Flocking in Fixed and Switching Networks

Herbert G. Tanner *Member, IEEE,* Ali Jadbabaie *Member, IEEE,*

George J. Pappas *Senior Member, IEEE*

Abstract

The work of this paper is inspired by the flocking phenomenon observed in [3]. We introduce a class of local control laws for a group of mobile agents that result in: (i) global alignment of their velocity vectors, (ii) convergence of their velocities to a common one, (iii) collision avoidance, and (iv) local minimization of the agents artificial potential energy. These are made possible through local control action by exploiting the algebraic graph theoretic properties of the underlying time-varying communication and sensor networks. The communication network links agents that instanteneously attempt to synchronize their velocity vectors, and the sensing network relays state information between agents in close proximity through local sensors. These two networks may not necessarily coincide, since the enabling physical mechanisms that bring them to life are fundamentally different. We show that even if these graphs switch arbitrarily, convergence and stability is preserved as long as connectivity is maintained.

Index Terms

Multi-agent systems, cooperative control, nonsmooth systems, algebraic graph theory.

I. Introduction

Over the last years, the problem of coordinating the motion of multiple autonomous agents has attracted significant attention. Research is motivated by recent advances in communication and computation, as well as inspiring links to problems in biology, social behavior, statistical physics, and computer graphics. Efforts have been directed in trying to understand how a group of

Herbert Tanner is with the Department of Mechanical Engineering, The University of New Mexico, MSC01 1150, Albuquerque, NM 87131, USA. Ali Jadbabaie and George Pappas are with the Department of Electrical and Systems Engineering, University of Pennsylvania, 200 South 33$^{\text{rd}}$ Street, Philadelphia, PA 19104, USA. Preliminary versions of this work have appeared in the 2003 IEEE Conference on Decision and Control [1], [2].

autonomous moving creatures such as flocks of birds, schools of fish, crowds of people [4], [5], or man-made mobile autonomous agents, can cluster in formations without centralized coordination.

Such problems have also been studied in ecology and theoretical biology, in the context of animal aggregation and social cohesion in animal groups (see for example [6], [7]). A computer model mimicking animal aggregation was proposed by [3]. Following the work of [3] several other computer models have appeared in the literature and led to creation of a new area in computer graphics known as *artificial life* [3], [8]. At the same time, several researchers in the area of statistical physics and complexity theory have addressed flocking and schooling behavior in the context of non-equilibrium phenomena in many-degree-of-freedom dynamical systems and self organization in systems of self-propelled particles [9]–[11]. Similar problems have become a major thrust in systems and control theory, in the context of cooperative control, distributed control of multiple vehicles and formation control; see for example [12]–[26]. The main goal of the above papers is to develop a decentralized control strategy so that a global objective, such as a tight formation with desired inter-vehicle distances, is achieved.

Reynolds [3] aimed at generating a computer animation model of the motion of bird flocks and fish schools. The author called the generic simulated flocking creatures "boids". The basic flocking model consists of three simple steering behaviors which describe how an individual agent maneuvers based on the positions and velocities its nearby flockmates:

- **Separation:** steer to avoid crowding local flockmates.
- **Alignment:** steer towards the average heading of local flockmates.
- **Cohesion:** steer to move toward the average position of local flockmates.

In Reynolds' model, each agent has direct access to the whole scene's geometric description, but flocking requires that it reacts only to flockmates within a certain small neighborhood around itself. The neighborhood is characterized by a distance and an angle, measured from the agent's direction of flight. Flockmates outside this local neighborhood are ignored. The neighborhood could be considered a model of limited perception (as by fish in murky water), or just the the region in which flockmates influence an agent's steering. The superposition of these three rules results in all agents moving as a flock while avoiding collisions.

Vicsek et al. [9] proposed a model which, although developed independently, turns out to be a special case of [3] where all agents move with the same speed (no dynamics), and only follow an alignment rule. In [9], each agent heading is updated as the average of the headings of the agent and its nearest neighbors, plus some additive noise. Numerical simulations in [9] indicate a coherent collective motion, in which the headings of all agents converge to a common value, a surprising result in the physics community that was followed by a series of papers. The first rigorous proof of convergence for Vicsek's model (in the noise-free case) was recently given by [20]. Generalizations of this model include a leader follower strategy, in which one agent acts as a group leader and the other agents would just follow the aforementioned cohesion/separation/alignment rules, resulting in leader following.

Motivation for this work comes primarily from the need to theoretically explain the flocking phenomenon of [3]. Flocking has been given many definitions [10], [12], [27], [28]; it is therefore understood quite differently by different authors. In this work we interpret Reynolds flocking model as a mechanism for achieving *velocity synchronization* and *regulation of relative distances* between agents in the same group. Under the assumption of connected (but arbitrarily switching) network topology, we construct local control laws that allow a group of mobile agents with double integrator dynamics to align their velocities, move with a common speed and achieve desired inter-agent distances while avoiding collisions with each other. We believe that these control laws capture the essense of Reynolds model, both in terms of the nature of local interactions and with respect to the overall objective.

We theoretically establish the stability properties of the interconnected closed loop system by combining results from classical and nonsmooth control theory, robot navigation, mechanics and algebraic graph theory. Stability is shown to rely on the connectivity properties of the graph that represents agent interconnections, in terms of not only asymptotic convergence but also convergence speed and robustness with respect to arbitrary changes in the interconnection topology. Exploiting modern results from algebraic graph theory, these properties are directly related to the topology of the network through the eigenvalues of the Laplacian of the graph. Collision avoidance and pairwise distance convergence is ensured through the application of a set of local artificial potential fields [29], [30]. Potential fields have been used frequently for collision avoidance in decentralized multi-agent systems [12], [18], [23]. Similar results regarding collective flocking behavior have been independently produced by [31], although the analysis techniques, both in terms of collision avoidance and velocity synchronization, are fundamentally different.

The approach in this paper is different from our earlier work [1], [2]: here we make a distinction between the *sensing* and the *communication* network. These two networks need not necessarily coincide, a fact that further motivates a nonsmooth approach to cooperative control design and analysis. Besides sensing and communication ranges not being equal, there are additional reasons why the two networks should be distinguished:

The *sensing* network is understood as set of interactions between agents, based on information that is conveyed by sensory data. Sensor range is typically limited, and therefore interactions are local. Despite the fact that sensor-triggered agent interactions are time-varying, one may argue that agents can *predict* when other agents will move beyond sensor range, and therefore, phase out *smoothly* the effect of departing groupmates. Similarly, an approaching agent just within sensor range could be regarded as a not so critical event (collision is not imminent) and the effect of the closing distance could be taken into account gradually in decision making.

The *communication* network, however, described as the set of channels for exchange of information between agents, may undergo unpredictable changes. Velocity information is not easily sensed and will typically be communicated between mobile agents over radio channels. In addition, bandwidth limitations may dictate a certain network topology for information exchange. Just like in the case of internet connections, physical proximity may not imply network proximity. Especially when distant agents are to exchange information, there could be no indication when existing communication links will be lost, or new links established. Uncertainty calls for analysis methods that can account for arbitrary changes, both in terms of link identity, and in terms of switching frequency.

We start our analysis with the case where the two networks are the same and do not change with time. Each agent computes its control input based on a fixed set of network "neighbors." In this case, the control inputs for the agent are smooth and the stability analysis is based on the classic version of LaSalle's invariance principle, facilitated by the algebraic properties of the interconnection graph that allow the connectivity properties of the network to be reflected on the convergence estimate. In the second part of the paper, we distinguish between the two networks, and we allow the network topologies to switch with time. Sensing network changes are accomodated by continuous control signals, but communication network switches introduce control discontinuities and give rise to a discontinuous closed loop dynamical system. Nonsmooth analysis is used to establish stability, and important results are reviewed briefly in the Appendix.

