A CLOSED FORM APPROACH
TO THE VALUATION AND HEDGING OF BASKET AND SPREAD OPTIONS

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Abstract

We develop a new approach to valuing and hedging basket and spread options. We consider baskets of assets with potentially negative portfolio weights (spread options are a subclass of such basket options). The basket distribution is approximated using a generalized family of log-normal distributions. This approximation copes with negative basket values as well as negative skewness of the basket distribution, and it provides closed formulae for the option price and greeks. Numerical simulations show our approach provides a very close approximation to the option price, and performs remarkably well in terms of the hedging error. We analyze option price sensitivities with respect to the assets' volatilities and correlations; and explain the seemingly paradoxical phenomenon of negative volatility vegas.

1. Introduction

Basket options are options whose payoff depends on the value of a *basket*, i.e., a *portfolio of assets*. Equity index options and currency basket options are classical examples of basket options. However, basket options are becoming increasingly widespread in commodity and particularly energy markets.

Commodity basket options are over-the-counter (OTC) instruments tailored to suit the needs of a particular customer. For example, an oil company which owns a portfolio of energy products (crude oil, refined products, natural gas) protects the portfolio value by buying a suitable basket call option. An important subclass of basket options are *spread options*, where the underlying value is the *spread* (i.e. the difference) between the prices of two commodities. For example, a power company purchases natural gas (or other fuel) to produce and sell electricity. To protect its profits, the company buys a call option on the so-called *spark spread*, which is the difference between the electricity price and the natural gas price multiplied by the factor 0.68[1].

Other typical energy baskets are the so-called *crack spreads*, or refinery margins, which are the spreads between crude oil and a number of refined products (for example, a 3:2:1 crack spread is the difference between the crude oil price and the heating oil and unleaded gasoline prices). In agricultural markets, the *soybean crush spread* is the difference between the price of soy and two soy products, soy oil and soy meal. These are typical commodity baskets, comprising a number of different but closely related commodities.

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[1]This factor is obtained from converting therms (natural gas energy units) into MWh (MegaWatt hour) and assuming 50% generation efficiency, common in the UK.

Options on such baskets are sold mostly over-the-counter and are widely used by commodity producers and consumers. In recent years some exchange-traded commodity spread options have been introduced, for example two types of crack spread options are traded nowadays on the New York Mercantile Exchange (NYMEX)[2].

Valuing basket options is a challenging task because the underlying value is a weighted sum of individual asset prices. The common assumption of log-normality (and hence, the famous Black-Scholes formula) cannot be applied directly, because the sum of log-normal random variables is not log-normal. This problem also occurs in pricing Asian options, and the existing approaches to pricing basket options are those used for Asian options. The Wakeman method (Turnbull and Wakeman (1991)) is based on a log-normal approximation of the basket value distribution by moment-matching. Milevsky and Posner (1995) use the reciprocal Gamma distribution to approximate the basket distribution. Both approaches are attractive because they lead to a closed formula for the approximate option price. However, they are only applicable when all the portfolio (i.e. basket) weights are positive.

Another class of methods is used to value spread options. Here the log-normal distribution cannot be used even as an approximation, because spreads can (and often do) have negative values. One possible solution is to assume the distribution of the spread can be approximated by the normal distribution. This leads to the *Bachelier method*, applied by Shimko (1994) to spread options. However, option prices obtained by the Bachelier method are often significantly different to the real option prices or those obtained by Monte Carlo simulation. This is due to the poor approximation of the spread distribution by the normal distribution. Another, more successful approximation method is suggested by Kirk (1995) (inspired by the classical paper of Magrabe (1978)), who replaced the difference of asset prices by the ratio and adjusted the strike price. Pearson (1995) used a conditional argument to reduce a two-dimensional integral to a one-dimensional one, to obtain an semi-numerical approximation for the spread option price. Most of the mentioned methods provide closed formulae for the approximate option price, but can only deal with spreads between two assets. Carmona and Durrelman (2003) give a good overview of spread options and propose precise lower bounds to approximate spread option prices. Although they mention a possibility to extend their method for more than two assets, at present it can only deal with spread options.

Commodity market participants often deal with spreads and baskets comprising more than two assets, where some of the assets may have negative portfolio weights, because producers must purchase some "raw" commodity to produce their products. In soybean crush spread, soy enters with negative weight (raw material) and soy oil and soy meal with positive weights (products). Analogously, in crack spread the raw material is oil and has a negative weight, while unleaded gasoline and heating oil are the oil products and hence, have positive weights. More complex portfolios (baskets) are common in energy and agricultural markets and so are OTC options on them.

To our knowledge, there is no analytical approximation approach available for pricing and hedging of general basket options, i.e. those comprising several assets with potentially negative portfolio weights. Numerical or Monte Carlo methods are the only possibilities, but they may be slow and do not provide a closed formula for the option price - something of great value to practitioners. A closed formula for the option price (or its approximation) is not only easy to understand and implement, but it also leads to closed formulae for the option greeks (i.e. sensitivities to the model parameters, such as volatilities), which then can be quickly and accurately evaluated. This is essential for hedging the option and managing an option portfolio. Moreover, a closed formula for an option price can be inverted to imply the assets' volatilities and correlations.

We propose a method which deals with general basket and spread options and provides a closed formula approximation to the option price and the greeks. Our approach is multi-factor, i.e. it assumes a separate

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[2]For descriptions of these options see *www.nymex.com*

price processes for each asset[3], and the dependencies between these processes are quantified by correlations. Our approach is closely related to the log-normal approximation, which is generally considered a better approximation for the basket value than e.g. the normal distribution. In this respect our approach is close in spirit to the Wakeman method, but is able to deal with negative basket values. Moreover, the log-normality assumption means the classical tools of option pricing such as the Black-Scholes formula are readily applicable.

Here we consider baskets of futures or forward contracts[4] on different (but related) commodities. Such basket options are very common in commodity markets, and certainly more common than basket options on physical commodities (however, our approach can be readily extended to such basket options). For example, all exchange-listed spread options are written on baskets (or spreads) of futures.

We assume that the futures in the basket and the basket option mature at the same time. In practice, different commodity futures have different expiration schedules, and a typical basket option matures just before the earliest expiring futures or forward contract in the basket.

In commodity markets, companies often purchase the so-called *calendar strips of options* to protect their profits during extended periods of time. A strip of basket or futures options is a collection of options maturing every calendar month for e.g. a year (a strip of 12 options) or six months (a strip of 6 options), together with the underlying futures contracts. Often in practice, each option in the strip is priced as a separate option, each with its own set of parameters. Theoretically, however, to price calendar strips of options, forward curve models are needed, i.e. models describing the simultaneous dynamics of futures prices with different maturities. An output of such a model is a collection of futures price volatilities (the so-called *volatility forward curve*), which are then used for pricing a calendar strip of options. This subject, however, is outside the scope of this paper; here we shall focus on valuing and hedging one option on a basket of futures, all maturing at the same time.

2. The model

Consider a basket of futures, whose prices $(F_i(t))_i$ follow correlated Geometric Brownian Motions. The basket value at time t is given by

$$B(t) = \sum_{i=1}^{N} a_i F_i(t),$$

where a_i is the weight (possibly negative) corresponding to the asset i.

A general basket can have negative values, which makes a direct approximation of its distribution by the log-normal distribution impossible. Moreover, the basket distribution can be negatively skewed, while the log-normal distribution always has positive skewness. However, a "family" of log-normal distributions (obtained from the ordinary log-normal distribution by shifting it along x-axis and/or reflecting it across y-axis) is well-suited to approximating general basket distributions.

Under the assumption of Geometric Brownian Motion dynamics for the asset prices, the first few moments of the basket value can be easily calculated. The moment matching plays a key role in choosing the appropriate approximating distribution and estimating its parameters. Finally, by applying the Black-Scholes model, we calculate the European option price, which is then used to obtain closed formulae for the greeks.

