|p_1-p_2\right|, a.e.t\,;$$

(iii) $\mu_i(t,x), \beta_i(t,x), \lambda_1(t,x), \lambda_2(t,x)$ are continuous in $\overline{Q_i}$ such that

$$0 \le \overline{\mu_i} \le \mu_i(t,x) \le\ll \infty,\ 0 \le \overline{\lambda_1} \le \lambda_1(t,x) < \infty,\ \ 0 \le \lambda_2(t,x) \le \overline{\lambda_2} < \infty,\ 0 \le \beta_i(t,x) \le \overline{\beta_i} < \infty.$$

Lemmas and corollaries

In this section, we will prove the convergence of Q_{it}. In order to prove the convergence of Q_{it}, we first give several lemmas. Our first lemma shows that the continuous approximation has bounded second moments in a strong sense.

Lemma 3.1 Under the assumptions (i)-(iii), then $E\sup\limits_{0\le t\le T}\left|Q_{it}\right|^2 \le C(i=1,2).$ (3)

Proof. Applying It^o formula to $\left|Q_{1t}\right|^2$ yields

$$\begin{aligned}\left|Q_{1t}\right|^2 = \left|Q_{10}\right|^2 &- 2\int_0^t\int_\Gamma [k\nabla Q_{1s}\cdot\nabla Q_{1s} + (\mu_1(s,x)-\beta_1(s,x))Z_{1s}Q_{1s}]dxds \\ &-2\int_0^t\int_\Gamma(\lambda_1(s,x)Z_{2s}Q_{2s}Z_{1s}Q_{1s})dxds + 2\int_0^t\int_\Gamma(f(s,x)Z_{1s}Q_{1s})dxds \\ &+2\int_0^t\int_\Gamma Q_{1s}g(s,x,Z_{1s})dxdw_s + \int_0^t\left\|g(s,x,Z_s\right\|_2^2 ds\end{aligned} \tag{4}$$

Using condition(ii)yields

$$\begin{aligned}E\sup_{0\le s\le t}\left|Q_{1s}\right|^2 \le E\left|Q_{10}\right|^2 &+ (\overline{\beta_1}+\overline{\mu_1}+1)\int_0^t E\sup_{0\le s\le t}\left|Q_{1s}\right|^2 ds + (\overline{\beta_1}+\overline{\mu_1}+2K^2)\int_0^t\left|Z_{1s}\right|^2 ds \\ &+\overline{\lambda_1}\int_0^t E\sup_{0\le s\le t}\left|Q_{2s}\right|^2\left|Q_{1s}\right|^2 ds + \overline{\lambda_1}\int_0^t E\left|Z_{2s}\right|^2\left|Z_{1s}\right|^2 ds \\ &+2E\sup_{0\le s\le t}\int_0^s\int_\Gamma Q_{1r}g(r,x,Z_{1r})dw_{1r}dx.\end{aligned} \tag{5}$$

By Burkholder-Davis-Gundy's inequality(see, for example, [6]), for some positive constant $K_1 > 0$. Thus, it follows from(5):

$$E\sup_{0\le s\le t}|Q_{1s}|^2 \le 4(\overline{\beta_1}+\overline{\mu_1}+K^2+K_1K^2+1)\int_0^t E\sup_{0\le r\le s}|Q_{1r}|^2 ds$$
$$+4\overline{\lambda_1}\int_0^t E\sup_{0\le r\le s}|Q_{2r}|^2|Q_{1r}|^2 ds + 2E|Q_{10}|^2,\ \forall t\in[0,T]$$

Now, Gronwall's lemma obviously implies the required result.

$$E\sup_{t\in[0,T]}|Q_{1t}|^2 \le C_1 \tag{6}$$

We can get (7) from the above methods of proof.

$$E\sup_{t\in[0,T]}|Q_{2t}|^2 \le C_2,\ C=\max(C_1,\ C_2) \tag{7}$$

Next, we show that the continuous time approximation remains close $\times A$ to the step function Z_{it} in a strong sense.

Lemma 3.2 Under the assumptions(i)-(iii), then $E\sup_{0\le t\le T}|Q_{it}-Z_{it}|^2 \le C' \triangle t(i=1,\ 2).$ (8)

The proof of this Lemma is an analogous to that of Lemma in [2].

Convergence with the Lipschitz condition

Theorem 4.1 Under the assumptions (i)-(iii), then $E\sup_{0\le t\le T}|y_{it}-Q_{it}|^2 \le C''\Delta t(i=1,\ 2).$ (9)

Proof. For any $t\in[0,T]$,

$$E\sup_{0\le s\le t}|y_{1s}-Q_{1s}|^2 \le (\overline{\beta_1}+\overline{\mu_1}+K)\int_0^t E\sup_{0\le r\le s}|y_{1r}-Q_{1r}|^2 ds + \overline{\lambda_1}\int_0^t E\sup_{0\le r\le s}|y_{1r}-Q_{1r}|^2|y_{2r}-Q_{2r}|^2 ds$$
$$+(\overline{\beta_1}+\overline{\mu_1}+\overline{\lambda_1}+K+K^2)\int_0^t E|y_{1s}-Z_{1s}|^2 ds + \overline{\lambda_1}E\int_0^t|y_{2s}-Q_{2s}|^2|y_{1s}-Q_{1s}|^2 ds$$
$$+2\mathrm{E}\sup_{0\le s\le t}\int_0^s(y_{1s}-Q_{1s},(g(s,x,y_{1s})-g(s,x,Z_{1s}))dw_s \tag{10}$$

Applying Lemma 3.2 we obtain a bound of the form

$$E\sup_{0\le s\le t}|y_{1s}-Q_{1s}|^2 \le D_1\Delta t + \overline{\lambda_1}TC_3C_4\Delta t + D_2\int_0^t E\sup_{0\le r\le s}|y_{1r}-Q_{1r}|^2 ds$$
$$+\overline{\lambda_1}\int_0^t E\sup_{0\le r\le s}|y_{2r}-Q_{2r}||y_{1r}-Q_{1r}|ds,$$

where $D_1 = 2(\overline{\beta_1}+\overline{\mu_1}+K+K^2+2K_1)TC_3, D_2 = 2(2\overline{\beta_1}+2\overline{\mu_1}+2K+K^2+2K_1)TC_3.$

$$E\sup_{0\le t\le T}|y_{1t}-Q_{1t}|^2 \le C_3\Delta t. \tag{11}$$

On applying the Gronwall inequality we can obtain (11), where $C_4 = D_1 \exp(D_2 T)$.

We can get(12) from the same proof procedure

$$E \sup_{0 \le t \le T} \left| y_{2t} - Q_{2t} \right|^2 \le C_4 \Delta t,\ C^{"} = \max(C_3,\ C_4). \tag{12}$$

It is easy to deduce that the following theorem is satisfied.

Theorem 4.2 Assume the preceding (i)-(iii) hold,the approximate solution will converge to the true solution of Eq(2) in the sense

$$\lim_{\Delta t \to 0} E\left[\sup_{0 \le t \le T} \left| y_{it} - Q_{it} \right|^2 \right] = 0(i = 1,\ 2). \tag{13}$$

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