II. Problem Formulation

Consider a group of N mobile agents. Each mobile agent is a dynamical system moving on the plane. Generalizations to three dimensions and more complex dynamics are possible, but for simplicity we let each agent be described by a double integrator:

$$\dot{r}_i = v_i \tag{1a}$$

$$\dot{v}_i = u_i \quad i = 1, \ldots, N\,, \tag{1b}$$

where $r_i = (x_i, y_i)^T$ is the position of agent i, $v_i = (\dot{x}_i, \dot{y}_i)^T$ its velocity and $u_i = (u_x, u_y)^T$ its acceleration inputs. Let the relative position vector between agents i and j be denoted $r_{ij} = r_i - r_j$. Agent i is steered via its acceleration input u_i which consists of two components (Figure 1):

$$u_i = \alpha_i + a_i\,. \tag{2}$$

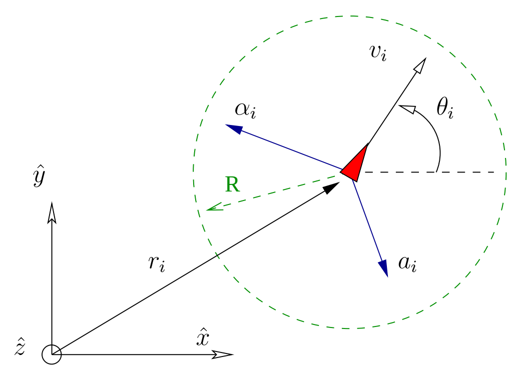

Fig. 1. The (planar) position of agent i is described by a vector r_i in some inertial coordinate frame. Angle θ_i characterizes the direction of its velocity vector, v_i. A spherical sensing region of radius R, centered at agent i, represents the area in which other agents are detected. The two components of the control input (2) can be thought of as "forces" (more accurately, accelerations) acting along different directions on the plane of motion.

The first component of (2), α_i aims at aligning the velocity vectors of all the agents and to make them move with a common speed and direction. Component a_i is thought to be a vector in the direction of the negated gradient of an artificial potential function, V_i. Thus, a_i contributes to collision avoidance and cohesion in the group. In Figure 1, R denotes the (spherical) sensing radius of agent i. Agents beyond this range are assumed not to affect a_i.

In our interpretation of Reynold's notion of flocking, a group of mobile agents is said to flock, when all agents attain the same velocity vector, and distances between the agents are time invariant. In this context, relative distance regulation is understood as convergence to a steady state, not necessarily common. In addition, we require that during the convergence phase agents should not collide with each other. A collision is assumed to have occured when the coordinates of two agents coincide. The problem here is to design the control input (2) so that in the group of mobile agents, velocities are synchronized and pair-wise distances stabilized, giving rise to an emergent cooperative behavior that resembles flocking. The control law sought for agent i is required to be "local" in the sense that it should not depend on the state of all other groupmates.

III. Coordination Strategy

In this section we introduce local control laws of the form of (2), which cause the group of mobile agents to flock asymptotically. The control laws are uniform for all agents and can accommodate a large class of artificial potential functions. The controller component α_i of agent i requires velocity information from a subset of the agent's flockmates denoted $\mathcal{N}_i$. Velocity information is thought to be transmitted over communication channels. This communication network is represented by a graph:

Definition 1 (Communication graph) *The communication graph, $\mathcal{G}_c = \{\mathcal{V}, \mathcal{E}_c\}$, is an undirected graph consisting of:*

- *a set of vertices (nodes), $\mathcal{V} = \{1, \dots, N\} \subset \mathbb{N}$, indexed by the agents in the group, and*
- *a set of edges, $\mathcal{E}_c = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid i \sim j\}$, containing unordered pairs of nodes that represent communication links.*

The communication network neighbors of agent i are assumed to belong to a set $\mathcal{N}_c(i)$:

$$\mathcal{N}_c(i) \triangleq \{j \mid (i,j) \in \mathcal{E}_c\} \subseteq \mathcal{V} \setminus \{i\}.$$

The second component of the control input for agent i, responsible for collision avoidance and group cohesion, a_i is computed using inter-agent *distance* information, provided by the agent sensor(s). Agents in distances smaller than R are affecting each other control inputs, and each such interaction is thought to have been caused by a link in the *sensing* graph of the group:

Definition 2 (Sensing graph) *The sensing graph, $\mathcal{G}_s = \{\mathcal{V}, \mathcal{E}_s\}$, is an undirected graph consisting of:*

- *a set of vertices (nodes), $\mathcal{V} = \{1, \ldots, N\} \subset \mathbb{N}$, indexed by the agents in the group, and*
- *a set of edges, $\mathcal{E}_s = \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid \|r_i - r_j\| \leq R\}$, containing unordered pairs of nodes that represent sensing links.*

Similarly, sensing network neighbors of agent i define a set $\mathcal{N}_s(i)$:

$$\mathcal{N}_s(i) \triangleq \{j \mid (i,j) \in \mathcal{E}_s\} \subseteq \mathcal{V} \setminus \{i\}.$$

The control input for agent i is now defined as:

$$u_i = \underbrace{- \sum_{j \in \mathcal{N}_c(i)} (v_i - v_j)}_{\alpha_i} \underbrace{- \sum_{j \in \mathcal{N}_s(i)} \nabla_{r_i} V_{ij}}_{a_i} . \tag{3}$$

Function V_{ij} depends on the distance between sensing neighbors and defined as follows,

Definition 3 (Potential function) *Potential V_{ij} is a differentiable, nonnegative, function of the distance $\|r_{ij}\|$ between agents i and j, such that*

1) $V_{ij}(\|r_{ij}\|) \rightarrow \infty$ *as* $\|r_{ij}\| \rightarrow 0$,
2) V_{ij} *attains its unique minimum when agents i and j are located at a desired distance.*

Extension to the case where agent volume is captured by a sphere of radius R_a is immediate: one needs to define V_{ij} as a function of the distance between the agent spheres, and re-write it as $V_{ij}(\|r_{ij}\| - 2R_a)$.

Definition 3 ensures that minimization of the inter-agent potential functions implies cohesion and separation in the group. An example of such a function is the following:

$$V_{ij} = \begin{cases} -a_1\|r_{ij}\| + \log(\|r_{ij}\|) + \frac{a_2}{\|r_{ij}\|}, & \text{if } \|r_{ij}\| < R \\ -a_1 R + \log(R) + \frac{a_2}{R}, & \text{if } \|r_{ij}\| \geq R \end{cases}, \tag{4}$$

with $a_1 = \frac{1}{r_{min}+R}$, $a_2 = \frac{R\, r_{min}}{r_{min}+R}$, the graph of which is shown in Figure 2. Having defined V_{ij} we can now express agent i total potential as:

$$V_i = \sum_{j \in \mathcal{N}_i} V_{ij}(\|r_{ij}\|), \tag{5}$$

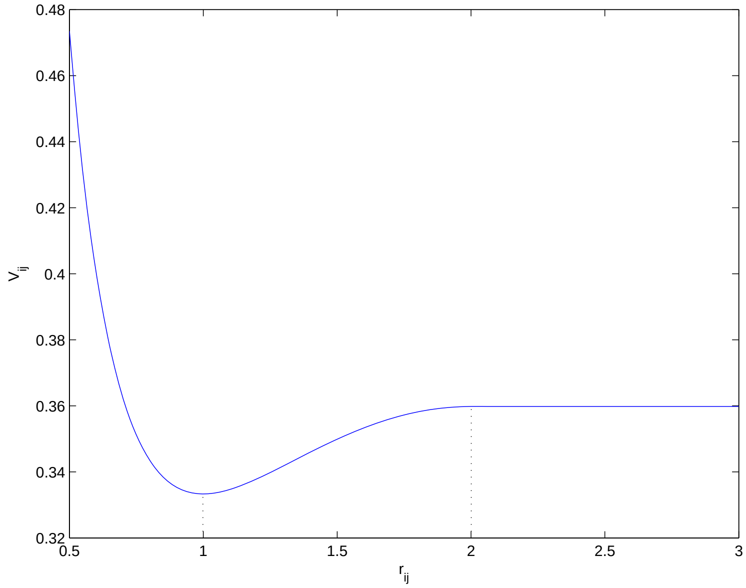

Fig. 2. Example of the inter-agent artificial potential function defined by (4); $R = 2$, $r_{min} = 1$.