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[3]One-factor approaches to spread options assume some stochastic process directly for the spread value, without separately modelling the individual price processes and dependencies between them, see e.g. Wilcox (1990).

[4]Under the assumption of deterministic interest rates, futures and forward prices coincide, see e.g. Bjork (1999).

2.1. Basket Distribution

Under the risk-adjusted probability measure, the futures prices are martingales, hence the stochastic differential equation for $F_i(t)$ is

$$\frac{dF_i(t)}{F_i(t)} = \sigma_i dW_i(t), \quad i = 1, 2, ..., N \tag{1}$$

where $F_i(t)$ is the price of the futures contract i at time t, N is the number of assets in the basket, σ_i is the volatility of futures i, $W_i(t)$ and $W_j(t)$ are the Brownian motions driving futures i and j with correlation $\rho_{i,j}$.

The dynamics (1) implies that the distribution of the futures prices is log-normal. Although the sum of log-normal random variables is not log-normal, the log-normal distribution approximates the distribution of such sum quite well, and certainly much better than the normal distribution (at least for a relatively small number of summands). This has already been noted by Mitchell (1968), who studied the accuracy of the log-normal approximation for the sum of log-normal random variables in the optical context, i.e. a ray propagating through a medium with randomly varying index of refraction. Studies from many other areas of science, ranging from physics to life sciences to economics (see e.g. Aitchison and Brown (1957), Crow and Shimizu (1988), Limpert, Stahel and Abbt (2001)) have confirmed the high accuracy of the log-normal approximation for the sum of log-normal random variables.

Motivated by these studies, we choose the log-normal distribution for approximating the basket distribution. However, for baskets containing negative weights the log-normal approximation cannot be used directly, due to potentially negative values and negative skewness. Let us illustrate this in the following two examples. We consider two hypothetical baskets with two assets (i.e. spreads), for both we assume that the (constant) interest rate r is 3 % per annum and the time to maturity T is 1 year. We denote F_0 the vector of futures prices at time t=0, σ is the vector of annualized futures volatilities, a is the vector of weights and ρ the correlation between the assets.

Figures 1 and 2 show the simulated distribution of the terminal basket values (at time T) under the risk-adjusted probability measure.
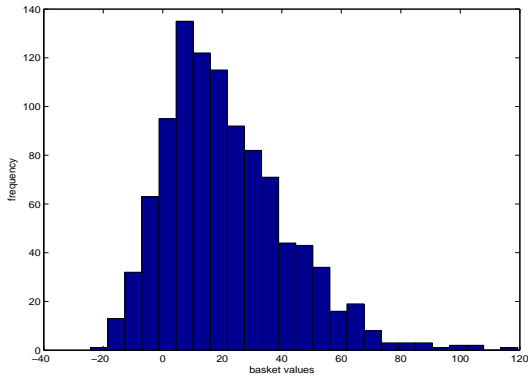

Figure 1: Basket 1 distribution at $T = 1$ year. F_0=[100;120], σ=[0.2;0.3], ρ=0.9, a=[-1;1].
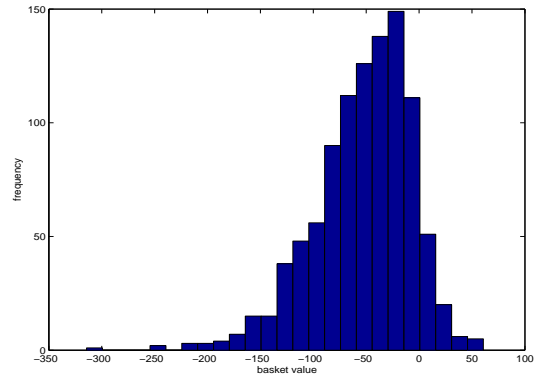

Figure 2: Basket 2 distribution at $T = 1$ year. F_0=[150;100], σ=[0.3;0.2], ρ=0.3, a=[-1;1].

Note that the "shape" of the distribution still resembles the log-normal distribution, due to the skewness and one large tail, only in the first figure it is shifted along the horizontal axis to the left by approximately 25 units, and in the second figure it is reflected to the vertical axis and then shifted along horizontal axis to the right by approximately 60 units. Such examples give us the idea to use the so-called *shifted* and *negative log-normal distributions* to approximate distributions such as in Figures 1 and 2.

Recall that the probability density function (p.d.f.) of the regular log-normal distribution is given by

$$f(x) = \frac{1}{sx\sqrt{2\pi}} \exp\left(-\frac{1}{2s^2}(\log x - m)^2\right), x > 0. \tag{2}$$

The shifted log-normal distribution is defined by the probability density function

$$f(x) = \frac{1}{s(x-\tau)\sqrt{2\pi}} \exp\left(-\frac{1}{2s^2}(\log(x-\tau) - m)^2\right), x > \tau, \tag{3}$$

and we define the *negative log-normal distribution* by the probability density function

$$f(x) = \frac{-1}{sx\sqrt{2\pi}} \exp\left(-\frac{1}{2s^2}(\log -x - m)^2\right), x < 0. \tag{4}$$

The combination of the shift and the reflection in the y-axis gives rise to the *negative shifted log-normal distribution*, defined by its probability density function

$$f(x) = \frac{1}{s(-x-\tau)\sqrt{2\pi}} \exp\left(-\frac{1}{2s^2}(\log(-x-\tau) - m)^2\right), x < -\tau. \tag{5}$$

Everywhere m is the scale, s is the shape and τ is the location parameters.

Note that, if a random variable X has the (regular) log-normal distribution, then the random variable $Y = X + \tau$ has the shifted log-normal distribution and the random variable $Z = -X$ the negative log-normal distribution.

The shifted log-normal distribution can be used to approximate the distribution of the Basket 1, shown in Figure 1, since the basket has positive skewness and can have negative values. Figures 3 (empirical and approximating p.d.f.) and 4 (the QQ-plot) show that the shifted log-normal distribution approximates the Basket 1 distribution very well (apart from slight deviations in one tail - an observation also noted by Mitchell (1968)).
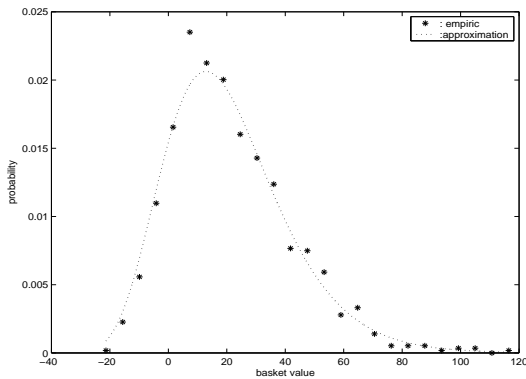

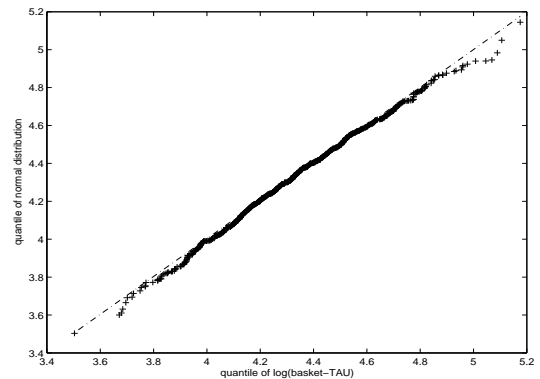Figure 3: Approximating and empirical distributions of Basket 1.

Figure 4: QQ-plot of approximating vs. empirical distributions of Basket 1.

The Basket 2, shown in Figure 2 has negative skewness and can have positive values. Hence we choose the negative shifted log-normal as the approximating distribution. This approximation is again very good, as shown in Figures 5 and 6. Such observations lead us to use the regular, shifted, negative and negative shifted log-normal distributions for approximating the basket distribution.