IV. Fixed Interconnection Topology

If the interconnection topology of the group is represented by a time invariant but connected graph, then control laws (3) create an asymptotically stable equilibrium manifold on which the group satisfies the conditions for flocking as described above. Each agent maintains the same set of neighbors, implying that the neighboring graph is constant. The main consequence of time invariance is that the mechanical energy of the group is differentiable, the agent control laws are smooth and classic Lyapunov theory can be applied.

For analysis purposes, we will define a dynamical system derived from (1)-(2) by stacking the position and velocity vectors. This system will have the vector $(\bar{r}, v)$ as its state, where $\bar{r} = (B_{K_N} \otimes I)r$ is the stack vector of all *relative* positions between agents, r is the stack vector of agent positions, v is the stack vector of all agent velocities, $\otimes$ denotes the Kronecker matrix product, B_{K_N} is the incidence matrix of the complete graph with N vertices K_N, and I is the identity matrix of appropriate dimension (in the sequel, we will use I_2). The system dynamics are then expressed as:

$$\dot{\bar{r}} = (B_{K_N} \otimes I_2) v \tag{6a}$$

$$\dot{v} = u, \tag{6b}$$

where u is the stack vector of all agent inputs, defined in (2).

Consider the following nonnegative function:

$$W(\bar{r}, v) = \frac{1}{2} \sum_{i=1}^{N} (V_i + v_i^T v_i). \tag{7}$$

Using LaSalle's invariance principle we can show that the closed loop system of agents (6) flocks, provided that the neighboring graph is connected:

Theorem 1 (Flocking in a fixed network) *Consider a system of N mobile agents with dynamics* (6)*, each steered by control law* (3) *and assume that the communication and sensing graphs are connected. Then all agent velocity vectors become asymptotically the same, collisions between agents are avoided, and the system approaches a configuration that locally minimizes all agent potentials.*

Proof: Since the sensing graph is connected, by definition there is a (sensing) path from every vertex to every other vertex. The graph's diameter, therefore, cannot be larger than $N-1$. This implies that the largest distance between any two agents in the graph, (by the triangle inequality) should be smaller than $(N-1)R$. As a result, $\sum_{(i,j)\in\mathcal{V}\times\mathcal{V}} \|r_{ij}\| \leq \frac{N(N-1)^2 R}{2}$. Thus, $\bar{r}$ always evolves in a closed and bounded set. Similarly, the level sets of W define compact sets in the space of agent velocities: $W \leq c \Rightarrow \sum_i v_i^2 \leq c \Rightarrow \|v_i\|^2 \leq c$. Consequently, the set

$$\Omega = \{(\bar{r}, v) \mid \sqrt{\|\bar{r}\|^2 + \|v\|^2} \leq c + \frac{N(N-1)^2 R}{2}\} \tag{8}$$

is compact. The derivative of W defined in (7) is:

$$\dot{W} = \frac{1}{2} \sum_{i=1}^{N} \dot{V}_i - \sum_{i=1}^{N} v_i^T \left(\sum_{j\sim i} (v_i - v_j) + \nabla_{r_i} V_i \right). \tag{9}$$

Note however that due to the symmetric nature of V_{ij},

$$\frac{1}{2} \sum_{i=1}^{N} \dot{V}_i = \sum_{j\sim i} \dot{r}_{ij}^T \nabla_{r_{ij}} V_{ij} = \sum_{j\sim i} (\dot{r}_i^T \nabla_{r_{ij}} V_{ij} - \dot{r}_j^T \nabla_{r_{ij}} V_{ij})$$

$$= \sum_{j\sim i} (\dot{r}_i^T \nabla_{r_i} V_{ij} + \dot{r}_j^T \nabla_{r_j} V_{ij}) = \sum_{i=1}^{N} \dot{r}_i^T \nabla_{r_i} V_i. \tag{10}$$

Thus, (9) simplifies to

$$\dot{W} = \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i - \sum_{i=1}^{N} v_i^T \left(\sum_{j\sim i} (v_i - v_j) + \nabla_{r_i} V_i \right) = -\sum_{i=1}^{N} v_i^T \sum_{j\sim i} (v_i - v_j) = -v^T (L_c \otimes I_2) v,$$

where v is the stack vector of all agent (two dimensional) velocity vectors, L_c is the Laplacian of the communication graph. Expanding the quadratic form involving the Kronecker product, $\dot{W}$ can be written:

$$\dot{W} = -v^T \left(L_c \otimes \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] + L_c \otimes \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right) v = -v_x^T L_c v_x - v_y^T L_c v_y, \tag{11}$$

where v_x and v_y are the stack vectors of the components of the agent velocities along $\hat{x}$ and $\hat{y}$ directions (Figure 1), respectively.

For a connected communication graph, L_c is positive semidefinite and the eigenvector associated with the single zero eigenvalue is the N-dimensional vector of ones. This means that $\dot{W}$ is negative semi-definite, will only be zero whenever both v_x and v_y belong to $\mathrm{span}\{\mathbf{1}\}$, implying that all agent velocities have the same components and are therefore equal. It follows immediately that $\dot{\bar{r}} = 0$.

The negative semi-definiteness of $\dot{W}$ also ensures the invariant properties of Ω, by selecting c to be equal to the value of W at initial time. In addition, it establishes collision avoidance since boundedness of W implies boundedness for every V_{ij}, and thus places a lower bound on all relative distances $\|r_{ij}\|$. Applying LaSalle's invariance principle to the system described by the vector field $(\dot{\bar{r}}, \dot{v})$, it follows that if the initial conditions of the system lie in Ω, its trajectories will converge to the largest invariant set inside the region $S = \{v \mid \dot{W} = 0\}$. Note that Ω can be made arbitrarily large, ensuring semi-global asymptotic stability of the invariant set. In S, the agent velocity dynamics are

$$\dot{v} = -\begin{bmatrix} \nabla_{r_1} V_1 \\ \vdots \\ \nabla r_N V_N \end{bmatrix} = -(B_s \otimes I_2) \begin{bmatrix} \vdots \\ \nabla_{r_{ij}} V_{ij} \\ \vdots \end{bmatrix},$$

where B_s is the incidence matrix of the sensing graph. The above, by a slight abuse of notation, can be expanded to

$$\dot{v}_x = -B_s [\nabla_{r_{ij}} V_{ij}]_x, \qquad \dot{v}_y = -B_s [\nabla_{r_{ij}} V_{ij}]_y.$$

Thus, both $\dot{v}_x$ and $\dot{v}_y$ belong in the range of B_s. For a connected sensing graph, $\mathrm{range}(B_s) = \mathrm{span}\{\mathbf{1}\}^\perp$ and therefore

$$\dot{v}_x,\ \dot{v}_y \in \mathrm{span}\{\mathbf{1}\}^\perp.$$

On the other hand, in the invariant set within S

$$v_x, v_y \in \text{span}\{\mathbf{1}\} \Rightarrow \dot{v}_x,\ \dot{v}_y \in \text{span}\{\mathbf{1}\},$$

which leads to contradiction unless

$$\dot{v}_x, \dot{v}_y \in \text{span}\{\mathbf{1}\} \cap \text{span}\{\mathbf{1}\}^{\perp} \equiv \{0\}.$$

This means that the agents velocities do not change in steady state and that the potential V_i of each agent i is (locally) minimized. ■

Corollary 1 (Distance setpoint stabilization) *If the sensing graph is a tree, then inter-agent distances can be stabilized to desired setpoints.*

Proof: For a tree, the number of edges is $N-1$ and thus B_s is full rank. In this case,

$$(B_s \otimes I_2)\begin{bmatrix} \vdots \\ \nabla_{r_{ij}} V_{ij} \\ \vdots \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \vdots \\ \nabla_{r_{ij}} V_{ij} \\ \vdots \end{bmatrix} = 0,$$

Let r_d be the configuration where V_{ij} attains its unique minimum. Then $\frac{\partial V_{ij}}{\partial \|r_{ij}\|} = 0$ implies that $\|r_{ij}\| = r_d$. ■

Corollary 2 (Convergence speed) *Velocity synchronization is accelerated as as the algebraic connectivity of the neighboring graph increases.*

Proof: Let us decompose the velocities v_x and v_y into two components

$$v_x = v_{x_p} \oplus v_{x_n}, \text{ where } v_{x_p} \in \text{span}\{\mathbf{1}\}, v_{x_n} \in \text{span}\{\mathbf{1}\}^{\perp},$$

$$v_y = v_{y_p} \oplus v_{y_n}, \text{ where } v_{y_p} \in \text{span}\{\mathbf{1}\}, v_{y_n} \in \text{span}\{\mathbf{1}\}^{\perp}.$$

Then from (11), since $L_c = B_c B_c^T$ we have that

$$\dot{W} = -v_{x_n}^T L_c v_{x_n} - v_{y_n}^T L_c v_{y_n}.$$

For a connected communication graph $\mathcal{G}_c$, B_c^T is full rank in $\text{span}\{\mathbf{1}\}^{\perp}$ and therefore,

$$\dot{W} \leq -\lambda_2(\|v_{x_p}\|^2 + \|v_{y_p}\|^2) = -\lambda_2 \|v_p\|^2$$

where λ_2 is the second smallest eigenvalue of the Laplacian, and $\|v_p\|$, expresses the magnitude of velocity misalignments. It is known that the addition of a new edge in a graph generally increases the eigenvalues of the Laplacian [32]. Hence, increasing the connectivity of the neighboring graph results to faster convergence. ■

V. SWITCHING INTERCONNECTION TOPOLOGY

One of the most interesting characteristics of the control scheme is that its stability is not affected by changes in the communication network. In this section we relax the assumption that communication network is fixed. Communication links between agents can be established or lost arbitrarily, at any time. Since the α_i compenent of the control law for agent i depends on its communication network neighbors ((3)), topology switches will introduce discontinuous changes in the closed loop dynamics of agent i.

The stability of the discontinuous dynamics will be analyzed using differential inclusions [33] and nonsmooth analysis [34]. A brief review of nonsmooth analysis and stability is given in the Appendix. In a switching interconnection topology, the agent dynamics can be expressed by means of a differential inclusion:

$$\dot{r}_i = v_i \tag{12a}$$

$$\dot{v}_i \in^{\text{a.e}} K[u_i] \quad i = 1, \dots, N\,, \tag{12b}$$

where $K[\cdot]$ is a differential inclusion (see Appendix) and a.e stands for "almost everywhere".

The dynamical system which we will analyze for stability is derived from (12), in exactly the same way as (6):

$$\dot{\bar{r}} = (B_{K_N} \otimes I_2) v \tag{13a}$$

$$\dot{v} \in^{\text{a.e}} K[u], \tag{13b}$$

where B_{K_N} is the incidence matrix of the complete graph with N vertices, K_N. Note that we do not make any assumption on the uniqueness of the solutions of (13).

Stability analysis is performed in this case using the same Lyapunov-like function (7):

$$W(\bar{r}, v) = \frac{1}{2} \sum_{i=1}^{N} (V_i + v_i^T v_i).$$

and the same expression for the control input (3):

$$u_i = -\sum_{j \in \mathcal{N}_c(i)} (v_i - v_j) - \sum_{j \in \mathcal{N}_s(i)} \nabla_{r_i} V_{ij},$$

only now, since the set $\mathcal{N}_c(i)$ can change arbitrarily, u_i will be a discontinuous function of time. The closed loop system, therefore, consists of a set of discontinuous differential equations, and stability analysis will be based on a nonsmooth version of LaSalle's invariance principle (Theorem 4 [35]).

We can now generalize Theorem 1 to the case where the commmunication topology switches arbitrarily between *connected* communication graph; the assumption on the connectivity of the communication graphs is essential in establishing the convergence of all velocities to a common vector:

Theorem 2 (Flocking in networks with switching) *Consider a system of N mobile agents with dynamics* (13)*, each steered by control law* (3) *and assume that at every time instant, the communication and sensing graphs are connected. Then all pairwise velocity differences converge asymptotically to zero, collisions between the agents are avoided, and the system approaches a configuration that locally minimizes all agent potentials.*

Proof: The fact that the system evolves in a compact set (defined in (8)) is derived from the connectivity assumption on the sensing and communication graphs, exactly as in the proof of Theorem 1. The invariant properties of Ω will be established in the sequel once W is shown to be non-increasing.

The Lyapunov-like function W is still differentiable, but now its derivative along the system's trajectories is not a quantity that can be exactly evaluated at the switching instants. This is because then we do not know exactly the value of $\dot{v}$; we can only ensure that $\dot{v} \in^{\text{a.e.}} K[u]$. The right hand side of the differential inclusion in (13) can be expanded as follows:

$$\begin{aligned} \dot{\bar{r}} &= (B_{K_N} \otimes I_2) v \\ \dot{v} &\in^{\text{a.e}} K[-(L_c \otimes I_2)v] - \begin{pmatrix} \nabla_{r_1} V_1 \\ \vdots \\ \nabla_{r_N} V_N \end{pmatrix}. \end{aligned}$$

Let ϕ_v be an arbitrary element of $K[-(L_c \otimes I_2)v]$. The *generalized* derivative of W, along a vector ϕ belonging in the set given by the right hand side of (13), will then be expressed as:

$$W^{\circ}(\bar{r}, v; \phi) = \frac{1}{2} \sum_{i=1}^{N} \dot{V}_i + v^T \phi_v - \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i,$$

which, using (10), becomes:

$$W^{\circ}(\bar{r}, v; \phi) = \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i + v^T \phi_v - \sum_{i=1}^{N} v_i^T \nabla_{r_i} V_i = v^T \phi_v. \tag{14}$$

Ryan's Theorem (c.f. Appendix, Theorem 4) examines the worst case for the rate of change of W:

$$m(r,v) = \max\{W^{\circ}(r,v;\phi) \mid \phi \in (\overset{v}{K[u]})\} = \max_{\phi_v \in K[-(L_c \otimes I_2)v]} \{v^T \phi_v\}.$$

Theorem 1 in Reference [36] enables us to write:

$$v^T K[-(L_c \otimes I_2)v] = K[-v^T (L_c \otimes I_2)v],$$

From Definition 4 it follows that

$$m(r,v) \in \max \overline{\mathrm{co}}\{-v^T (L_c \otimes I_2)v\}.$$

For a *connected* communication graph $\mathcal{G}_c$, L_c is positive semi-definite and therefore all quadratic forms of the type $-v^T(L_c \otimes I_2)v$ are nonpositive, regardless of the topology of the graph. Convex hulls of nonpositive numbers are nonpositive intervals, and thus $m(r,v)$ cannot be negative. The largest value it can have is zero. Rewriting v^T as:

$$v^T = \begin{pmatrix} v_{1x} & v_{1y} & v_{2x} & v_{2y} & \cdots & v_{Nx} & v_{Ny} \end{pmatrix},$$

we have that

$$-v^T(L \otimes I_2)v = v_x^T L_c v_x + v_y^T L_c v_y,$$

which implies that $m(r,v) = 0$ if and only

$$v_x = c_x \mathbf{1}_N \qquad\qquad v_y = c_y \mathbf{1}_N, \tag{15}$$

where $c_x, c_y \in \mathbb{R}$.