At the time of writing a basket option, the basket parameters (weights, initial asset prices, volatilities, correlations, the interest rate and the option's strike and time to maturity) are fixed. However, it is impossible
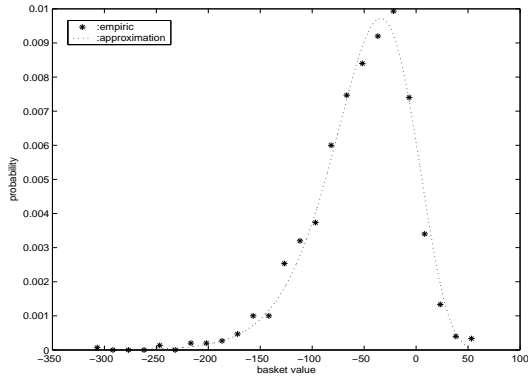
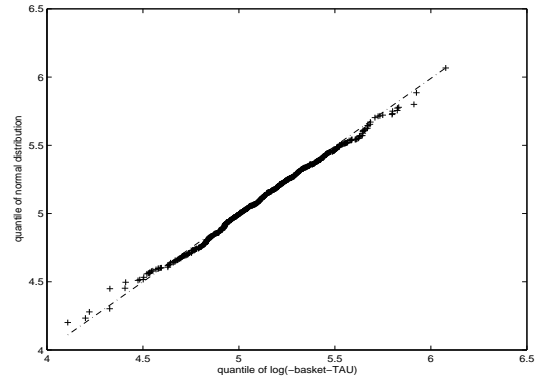Figure 5: Approximating and empirical distributions of Basket 2.

Figure 6: QQ-plot of approximating vs. empirical distribution of Basket 2.

to tell at a glance what is the shape the terminal basket distribution and so, which of the possible approximating distributions (2), (3), (4) or (5) to use. The two parameters: the skewness and the location of the basket distribution at time T - allow us to select the correct approximating distribution. To calculate these parameters, we use the moment matching procedure.

2.2. Parameters estimation

If the dynamics of the assets in the basket is given by (1), then calculations show that the first three moments of the basket on the maturity date T are

$$EB(T) = \sum_{i=1}^{N} a_i F_i(0), \tag{6}$$

$$E\big(B(T)\big)^2 = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j F_i(0) F_j(0) \exp\left(\rho_{i,j}\sigma_i\sigma_j T\right), \tag{7}$$

$$E\big(B(T)\big)^3 = \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j a_k F_i(0) F_j(0) F_k(0) \exp\left[(\rho_{i,j}\sigma_i\sigma_j + \rho_{i,k}\sigma_i\sigma_k + \rho_{j,k}\sigma_j\sigma_k)T\right]. \tag{8}$$

In terms of the first three moments, the skewness of basket can be calculated as

$$\eta_{B(T)} = \frac{E\big[B(T) - EB(T)\big]^3}{s_{B(T)}^3}, \tag{9}$$

where $s_{B(T)} = \sqrt{EB^2(T) - (EB(T))^2}$ is the standard deviation of the basket value at time T.

For each of the distributions (2), (3), (4) and (5), we can also derive the first three moments in terms of the parameters m, s, τ. For example, for the shifted log-normal distribution (3) the first three moments are

$$M_1(T) = \tau + \exp\left(m + \frac{1}{2}s^2\right) \tag{10}$$

$$M_2(T) = \tau^2 + 2\tau \exp\left(m + \frac{1}{2}s^2\right) + \exp\left(2m + 2s^2\right) \tag{11}$$

$$M_3(T) = \tau^3 + 3\tau^2 \exp\left(m + \frac{1}{2}s^2\right) + 3\tau \exp\left(2m + 2s^2\right) + \exp\left(3m + \frac{9}{2}s^2\right) \tag{12}$$

The parameters of the approximating distribution, e.g. the shifted log-normal, are estimated by matching the first three moments of the basket (6), (7), and (8) with the first three moments of the shifted log-normal

distribution (10), (11) and (12). This amounts to solving a nonlinear system of three equations with three unknowns (m, s and τ). If the approximating distribution is chosen negative (shifted) log-normal, then the distribution parameters are obtained by solving the same nonlinear equation system, only the moments M_1 and M_3 in (10) and (12) are replaced by by $-M_1$ and $-M_3$.

The choice of the approximating distribution (regular (shifted) or negative (shifted) log-normal) depends on the distribution skewness: if the skewness is positive, the regular or shifted log-normal distribution should be chosen as an approximating distribution. If the skewness is negative, then the approximating distribution is the negative or negative shifted log-normal.

The location parameter τ determines the shift of the approximating distribution. However, the regular log-normal distribution often provides a better fit than the shifted distribution, even when $\tau \neq 0$. This happens when the basket distribution is positively skewed and basket value cannot be negative. We illustrate this on the example of the Basket 3 with the parameters $F_0 = (110, 90)$, $\sigma = (0.3, 0.2)$, $a = (0.7, 0.3)$, $\rho = 0.9$. Its terminal distribution (at time T) is shown in Figure 7.

If we match the moments of this basket to the shifted log-normal distribution, the estimate of the location parameter τ is 34, so it seems that the shifted log-normal distribution provides the right approximation. However, Figures 8, 9 and the QQ-plots in Figures 10 and 11 suggest that the regular log-normal distribution provides a better fit.
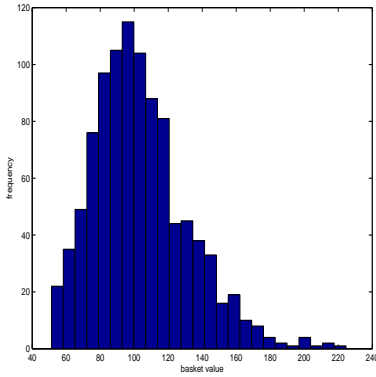

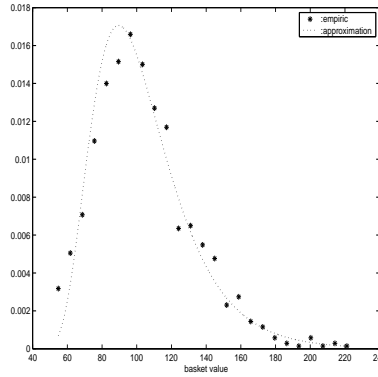Figure 7: Histogram of Basket 3 terminal values (at time T).

Figure 8: Shifted log-normal approximation and empirical distribution of Basket 3.
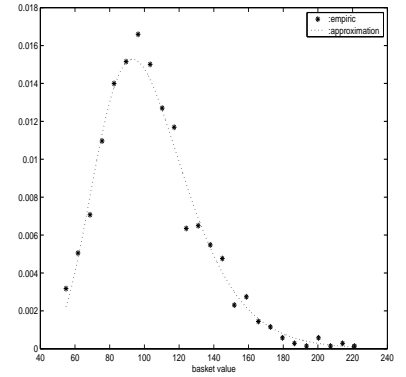
Figure 9: Regular log-normal approximation and empirical distribution of Basket 3.
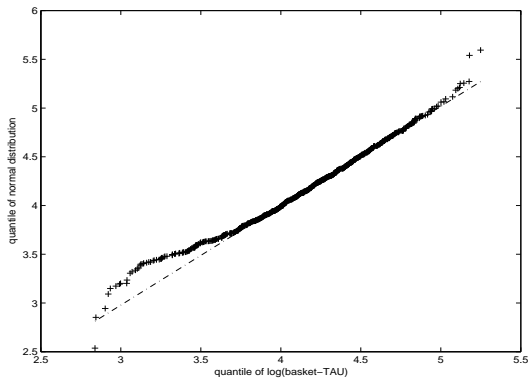

Figure 10: QQ-plot of shifted log-normal vs. empirical distribution of Basket 3.
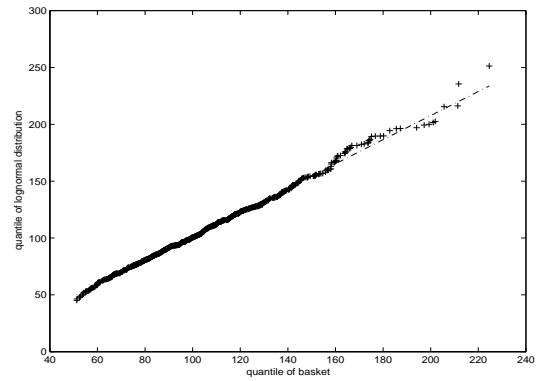
Figure 11: QQ-plot of regular log-normal vs. empirical distribution of Basket 3.

So for a positively skewed basket distribution, we choose the shifted log-normal approximation *only*

when $\tau < 0$, to cope with negative basket values, and when $\tau \geq 0$ we suggest to use the regular log-normal distribution[5]. Analogously, for negatively skewed distributions, we use the negative shifted log-normal distribution only when $\tau < 0$, and otherwise we use the negative log-normal distribution. Table 1 summarizes the choice of the approximating distribution for different parameter combinations.

Skewness	$\eta > 0$	$\eta > 0$	$\eta < 0$	$\eta < 0$
Location parameter	$\tau \geq 0$	$\tau < 0$	$\tau \geq 0$	$\tau < 0$
Approximating distribution	regular	shifted	negative	neg.shifted

Table 1: Choice of the approximating distribution

3. Option valuation and the greeks

Recall that we consider baskets of contemporaneous futures contracts. The basket itself can then be considered as a futures contract (see e.g. Hull (2002)), and hence, options on it can be valued using the Black's (1976) formula. However, this is only possible if the distribution of the basket value is log-normal. So we shall reduce the problem of valuing options on general baskets (i.e. those having the shifted or negative shifted log-normal distribution) to valuing options on baskets having regular log-normal distribution.

Let the value of the basket 1 (B_1) be (regular) log-normally distributed with parameters m, s. Furthermore, let the basket 2 (B_2) have the following relationship with the basket 1:

$$B_2(T) = B_1(T) + \tau$$

where τ is a constant. The distribution of the basket 2 must be shifted log-normal with the parameters m, s, τ. On the maturity date T, the payoff of a call option on the basket 2 is

$$\left(B_2(T) - X\right)^+ = \left((B_1(T) + \tau) - X\right)^+ = \left(B_1(T) - (X - \tau)\right)^+.$$

This is the payoff of a call option on the basket 1 with the same maturity date T and the strike price $(X - \tau)$, and such a call option can be valued by the Black's formula.

Next, suppose that the basket 3 (B_3) has the following relationship to the basket 1:

$$B_3(T) = -B_1(T).$$

The distribution of the basket 3 must be negative log-normal with parameters m, s. On the maturity date T, the payoff of a call option on the basket 3 is

$$\left(B_3(T) - X\right)^+ = \left((-X) - B_1(T)\right)^+.$$

This is the payoff of a put option on the basket 1 on the maturity date T with the strike price $(-X)$, and to value such a put option, the Black's formula can be applied again.

These arguments lead to the following closed form formulae for the price of the basket call option with the strike price X and the time of maturity T:

- Using the regular log-normal approximation (Wakeman method)

$$c = \exp(-rT)[M_1(T)N(d_1) - XN(d_2)] \tag{13}$$

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[5]Note that in this case our method is equivalent to the Wakeman method (Turnbull and Wakeman (1991))

$$\text{where} \quad d_1 = \frac{\log(M_1(T)) - \log X + \frac{1}{2}V^2}{V}$$

$$d_2 = \frac{\log(M_1(T)) - \log X - \frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{M_2(T)}{(M_1(T))^2}\right)}$$

- Using the shifted log-normal approximation

$$c = \exp(-rT)[(M_1(T) - \tau)N(d_1) - (X - \tau)N(d_2)] \tag{14}$$

$$\text{where} \quad d_1 = \frac{\log(M_1(T) - \tau) - \log(X - \tau) + \frac{1}{2}V^2}{V}$$

$$d_2 = \frac{\log(M_1(T) - \tau) - \log(X - \tau) - \frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{M_2(T) - 2\tau M_1(T) + \tau^2}{(M_1(T) - \tau)^2}\right)}$$

- Using the negative log-normal approximation

$$c = \exp(-rT)[-XN(-d_2) + M_1(T)N(-d_1)] \tag{15}$$

$$\text{where} \quad d_1 = \frac{\log(-M_1(T)) - \log(-X) + \frac{1}{2}V^2}{V}$$

$$d_2 = \frac{\log(-M_1(T)) - \log(-X) - \frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{M_2(T)}{(M_1(T))^2}\right)}$$

- Using the negative shifted log-normal approximation

$$c = \exp(-rT)[(-X - \tau)N(-d_2) + (M_1(T) + \tau)N(-d_1)] \tag{16}$$

$$\text{where} \quad d_1 = \frac{\log(-M_1(T) - \tau) - \log(-X - \tau) + \frac{1}{2}V^2}{V}$$

$$d_2 = \frac{\log(-M_1(T) - \tau) - \log(-X - \tau) - \frac{1}{2}V^2}{V}$$

$$V = \sqrt{\log\left(\frac{M_2(T) + 2\tau M_1(T) + \tau^2}{(M_1(T) + \tau)^2}\right)}$$

Everywhere $M_1(T)$ and $M_2(T)$ denote the first two moments of the basket on the maturity date T (given in (6) and (7)) and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Note that, for the regular and negative log-normal distributions, a quantity analogous to the basket "volatility" can be defined as $\sigma_{B(T)} = \frac{V}{\sqrt{T}}$. For the shifted and negative shifted log-normal distributions,

such analogy does not hold, since in these cases the quantity $\frac{V}{\sqrt{T}}$ is the "volatility" of the basket value minus the shift τ. Generally, we believe that it does not make sense to define the "volatility" of a general basket or a spread. In one-factor approaches, the dynamics of a basket is explicitly modelled (by e.g. a continuous-time diffusion driven by a Brownian motion), and hence the volatility is specified. Our approach is multi-factor, so we do not assume any dynamics directly for the basket value, but only for the underlying assets; hence, the basket "volatility" is not defined in this way. To define the volatility as the standard deviation of the log-returns or relative returns on a basket is also impossible, because for general baskets the log-returns cannot be defined at all (due to negative values), and the relative returns may explode, since the basket value can be arbitrary close to zero. However, some authors have attempted to define such a quantity; for example, Kirk (1995) defines the spread volatility as

$$\sigma_{spread} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho}, \tag{17}$$

where $\sigma_{1,2}$ are the volatilities of the assets in the spread and ρ is the correlation coefficient. Note that this is not the volatility in traditional sense, but the standard deviation of the difference between the assets' log-returns, which is not the spread return. In fact, using the expression (17) to price options can lead to underpricing (as we shall see in the next section), because the standard deviation of the spread relative returns can be much higher, due to spread values close to 0. So we avoid the use of the term "basket volatility" or "spread volatility" and derive the basket option price in terms of the basket moments.

To manage risks during the lifetime of an option, traders monitor changes in the option price caused by changes in the underlying values or the model parameters, such as volatilities. These changes are quantified by the option's *greeks* (which are the partial derivatives of the option price with respect to the parameters), which play a crucial role in hedging the option and managing risks associated with options portfolio. The advantage of having a closed formula for the option price is that it also leads to the closed formulae for the greeks, which can be quickly and accurately evaluated.

We can differentiate equations (13)-(16) with respect to the underlying asset prices, time to maturity, volatilities, correlations and the interest rate, to obtain closed formulae for option's deltas (Δ), theta (Θ), vegas (ϑ) and rho (ρ). There is a subtlety, however, since the formulae for the option price in the case of the shifted and negative shifted log-normal distribution contain the parameter τ, which also implicitly depends on F_i, σ_i, ρ and T. So when we differentiate formulae (13)-(16), we must take this dependence into account.