Applying Theorem 4 to the system described by the vector field $(\dot{\bar{r}}, \dot{v})$, it follows that for initial conditions in Ω, the Filippov solutions of the system converge to a subset of $\{v \mid v_x, v_y \in \mathrm{span}\{\mathbf{1}\}\}$. Equation (15) implies that for any two agents i and j,

$$\dot{r}_{ij} = v_i - v_j = 0.$$

In the set $\{v \mid v_x, v_y \in \mathrm{span}\{\mathbf{1}\}\}$ the acceleration dynamics reduces to

$$\dot{v} = (B_s \otimes I_2) \begin{bmatrix} \vdots \\ (\nabla_{r_{ij}} V_{ij}) \\ \vdots \end{bmatrix},$$

which implies that both $\dot{v}_x$ and $\dot{v}_y$ belong to the range of the incidence matrix B_s of the sensing graph $\mathcal{G}_s$. For a connected sensing graph, $\mathrm{range}(B_c) = \mathrm{span}\{\mathbf{1}\}^\perp$ and therefore

$$\dot{v}_x, \dot{v}_y \in \mathrm{span}\{\mathbf{1}\} \cap \mathrm{span}\{\mathbf{1}\}^\perp \equiv \{0\}. \tag{16}$$

From the above we conclude that the potential V_i of each agent is minimized. ■

Maintaining connectivity in the group while the network topology is switching based on the distance between the agents is a major issue. In the present analysis, this assumption is instrumental in showing the stability of the flocking motion of the group. The nonsmooth invariance theorem of Ryan [35], Theorem 4, does not require Ω to be compact, however the compactness and invariance of Ω implies the necessary *precompactness* of the solutions. If connectivity is lost, one cannot guarantee that $r_{ij} \in \Omega$ and thus stability may not be guaranteed.

VI. Numerical Simulations

This section presents the results of a numerical implementation of the proposed control scheme on a group of ten mobile agents. The number of agents in the group was kept that small for clarity of presentation. We investigate both the case of fixed communication topology and the case of where communication links switch arbitrarily. In both cases, sensing network links are distance dependent. Convergence is verified in cases, and case related characteristics are identified.

The case of fixed communication topology is investigated first. A group of ten mobile agents with dynamics (1) is initialized with random initial (x, y) positions in a rectangular area of 6.25 m^2 centered at the origin. Velocities were also randomly selected with magnitudes in the $(0, 1)$ m/s range, and with arbitrary directions. Randomly generated adjacency matrices defined connected sensing and communication graphs. Then the group motion evolves according to the closed loop system (1)-(3), and successive snapshots of this evolution are captured in Figure 3, for a time period of 100 simulation seconds. The particular time instant where the snapshot was taken is recorded below each frame. In Figure 3 the position of the agents is depicted by black dots and interconnections are represented by line segments connecting the agent locations. The path of each agent is shown by a dotted line and agent velocities are given as small arrows, which are scaled up at steady state to show how the vectors have been synchronized.

We next investigate a scenario where the communication topology changes arbitrarily. The agents are randomly initialized within the same range of positions and velocities. The integration

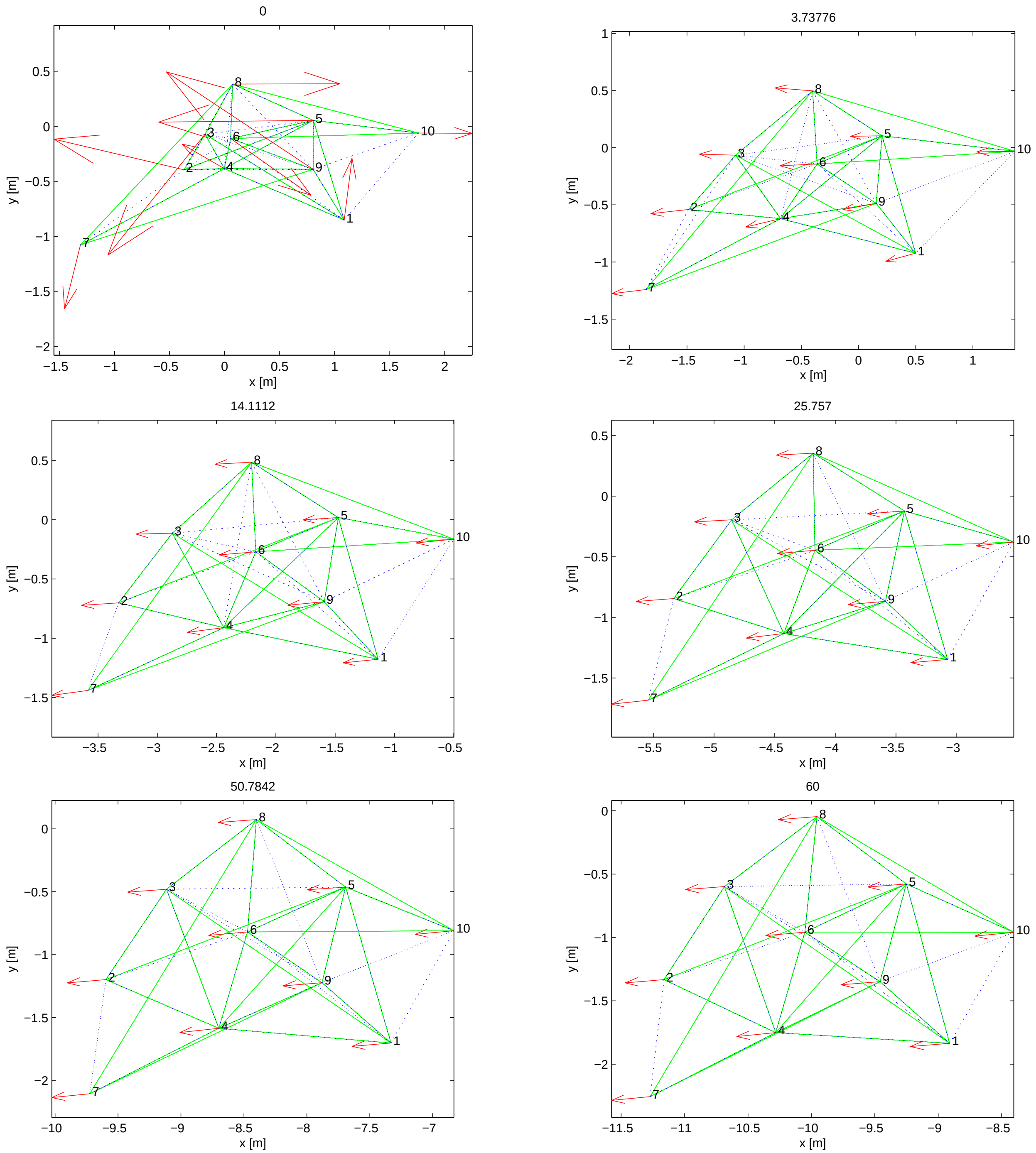

Fig. 3. Successive simulation time snapshots of flocking with fixed communication network topology. The figure on the upper left corner shows the initial condition. The steady state is shown in the bottom right figure. The timestamp of every snapshot is given on top of the corresponding figure.

period is now extended to 100 seconds to examine the effect of topology changes at steady state. Each call to the dynamic equation MATLAB function by the numerical integration function (`ode45`) can initiate a random switch to a completely different connected communication graph. Such switching happens with a given probability, but it is *not* otherwise restricted (for instance, in terms of dwell time). Figure 4 describes the evolution of a group of ten agents, where the communication topology is switching in the aforementioned manner. Once again, we depict the communication edges in solid (green) line segments and the sensing edges in dotted (blue) segments. Each snapshot shows a different communication graph, although the topology could have undergone several changes between these two time instants. The rate of change of the communication network can be seen in Figure 5. Figure 7 gives the time history of agent velocities. Convergence is fast, probably because with the network neighbors changing, an agent can have access to the velocities of a large set of its groupmates, rather than a restricted set of constant neighbors. Frequent topology switchings produce transients, but stability and overall convergence trend is evident.

Velocity synchronization in both cases is demonstrated in Figures 6-7. While Figures 3-4 have shown that all agents eventually move in the same direction, Figures 6-7 establish the convergence of agent speeds as well.