Although there is no closed formula for τ in terms of the model parameters, it is possible to obtain such closed formulae for the partial derivatives of τ with respect to F_i, σ_i, ρ, T. Recall that m, s and τ were obtained by the moment matching procedure, which amounts to solving a nonlinear system of equations, such as (10)-(12) for the shifted log-normal distribution. A similar system can be written down for the partial derivatives of the first three basket moments M_1, M_2, M_3 with respect to the model parameters, and then solved for the partial derivatives of the parameters m, s and τ. For example, to obtain the option's deltas, we differentiate the call price and the first three moments with respect to the asset prices F_i. The derivative of the call price with respect to F_i contains $\partial\tau/\partial F_i$, which can be found by solving the following (linear) equation system:

$$\begin{bmatrix} 1 & \alpha & s \\ 2(\tau + \alpha) & 2\tau\alpha + \beta & 2s(\tau\alpha + 2\beta) \\ 3(\tau^2 + 2\tau\alpha + \beta) & 3(\tau^2\alpha + 2\tau\beta + \gamma) & 3s(\tau^2\alpha + 4\tau\beta + 3\gamma) \end{bmatrix} \times \begin{bmatrix} \partial\tau/\partial F_i \\ \partial m/\partial F_i \\ \partial s/\partial F_i \end{bmatrix} = \begin{bmatrix} \partial M_1/\partial F_i \\ \partial M_2/\partial F_i \\ \partial M_3/\partial F_i \end{bmatrix},$$

where $\alpha = \exp(m + 1/2s^2)$, $\beta = \exp(2m + 2s^2)$ and $\gamma = \exp(3m + 9/2s^2)$. In the negative shifted log-normal case, the derivative $\partial M_1/\partial F_i$ changes to $-\partial M_1/\partial F_i$ and $\partial M_3/\partial F_i$ to $-\partial M_3/\partial F_i$. The partial derivatives $\partial\tau/\partial\sigma_i$, $\partial\tau/\partial\rho$ and $\partial\tau/\partial T$ are obtained analogously.

Due to these rather involved computations, the final closed formulae for the basket option's greeks are somewhat cumbersome (although easily implementable), and hence are reported in the Appendix.

In the next section we shall evaluate the model's performance by comparing option prices obtained by our method to those obtained by other methods (whenever possible) and Monte Carlo simulations. We shall also investigate the performance of the delta hedging and analyze the behavior of the greeks, especially those related to the volatilities and correlations, i.e. vegas.

To conclude this section, we summarize the algorithm for pricing general basket options.

1. Given the basket parameters (initial asset prices, volatilities, correlations and weights), as well as the time to the option's maturity and the interest rate, compute the first three moments of the terminal basket value according to equations (6), (7) and (8), and the basket skewness according to the formula (9).

2. If the basket skewness η is positive, the first guess of the approximating distribution is shifted log-normal; if the skewness is negative, it is the negative shifted log-normal distribution.

3. Match the moments of the distribution chosen in the step 2 to the basket's moments, to find the parameters m, s and τ.

4. Adjust the choice of the approximating distribution on the basis of the skewness η and the shift parameter τ according to the Table 1.

5. Compute the European call basket option price according to the appropriate formula (13)-(16), and the option's greeks according to the corresponding formulae given in Appendix.

4. Simulation study

We apply our approach to a number of hypothetical baskets, chosen so that all possible approximating distributions occur. The parameters of the test baskets are given in the Table 2 and the European call option prices on these baskets are given in the Table 3. The sign "-" means that the corresponding method cannot be applied to a particular basket.

	Basket 1	Basket 2	Basket 3	Basket 4	Basket 5	Basket 6
Futures price F_0	[100,120]	[150,100]	[110,90]	[200,50]	[95,90,105]	[100,90,95]
Volatility σ	[0.2,0.3]	[0.3,02]	[0.3,0.2]	[0.1;0.15]	[0.2,0.3,0.25]	[0.25,0.3,0.2]
Weights a	[-1,1]	[-1,1]	[0.7,0.3]	[-1,1]	[1,-0.8,-0.5]	[0.6,0.8,-1]
Correlation ρ	$\rho_{1,2}=0.9$	$\rho_{1,2}=0.3$	$\rho_{1,2}=0.9$	$\rho_{1,2}=0.8$	$\rho_{1,2}=\rho_{2,3}=0.9$ $\rho_{1,3}=0.8$	$\rho_{1,2}=\rho_{2,3}=0.9$ $\rho_{1,3}=0.8$
Strike price X	20	-50	104	-140	-30	35
Skewness η	$\eta>0$	$\eta<0$	$\eta>0$	$\eta<0$	$\eta<0$	$\eta>0$
Location parameter τ	$\tau<0$	$\tau<0$	$\tau>0$	$\tau>0$	$\tau<0$	$\tau<0$

Table 2: Basket parameters

Baskets 1,2 and 4 are spreads. Basket 3 is a "regular" basket, consisting of two assets with positive weights, so for this basket our approach reduces to the Wakeman method. Baskets 5 and 6 consist of three assets, some with negative portfolio weights. For these general baskets there is no closed form approach, apart from the one presented here, so we can only compare the corresponding option prices to the Monte Carlo simulations.

We chose high correlations between the assets to imitate realistic commodity baskets, where the assets in the basket are closely related. The assets' volatilities are also quite high (20-30%), again to reflect volatility levels typical for commodities. The options on baskets 1, 2, 5 and 6 are (nearly) at-the-money and 3 and 4 are out-of-the-money. The approximating distributions are chosen on the basis of the basket skewness η and

Method	Basket 1	Basket 2	Basket 3	Basket 4	Basket 5	Basket 6
Our approach	7.751 (shifted)	16.910 (neg.shifted)	10.844 (regular)	1.958 (neg.regular)	7.759 (neg.shifted)	9.026 (shifted)
Bachelier	8.052	17.237	-	2.121	-	-
Kirk	7.734	16.678	-	1.507	-	-
Monte Carlo	7.744 (0.014)	16.757 (0.023)	10.821 (0.018)	1.966 (0.005)	7.730 (0.01)	9.022 (0.015)

Table 3: Call option prices

the estimated shift τ, and are reported in the Table 3.

The Monte Carlo simulation is repeated 1000 times for each basket, to obtain the means and standard errors of call prices (which are given in parenthesis in the last row of the Table 3).

Call prices obtained by our approach are remarkably close to those obtained by the Monte Carlo simulations, and in all cases but one (Basket 2) are within the 95% Monte Carlo confidence bounds. This is not an inherent feature of the negative shifted log-normal approximation: for the Basket 5 the same approximating distribution was chosen and the call price is close to the Monte Carlo result. Note that our call prices in all cases are greater than those obtained by the Kirk method, and lower than the prices resulting from the Bachelier method. Recall that the Kirk method uses the spread volatility given by (17) to price options, and hence, leads to too low option prices, as discussed in the previous section. The Bachelier method performs poorly and significantly overestimates the call prices. The Kirk method performs better, but is inapplicable to Baskets 5 and 6, whereas our method provides a very close approximation to the call price. Moreover, for Basket 4, our price is much closer to the Monte Carlo result than that obtained by the Kirk method. In this case (skewness of basket is negative, and the estimate of τ is positive) the Kirk method significantly underestimates the call price.

Next, we investigate the performance of our method on the basis of delta-hedging the option. We generate price paths of the basket assets from the time of writing the option until maturity, and on each hypothetical day we calculate the option's deltas with respect to each asset. We then re-adjust daily the hedging portfolio according to the deltas. We define the hedge error as the difference between the option price and the discounted hedge cost (i.e. the cost of maintaining the delta-hedged portfolio). If the hedging scheme works perfectly, the hedge cost would be exactly equal to the theoretical option price and the hedge error would be zero. In practice it is not zero due to the model error and discrete (e.g. daily) hedging. We expect the hedge error and its standard deviation to decrease when the hedge interval decreases, i.e. when hedging is done more frequently.