VII. Conclusions

In this paper we introduce a local control law for a group of mobile agents that allows them to stabilize their pairwise distances, avoid collisions and move as a coherent group having a common velocity vector. Agents communicate their velocity vectors over a time-varying communication graph and sense their relative distances to neighbors that are within a certain range. Each agent control law is based on a combination of a component that aligns its velocities with the groupmates it is communicating with, and an artificial potential-based component that regulates distances with nearest neighbors. We show that the behavior induced by our control law is robust to arbitrary changes in the sensing and communication networks, as long as these remain connected during the motion. We prove that agent potential functions are locally minimized and velocity vectors converge asymptotically to a common vector, by exploiting the algebraic connectivity of the underlying sensing and communication graphs.

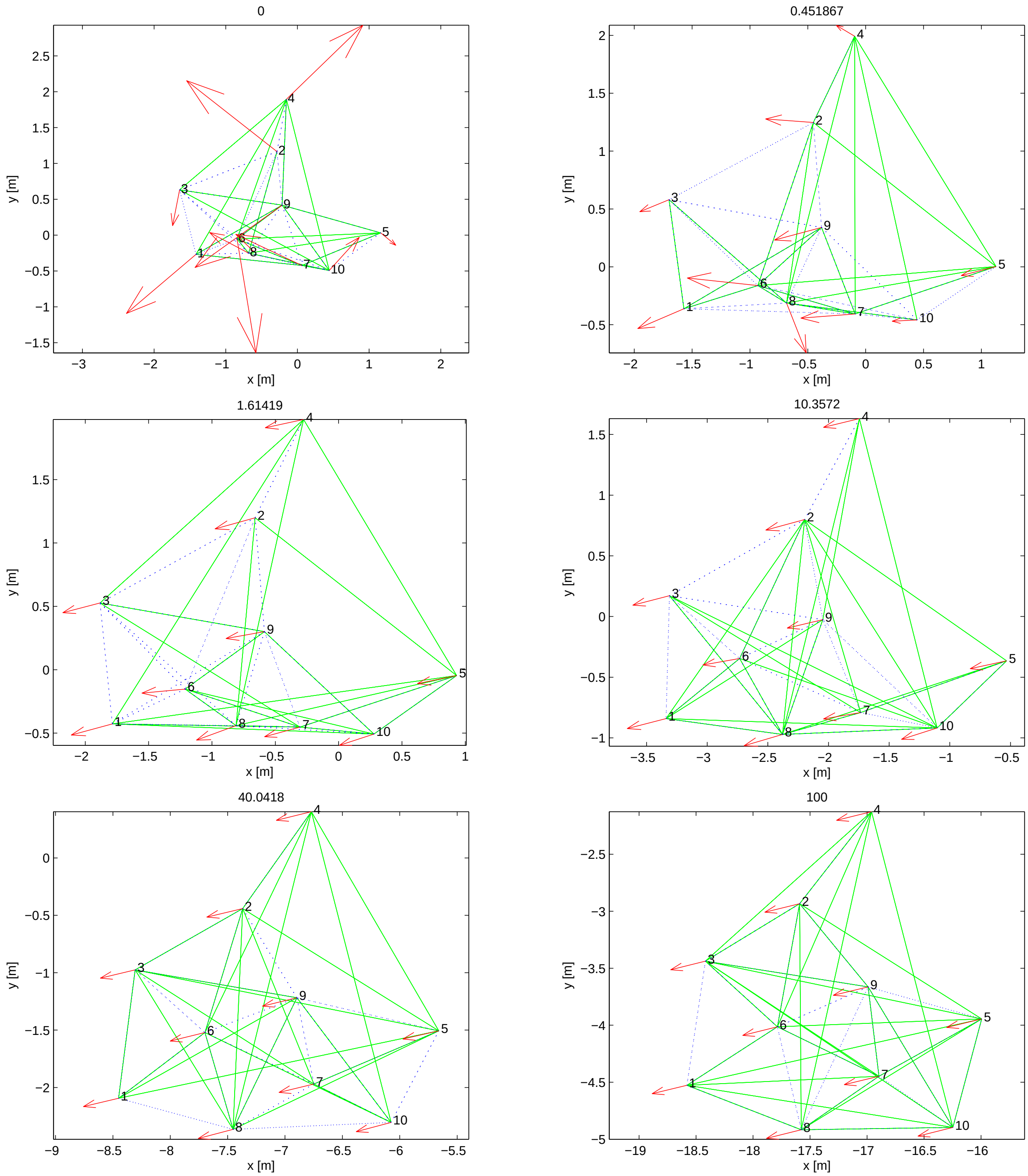

Fig. 4. Successive simulation time snapshots of flocking with dynamic interconnection topology. The top left figure shows the initial condition; bottom right gives the position after 100 simulation seconds. The time stamp of each snapshot is shown on top of the corresponding figure.

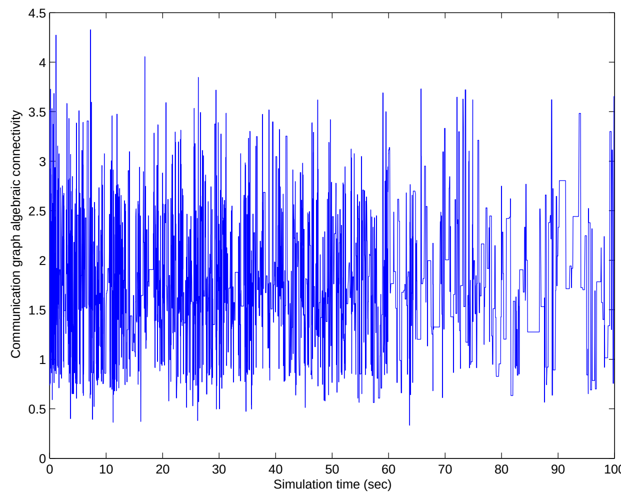

Fig. 5. Variation of the communication graph algebraic connectivity over time, when the graph is randomly switching.

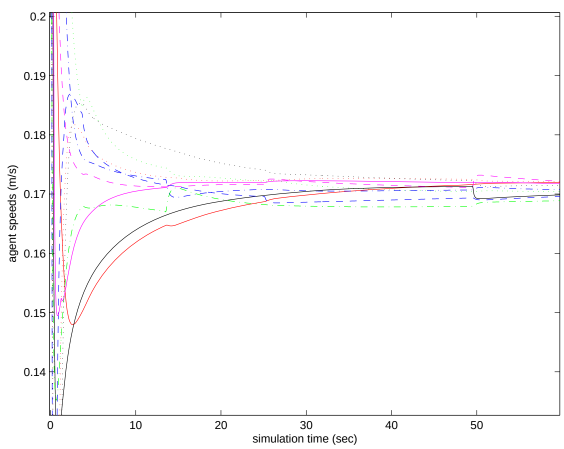

Fig. 6. Convergence of speeds with fixed communication network topology.

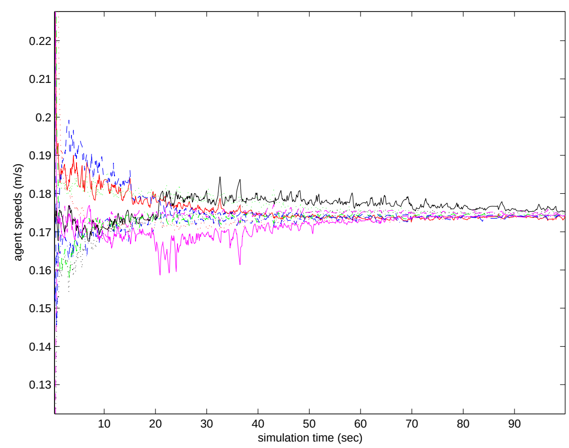

Fig. 7. Convergence of speeds under switching communication network topology.

ACKNOWLEDGMENT

This work has been partially supported by ARO MURI DAAD 19-02-01-0383. The authors will also like to thank Francesco Bullo, Jorge Cortés and Anurag Ganguli, for their constructive comments on nonsmooth analysis issues.

REFERENCES

[1] H. G. Tanner, A. Jadbabaie, and G. J. Pappas, "Stable flocking of mobile agents, Part I: Fixed topology," in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2010–2015.