We investigate the delta-hedging performance of our approach on the example of two baskets, one a spread with the parameters $F_0 = [100, 110]$, $\sigma = [0.1, 0.15]$, $a = [-1, 1]$, $\rho = 0.9$, $X = 10$, and the other one a general basket with parameters $F_0 = [95, 90, 105]$, $\sigma = [0.2, 0.3, 0.25]$, $a = [1, -0.8, -0.5]$, $\rho_{1,2} = \rho_{2,3} = 0.9$, $\rho_{1,3} = 0.8$, $X = -30$. For both examples the interest rate is 3% per annum and the time to maturity is one year. For each basket 1000 price paths were generated and the hedge errors computed for each price path.

In Figures 12 and 13 we plot the ratio of the hedge error standard deviation to the call price vs. the hedge interval. Both figures show that this ratio (and so, the standard deviation of the hedge error) decreases together with the hedge interval, as we expect. This is also the case for the mean hedge error. For both examples, the mean hedge error is around 5% for daily hedging.

For multi-asset derivatives, the analysis of vegas with respect to the individual volatilities and particularly the inter-asset correlations becomes essential. A typical behavior of the correlation vega is shown in
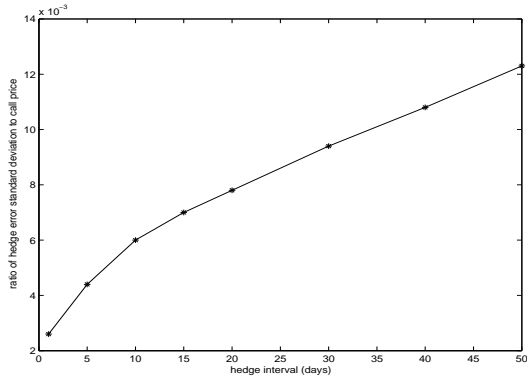
Figure 12: Hedge error vs. hedge interval. The basket parameters are F_0=[100,110], σ=[0.1,0.15], a=[-1,1], ρ=0.9, X=10.
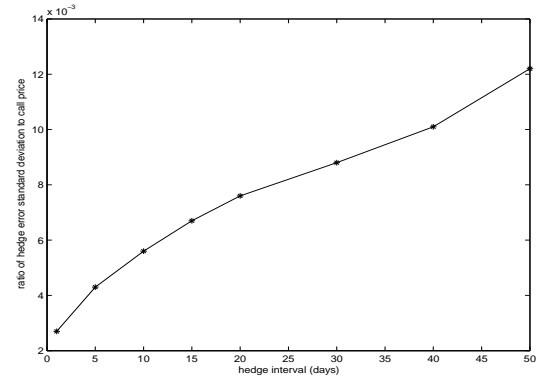
Figure 13: Hedge error vs. hedge interval. The basket parameters are F_0=[95,90,105], σ=[0.2,0.3,0.25], a=[1,-0.8,-0.5], $\rho_{1,2}$ = $\rho_{2,3}$=0.9, $\rho_{1,3}$=0.8, X=-30.

Figures 14-16. The underlying value is a spread with the parameters $F_0 = [100, 110]$, $a = [-1, 1]$, $\sigma = [0.1, 0.15]$, $r = 0.03$, $T = 1$. The terminal basket distribution is approximated by the shifted log-normal. In this example, the correlation vega is negative, i.e. increasing correlation leads to decreasing option price.
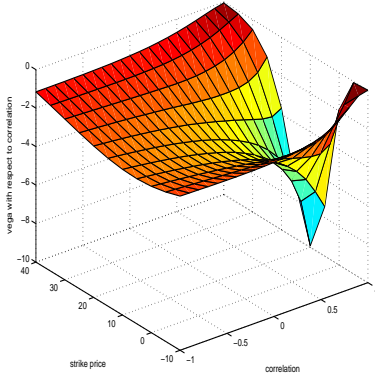

Figure 14: Correlation vega vs. correlations and strike prices.
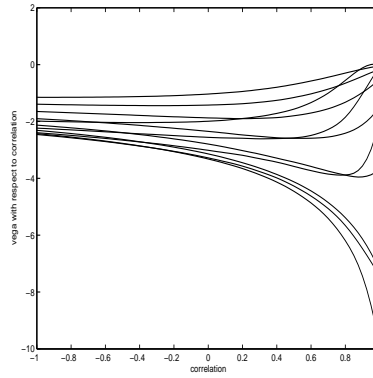
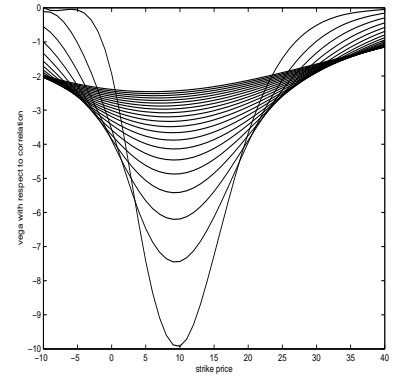Figure 15: Correlation vega vs. correlations for different strike prices

Figure 16: Correlation vega vs. strike prices for different correlations

A characteristic feature of the Black-Scholes model is that the vega (i.e. the sensitivity of the option price with respect to the underlying asset's volatility) is always positive, i.e. the option price can only increase if the volatility increases. This is not the case for basket and spread options. All vegas (i.e. with respect to volatilities and correlations) can be positive as well as negative. The first two plots in Figure 17 show the volatility vegas for the spread option characterized by $F_0 = [100, 110]$, $a = [-1, 1]$, $\rho = 0.8$, $r = 0.03$, $T = 1$, $X = 10$, versus various volatility levels. Note that there are regions of volatilities where one or both volatility vegas are negative. These regimes are also visible in the right plot of Figure 17 (where we plot the spread call price vs. the volatilities σ_1 and σ_2) and in Figure 18, showing slices of the surface plots. ¿From these plots it is clear that the call price does not necessarily increase with increasing individual volatilities. This does not contradict the Black-Scholes theory: in such regions, increase in one of the volatilities leads to a lower "variability" of the spread, which ultimately drives down the spread option price.
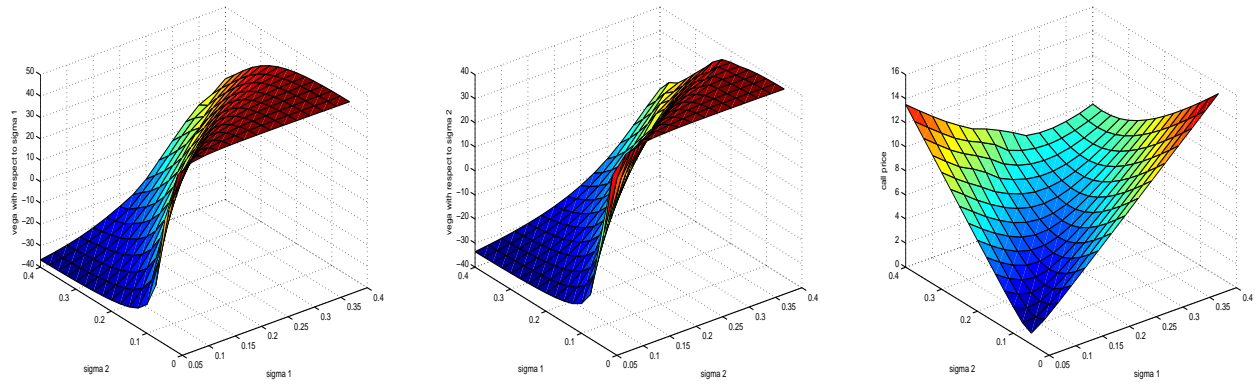
Figure 17: Volatility vegas for different volatilities σ_1 and σ_2, and call price vs. σ_1 and σ_2.
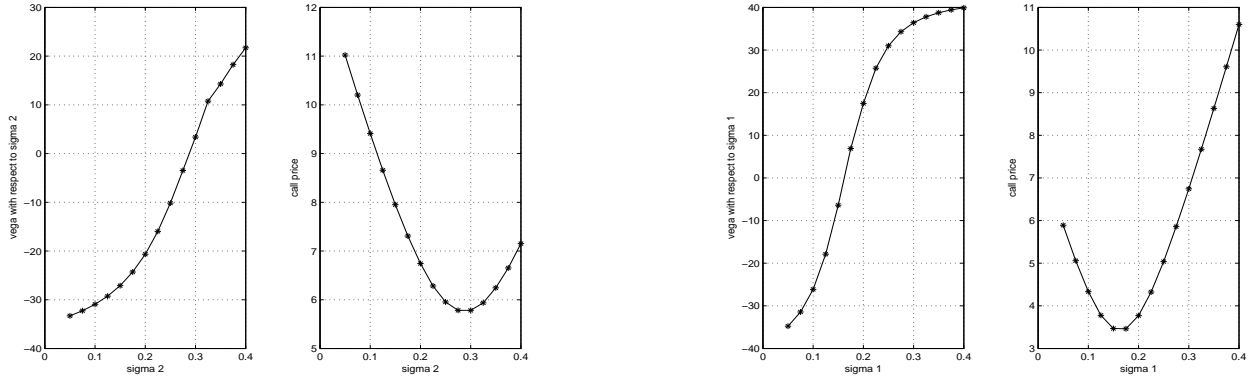

Figure 18: Volatility vega and call price for $\sigma_1 = 0.3$ (left) and $\sigma_2 = 0.2$ (right).

5. Conclusions and future work

We have introduced a new approach for pricing and hedging general basket and spread options. Our approach is based on approximating the basket distribution by a "family" of log-normal distributions: regular, shifted, negative or negative shifted log-normal. Such a log-normal approximation means that the widely used tools of option pricing such as the Black-Scholes formula are readily applicable. Our method leads to a closed-form solution for the European option price and option's greeks; it can be easily understood and implemented by market practitioners. Moreover, using our closed formula for the option price, inter-commodity correlations can be implied from the market prices of spread and basket options. To our knowledge, ours is the only existing analytic approach to general basket options - typical derivatives in many commodity markets. Numerical simulations have shown that the option prices obtained by our method approximate the prices resulting from Monte Carlo simulations remarkably well, and the delta-hedging performance of our method is also very good.

Here we considered baskets of futures or forward contracts. Our approach can be easily extended to baskets of physical commodities, by specifying the dynamics of spot commodity prices under the risk-adjusted probability measure. This, however, requires specification of the parameters such as the convenience yield and the market price of commodity risk, which have to be calibrated from market data. The extension of our method to physical commodity baskets will be reported shortly.

An important feature of energy markets is that most delivery contracts are priced on the basis of an average price over a certain period. Hence, most energy derivatives (also basket and spread options) are Asian-style. So Asian basket options (that is, an Asian option on a basket of assets) need to be considered

as well. Extension of our approach for valuation and hedging of Asian basket option is a topic of future research.

Acknowledgments

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Appendix

The closed form expressions for the greeks are given by:
Using the regular log-normal approximation (Wakeman method)

$$\Delta_i = \frac{\partial c}{\partial F_i} = \exp\left(-rT\right)\left[\frac{\partial M_1}{\partial F_i}N(d_1) + Xn(d_2)\frac{\partial V}{\partial F_i}\right]$$

$$\Theta = \frac{\partial c}{\partial T} = -rc + \exp\left(-rT\right)Xn(d_2)\frac{\partial V}{\partial T}$$

$$\vartheta_{i,i} = \frac{\partial c}{\partial \sigma_i} = \exp\left(-rT\right)Xn(d_2)\frac{\partial V}{\partial \sigma_i}$$

$$\vartheta_{i,j} = \frac{\partial c}{\partial \rho_{i,j}} = \exp\left(-rT\right)Xn(d_2)\frac{\partial V}{\partial \rho_{i,j}}, \ \ i \neq j$$

$$\rho = \frac{\partial c}{\partial r} = -Tc$$

$$\text{where} \quad \frac{\partial V}{\partial F_i} = \frac{1}{2M_1 M_2 V}\left[M_1\frac{\partial M_2}{\partial F_i} - 2M_2\frac{\partial M_1}{\partial F_i}\right]$$

$$\frac{\partial V}{\partial \sigma_i} = \frac{1}{2M_2 V}\cdot\frac{\partial M_2}{\partial \sigma_i}$$

$$\frac{\partial V}{\partial \rho_{i,j}} = \frac{1}{2M_2 V}\cdot\frac{\partial M_2}{\partial \rho_{i,j}}, \ \ i \neq j$$

$$\frac{\partial V}{\partial T} = \frac{1}{2M_2 V}\cdot\frac{\partial M_2}{\partial T}$$

Using the shifted log-normal approximation

$$\Delta_i = \frac{\partial c}{\partial F_i} = \exp\left(-rT\right)\left[\frac{\partial M_1}{\partial F_i}N(d_1) + (X-\tau)n(d_2)\frac{\partial V}{\partial F_i} + \left(N(d_2) - N(d_1)\right)\frac{\partial \tau}{\partial F_i}\right]$$

$$\Theta = \frac{\partial c}{\partial T} = -rc + \exp\left(-rT\right)\left[(X-\tau)n(d_2)\frac{\partial V}{\partial T} + \left(N(d_2) - N(d_1)\right)\frac{\partial \tau}{\partial T}\right]$$

$$\vartheta_{i,i} = \frac{\partial c}{\partial \sigma_i} = \exp\left(-rT\right)\left[(X-\tau)n(d_2)\frac{\partial V}{\partial \sigma_i} + \left(N(d_2) - N(d_1)\right)\frac{\partial \tau}{\partial \sigma_i}\right]$$

$$\vartheta_{i,j} = \frac{\partial c}{\partial \rho_{i,j}} = \exp\left(-rT\right)\left[Xn(d_2)\frac{\partial V}{\partial \rho_{i,j}} + \left(N(d_2) - N(d_1)\right)\frac{\partial \tau}{\partial \rho_{i,j}}\right], \ \ i \neq j$$

$$\rho = \frac{\partial c}{\partial r} = -Tc$$

$$\text{where} \quad \frac{\partial V}{\partial F_i} = \frac{1}{2(M_1 - \tau)(M_2 - 2\tau M_1 + \tau^2)V}\left[(M_1 - \tau)\frac{\partial M_2}{\partial F_i} + 2(\tau M_1 - M_2)\frac{\partial M_1}{\partial F_i}\right.$$
$$\left. + 2(M_2 - M_1^2)\frac{\partial \tau}{\partial F_i}\right]$$

$$\frac{\partial V}{\partial \sigma_i} = \frac{1}{2(M_1 - \tau)(M_2 - 2\tau M_1 + \tau^2)V}\left[(M_1 - \tau)\frac{\partial M_2}{\partial \sigma_i} + 2(M_2 - M_1^2)\frac{\partial \tau}{\partial \sigma_i}\right]$$

$$\frac{\partial V}{\partial \rho_{i,j}} = \frac{1}{2(M_1 - \tau)(M_2 - 2\tau M_1 + \tau^2)V}\left[(M_1 - \tau)\frac{\partial M_2}{\partial \rho_{i,j}} + 2(M_2 - M_1^2)\frac{\partial \tau}{\partial \rho_{i,j}}\right], \ \ i \neq j$$

$$\frac{\partial V}{\partial T} = \frac{1}{2(M_1 - \tau)(M_2 - 2\tau M_1 + \tau^2)V}\left[(M_1 - \tau)\frac{\partial M_2}{\partial T} + 2(M_2 - M_1^2)\frac{\partial \tau}{\partial T}\right]$$