[2] ——, “Stable flocking of mobile agents, Part II: Dynamic topology,” in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2016–2021.

[3] C. Reynolds, “Flocks, birds, and schools: a distributed behavioral model,” *Computer Graphics*, vol. 21, pp. 25–34, 1987.

[4] T. Vicsek, “A question of scale,” *Nature*, vol. 411, p. 421, May 24 2001.

[5] D. J. Low, “Following the crowd,” *Nature*, vol. 407, pp. 465–466, 2000.

[6] D. Grunbaum and A. Okubo, “Modeling social animal aggregations,” *Frontiers in Theoretical Biology*, vol. 100 of Lecture Notes in Biomathematics, pp. 296–325, 1994.

[7] G. Flierl, D. Grunbaum, S. Levin, and D. Olson, “From individuals to aggregations: the interplay between behavior and physics,” *Journal of Theoretical Biology*, vol. 196, pp. 397–454, 1999.

[8] D. Terzopoulos, “Artificial life for computer graphics,” *Communications of the ACM*, vol. 42, no. 8, pp. 32–42, 1999.

[9] T. Vicsek, A. Czirok, E. B. Jacob, I. Cohen, and O. Schochet, “Novel type of phase transitions in a system of self-driven particles,” *Physical Review Letters*, vol. 75, pp. 1226–1229, 1995.

[10] J. Toner and Y. Tu, “Flocks, herds, and schools: A quantitative theory of flocking,” *Physical Review E.*, vol. 58, pp. 4828–4858, 1998.

[11] H. Levine and W. J. Rappel, “Self organization in systems of self-propelled particles,” *Physical Review E*, vol. 63, pp. 208–211, 2001.

[12] N. Leonard and E. Fiorelli, “Virtual leaders, artificial potentials and coordinated control of groups,” in *IEEE Conference on Decision and Control*, Orlando, FL, 2001.

[13] J. Cortes, S. Martinez, T. Karatas, and F. Bullo, “Coverage control for mobile sensing networks,” *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, April 2004.

[14] R. O. Saber and R. M. Murray, “Distributed structural stabilization and tracking for formations of dynamic multi-agents,” in *IEEE Conference on Decision and Control*, Las Vegas, NV, 2002, pp. 209–215.

[15] J. H. Reif and H. Wang, “Social potential fields: A distributed behavioral control for autonomous robots,” *Robotics and Autonomous Systems*, vol. 27, pp. 171–194, 1999.

[16] J. A. Fax and R. M. Murray, “Graph Laplacians and stabilization of vehicle formations,” *15th IFAC Congress, Barcelona, Spain*, 2002.

[17] Y. Liu, K. M. Passino, and M. M. Polycarpou, “Stability analysis of m-dimensional asynchronous swarms with a fixed communication topology,” *IEEE Transactions on Automatic Control*, vol. 48, no. 1, pp. 76–95, February 2003.

[18] V. Gazi and K. M. Passino, “Stability analysis of swarms,” *IEEE Transactions on Automatic Control*, vol. 48, no. 4, pp. 692–696, April 2003.

[19] P. Tabuada, G. J. Pappas, and P. Lima, “Feasible formations of multi-agent systems,” in *Proceedings of the American Control Conference*, Arlington, VA, 2001.

[20] A. Jadbabaie, J. Lin, and A. S. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Transactions on Automatic Control*, vol. 48, no. 6, pp. 988–1001, July 2002.

[21] P. Ögren, M. Egerstedt, and X. Hu, “A control Lyapunov function approach to multi-agent coordination,” *IEEE Transactions on Robotics and Automation*, vol. 18, no. 5, October 2002.

[22] J. P. Desai, J. P. Ostrowski, and V. Kumar, “Modeling and control of formations of nonholonomic mobile robots,” *IEEE Transactions on Robotics and Automation*, vol. 17, no. 6, pp. 905–908, 2001.

[23] E. Justh and P. Krishnaprasad, “A simple control law for UAV formation flying,” Institute for Systems Research, Tech. Rep. 2002-38, 2002.

[24] R. Vidal, O. Shakernia, and S. Sastry, "Distributed formation control with omnidirectional vision based motion segmentation and visual servoing," *Robotics and Automation Magazine*, 2003, (to appear).

[25] A. Pant, P. Seiler, T. J. Koo, and K. Hedrick, "Mesh stability of unmanned aerial vehicle clusters," in *Proceedings of the American Control Conference*, Arlington, VA, June 2001, pp. 62–68.

[26] Z. Lin, M. Broucke, and B. Francis, "Local control strategies for groups of mobile autonomous agents," *IEEE Transactions on Automatic Control*, vol. 49, no. 4, pp. 622–629, April 2004.

[27] A. Barbosa, "The effects of predation risk on scanning and flocking behavior in dunlin," *Journal of Field Ornithology*, vol. 68, no. 4, pp. 607–612, 1997.

[28] R. O. Saber and R. M. Murray, "Flocking with obstacle avoidance: Cooperation with limited communication in mobile networks," in *Proceedings of the IEEE Conference on Decision and Control*, Maui, Hawaii, December 9-12 2003, pp. 2022–2028.

[29] O. Khatib, "Real-time obstacle avoidance for manipulators and mobile robots," *The International Journal of Robotics Research*, vol. 5, no. 1, pp. 396–404, Spring 1986.

[30] E. Rimon and D. Koditschek, "Exact robot navigation using artificial potential functions," *IEEE Transactions on Robotics and Automation*, vol. 8, no. 5, pp. 501–518, October 1992.

[31] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, September 2004.

[32] R. Merris, "Laplacian matrices of graphs: A survey," *Linear Algebra and its Applications*, vol. 197, pp. 143–176, 1994.

[33] A. F. Filippov, *Differential equations with discontinuous right-hand side*, ser. Mathematics and Its Applications (Soviet Series), F. M. Arscott, Ed. The Netherlands: Kluwer Academic Publishers, 1988.

[34] F. H. Clarke, *Optimization and Nonsmooth Analysis*, ser. Classics in applied mathematics; 5. Philadelphia: SIAM, 1990.

[35] E. P. Ryan, "An integral invariance principle for differential inclusions with applications in adaptive control," *SIAM Journal on Control and Optimization*, vol. 36, no. 3, pp. 960–980, May 1998.

[36] B. Paden and S. Sastry, "A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Transactions on Circuits and Systems*, vol. CAS-34, no. 1, pp. 73–82, 1987.

[37] C. Godsil and G. Royle, *Algebraic Graph Theory*. New York: Springer Graduate Texts in Mathematics # 207, 2001.

[38] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Transactions on Automatic Control*, vol. 39, no. 9, pp. 1910–1914, 1994.

[39] A. Bacciotti and F. Ceragioli, "Stability and stabilization of discontinuous systems and nonsmooth lyapunov functions," *ESAIM: Control Optimisation and Calculus of Variations*, vol. 4, pp. 361–376, 1999.

[40] F. Clarke, Y. Ledyaev, R. Stern, and P. Wolenski, *Nonsmooth Analysis and Control Theory*, ser. Graduate Texts in Mathematics; 178. New York: Springer, 1998.

APPENDIX I

ALGEBRAIC GRAPH THEORY

Stability analysis of the group of agents builds around several results on algebraic graph theory. This necessitates a brief introduction of related graph theoretic notation and terminology. The interested reader is referred to [37] for details.

An (undirected) *graph* $\mathcal{G}$ consists of a *vertex set*, $\mathcal{V}$, and an *edge set* $\mathcal{E}$, where an edge is an unordered pair of distinct vertices in $\mathcal{G}$. If $i, j \in \mathcal{V}$, and $(i, j) \in \mathcal{E}$, then i and j are said to be *adjacent*, or neighbors and we denote this by writing $i \sim j$. A graph is called *complete* if any two vertices are neighbors. The number of neighbors of each vertex is its *valency* or *degree*. A *path* of length r from vertex i to vertex j is a sequence of $r + 1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of a graph $\mathcal{G}$, then $\mathcal{G}$ is said to be *connected*.

The *valency matrix* $\Delta(\mathcal{G})$ of a graph $\mathcal{G}$ is a diagonal matrix with rows and columns indexed by $\mathcal{V}$, in which the (i, i)-entry is the valency of vertex i. An *orientation* of a graph $\mathcal{G}$ is the assignment of a direction to each edge, so that the edge (i, j) is now an arc from vertex i to vertex j. We denote by $\mathcal{G}^\sigma$ the graph $\mathcal{G}$ with orientation σ. The *incidence matrix* $B(\mathcal{G}^\sigma)$ of an oriented graph $\mathcal{G}^\sigma$ is the matrix whose rows and columns are indexed by the vertices and edges of $\mathcal{G}$ respectively, such that the i, j entry of $B(\mathcal{G}^\sigma)$ is equal to 1 if edge j is incoming to vertex i, -1 if edge j is outcoming from vertex i, and 0 otherwise. The symmetric matrix defined as:

$$L(\mathcal{G}) = B(\mathcal{G}^\sigma)B(\mathcal{G}^\sigma)^T$$

is called the *Laplacian* of $\mathcal{G}$ and is independent of the choice of orientation σ. It is known that the Laplacian captures many interesting properties of the graph. Among those, is the fact that L is always symmetric and positive semidefinite, and the algebraic multiplicity of its zero eigenvalue is equal to the number of connected components in the graph. For a connected graph, the n-dimensional eigenvector associated with the single zero eigenvalue is the vector of ones, $\mathbf{1}_n$. The second smallest eigenvalue, λ_2 is positive and is known as the algebraic connectivity of the graph, because it is directly related to how the nodes are interconnected.

In what follows, we will use graph theoretic terminology to represent the interconnections between the agents in the group. The connectivity properties of the induced graph will prove crucial for establishing the stability of the flocking motion of the group.

APPENDIX II
NONSMOOTH ANALYSIS AND SYSTEM STABILITY

The purpose of this section is to briefly introduce the mathematical machinery related to nonsmooth stability analysis. We begin with a definition of our notion of solutions of differential equations with discontinuous right hand sides:

Definition 4 ([36]) *Consider the following differential equation in which the right hand side can be discontinuous:*

$$\dot{x} = f(x) \tag{17}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is measurable and essentially locally bounded and n is finite. A vector function $x(\cdot)$ is called a solution of (17) *on $[t_0, t_1]$, where if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$*

$$\dot{x} = K[f](x)$$

where

$$K[f](x) \triangleq \overline{co}\{\lim_{x_i \to x} f(x_i) \mid x_i \notin M_f \cup M\}$$

where $M_f \subset \mathbb{R}^n$, $\mu(M_f) = 0$, $M \subset \mathbb{R}^n$, $\mu(M) = 0$.

In the above, $\mu(\cdot)$ denotes the measure of the set, and M_f the set where f is not differentiable. The set M can be arbitrary.

According to this definition, a trajectory $x(t)$ is considered a solution of the discontinuous differential equation (17) if its tangent vector, where defined, belongs in the convex closure of the limit of the vector fields defined by (17) in a decreasingly small neighborhood of the solution point. Being able to exclude a set of measure zero, is critical since one can thus define solutions even at points where the vector field in (17) is not defined.

A sligthtly more general definition of (maximal) solutions can be found in [35]:

Definition 5 ([35]) *Consider the autonomous initial-value problem*

$$\dot{x}(t) \in X\big(x(t)\big), \quad x(t) \in G, \quad x(t_0) = x^0, \tag{18}$$

where $G \neq \emptyset$ is an open subset of $\mathbb{R}^N$. The set-valued map $(x) \mapsto X(x) \subset \mathbb{R}^N$ in (18) *is assumed to be upper semicontinuous on $\mathbb{R} \times G$, with nonempty, convex, and compact values.*

This definition is sufficient to ensure that the solution is absolutely continuous on compact subintervals $I \in \mathbb{R}$ ($x(t) \in AC(I; G)$). Ryan defines solutions to be maximal if they cannot be extended any further in time:

Definition 6 ([35]) *A solution of* (18) *is said to be maximal if it does not have a proper right extension which is also a solution of* (18).

Then, it can be shown that all solutions of (18) can be thought to be maximal:

Proposition 1 ([35]) *Every solution of* (18) *can be extended to a maximal solution.*

A maximal solution is called precompact if it always stays in the closure of G:

Definition 7 ([35]) *A solution $x \in AC([t_0, \omega); G)$ of* (18) *is precompact if it is maximal and the closure $\mathrm{cl}\big(x([t_0, \omega))\big)$ of its trajectory is a compact subset of G.*

Lyapunov stability has been extended to nonsmooth systems [38], [39]. Establishing stability results in this framework requires working with generalized derivatives [34], whenever classical derivatives are not defined.

Definition 8 ([34]) *Let f be Lipschitz near a given point x and let w be any vector in a Banach space X. The generalized directional derivative of f at x in the direction w, denoted $f^\circ(x; w)$ is defined as follows:*

$$f^\circ(x; w) \triangleq \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tw) - f(y)}{t}$$

The generalized gradient, on the other hand, is generally a set of vectors, which reduces to the single classical gradient in the case where the function is differentiable:

Definition 9 ([34]) *The generalized gradient of f at x, denoted $\partial f(x)$, is the subset of X^* given by:*

$$\partial f(x) \triangleq \{\zeta \in X^* \mid f^\circ(x; w) \geq \langle \zeta, w \rangle, \, \forall w \in X\}$$

In the special case where X is finite dimensional, we have the following convenient characterization of the generalized gradient:

Theorem 3 ([40]) *Let $x \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near x. Let Ω be any subset of zero measure in $\mathbb{R}^n$, and let Ω_f be the set of points in $\mathbb{R}^n$ at which f fails to be differentiable. Then*

$$\partial f(x) \triangleq \mathrm{co}\{\lim_{x_i \to x} \nabla f(x_i) \mid x_i \notin \Omega, x_i \notin \Omega_f\}$$

A *weakly invariant set* is defined to be the set where at least one of the (possibly multiple) maximal solutions of (18) stays forever in the set:

Definition 10 ([35]) *Relative to* (18)*,* $S \subset \mathbb{R}^N$ *is said to be weakly invariant set if, for each* $x^0 \in S \cap G$*, there exists at least one maximal solution* $x \in AC([0,\omega);G)$ *of* (18) *with* $\omega = \infty$ *and with trajectory* $x([0,\omega))$ *in* S*.*

Shevitz and Paden [38] proposed a nonsmooth version of LaSalle's invariance principle, and Bacciotti and Ceragioli [39] have given an alternative nonsmooth characterization of the invariance principle which also applies to the case where uniqueness of solutions cannot be guaranteed. Here, we choose to apply the invariance principle introduced by [35] since, not only does it not require uniqueness of solutions, but also lifts the regularity requirement from the Lyapunov-like function.

Theorem 4 ([35]) *Let* $V : G \to \mathbb{R}$ *be locally Lipschitz. Define*

$$m : G \to \mathbb{R}, z \mapsto m(z) \triangleq \max\{V^\circ(z;\phi) \mid \phi \in X(z)\}.$$

Suppose that $U \subset G$ *is non-empty and that* $m(z) \leq 0$ *for all* $z \in U$*. If* x *is a precompact solution of* (18) *with trajectory in* U*, then for some constant* $c \in V(\mathrm{cl}(U) \cap G)$*,* x *approaches the largest weakly invariant set in* $\Sigma \cap V^{-1}(c)$*, where*

$$\Sigma = \{z \in \mathrm{cl}(U) \cap G \mid m(z) \geq 0\}.$$

In the above theorem, the invariant set is defined more generally, but this does not restrict our analysis. In essence, it allows the generalized time derivative of V to be positive on the boundary of U. From Definitions 9 and 4, one can see that $m(z)$ is nothing but the maximal element in $\dot{\tilde{V}}$ of [38], if V turns out to be regular.