Using the negative regular log-normal approximation

$$\Delta_i = \frac{\partial c}{\partial F_i} = \exp(-rT)\left[\frac{\partial M_1}{\partial F_i}N(-d_1) - Xn(-d_2)\frac{\partial V}{\partial F_i}\right]$$

$$\Theta = \frac{\partial c}{\partial T} = -rc - \exp(-rT)Xn(-d_2)\frac{\partial V}{\partial T}$$

$$\vartheta_{i,i} = \frac{\partial c}{\partial \sigma_i} = -\exp(-rT)Xn(d_2)\frac{\partial V}{\partial \sigma_i}$$

$$\vartheta_{i,j} = \frac{\partial c}{\partial \rho_{i,j}} = -\exp(-rT)Xn(-d_2)\frac{\partial V}{\partial \rho_{i,j}}$$

$$\rho = \frac{\partial c}{\partial r} = -Tc$$

$$\text{where} \quad \frac{\partial V}{\partial F_i} = \frac{1}{2M_1 M_2 V}\left[M_1\frac{\partial M_2}{\partial F_i} - 2M_2\frac{\partial M_1}{\partial F_i}\right]$$

$$\frac{\partial V}{\partial \sigma_i} = \frac{1}{2M_2 V}\frac{\partial M_2}{\partial \sigma_i}$$

$$\frac{\partial V}{\partial \rho_{i,j}} = \frac{1}{2M_2 V}\frac{\partial M_2}{\partial \rho_{i,j}}$$

$$\frac{\partial V}{\partial T} = \frac{1}{2M_2 V}\frac{\partial M_2}{\partial T}$$

Using the negative shifted log-normal approximation

$$\Delta_i = \frac{\partial c}{\partial F_i} = \exp(-rT)\left[\frac{\partial M_1}{\partial F_i}N(-d_1) + (-X-\tau)n(-d_2)\frac{\partial V}{\partial F_i} + \big(N(-d_1) - N(-d_2)\big)\frac{\partial \tau}{\partial F_i}\right]$$

$$\Theta = \frac{\partial c}{\partial T} = -rc + \exp(-rT)\left[(-X-\tau)n(-d_2)\frac{\partial V}{\partial T} + \big(N(-d_1) - N(-d_2)\big)\frac{\partial \tau}{\partial T}\right]$$

$$\vartheta_{i,i} = \frac{\partial c}{\partial \sigma_i} = \exp(-rT)\left[(-X-\tau)n(-d_2)\frac{\partial V}{\partial \sigma_i} + \big(N(-d_1) - N(-d_2)\big)\frac{\partial \tau}{\partial \sigma_i}\right]$$

$$\vartheta_{i,j} = \frac{\partial c}{\partial \rho_{i,j}} = \exp(-rT)\left[(-X-\tau)n(-d_2)\frac{\partial V}{\partial \rho_{i,j}} + \big(N(-d_1) - N(-d_2)\big)\frac{\partial \tau}{\partial \rho_{i,j}}\right], \ i \neq j$$

$$\rho = \frac{\partial c}{\partial r} = -Tc$$

$$\text{where} \quad \frac{\partial V}{\partial F_i} = \frac{1}{2(M_1+\tau)(M_2+2\tau M_1+\tau^2)V}\left[(M_1+\tau)\frac{\partial M_2}{\partial F_i} - 2(\tau M_1+M_2)\frac{\partial M_1}{\partial F_i}\right.$$
$$\left. - 2(M_2 - M_1^2)\frac{\partial \tau}{\partial F_i}\right]$$

$$\frac{\partial V}{\partial \sigma_i} = \frac{1}{2(M_1+\tau)(M_2+2\tau M_1+\tau^2)V}\left[(M_1+\tau)\frac{\partial M_2}{\partial \sigma_i} - 2(M_2 - M_1^2)\frac{\partial \tau}{\partial \sigma_i}\right]$$

$$\frac{\partial V}{\partial \rho_{i,j}} = \frac{1}{2(M_1+\tau)(M_2+2\tau M_1+\tau^2)V}\left[(M_1+\tau)\frac{\partial M_2}{\partial \rho_{i,j}} - 2(M_2 - M_1^2)\frac{\partial \tau}{\partial \rho_{i,j}}\right], \ i \neq j$$

$$\frac{\partial V}{\partial T} = \frac{1}{2(M_1+\tau)(M_2+2\tau M_1+\tau^2)V}\left[(M_1+\tau)(M_1+\tau)\frac{\partial M_2}{\partial T} - 2(M_2 - M_1^2)\frac{\partial \tau}{\partial T}\right]$$

Note that:

$$\frac{\partial M_1}{\partial \sigma_i} = 0, \quad \frac{\partial M_1}{\partial \rho_{i,j}} = 0, \quad \frac{\partial M_1}{\partial T} = 0, \quad \frac{\partial M_1}{\partial F_i} = a_i$$

$$\frac{\partial M_2}{\partial F_i} = 2a_i \sum_{j=1}^{N} a_j F_j(0) \exp\left(\rho_{i,j}\sigma_i\sigma_j T\right)$$

$$\frac{\partial M_2}{\partial \sigma_i} = 2a_i F_i(0) \sum_{j=1}^{N} a_j F_j(0) \rho_{i,j}\sigma_j T \exp\left(\rho_{i,j}\sigma_i\sigma_j T\right)$$

$$\frac{\partial M_2}{\partial \rho_{i,j}} = 2a_i a_j F_i(0)F_j(0)\sigma_i\sigma_j T \exp\left(\rho_{i,j}\sigma_i\sigma_j T\right), i \neq j$$

$$\frac{\partial M_2}{\partial T} = \sum_{j=1}^{N}\sum_{i=1}^{N} a_i a_j F_i(0)F_j(0)\rho_{i,j}\sigma_i\sigma_j \exp\left(\rho_{i,j}\sigma_i\sigma_j T\right)$$

$$\frac{\partial M_3}{\partial F_i} = 3a_i \sum_{j=1}^{N} a_j a_k F_j(0)F_k(0) \exp\left(\rho_{i,j}\sigma_i\sigma_j T + \rho_{i,k}\sigma_i\sigma_k T + \rho_{j,k}\sigma_j\sigma_k T\right)$$

$$\frac{\partial M_3}{\partial \sigma_i} = 3a_i F_i(0) \sum_{k=1}^{N}\sum_{j=1}^{N} a_j a_k F_j(0)F_k(0)\left(\rho_{i,j}\sigma_j T + \rho_{i,k}\sigma_k T\right) \exp\left(\rho_{i,j}\sigma_i\sigma_j T + \rho_{i,k}\sigma_i\sigma_k T + \rho_{j,k}\sigma_j\sigma_k T\right)$$

$$\frac{\partial M_3}{\partial \rho_{i,j}} = 6a_i a_j F_i(0)F_j(0)T\sigma_i\sigma_j \sum_{k=1}^{N} a_k F_k(0) \exp\left(\rho_{i,j}\sigma_i\sigma_j T + \rho_{i,k}\sigma_i\sigma_k T + \rho_{j,k}\sigma_j\sigma_k T\right), i \neq j$$

$$\frac{\partial M_3}{\partial T} = \sum_{k=1}^{N}\sum_{j=1}^{N}\sum_{i=1}^{N}(\rho_{i,j}\sigma_i\sigma_j + \rho_{i,k}\sigma_i\sigma_k + \rho_{j,k}\sigma_j\sigma_k)a_i a_j a_k F_i(0)F_j(0)F_k(0)$$
$$\exp\left(\rho_{i,j}\sigma_i\sigma_j T + \rho_{i,k}\sigma_i\sigma_k T + \rho_{j,k}\sigma_j\sigma_k T\right)$$

Everywhere $N(\cdot)$ denotes the cumulative distribution function and $n(\cdot)$ the density of the standard normal distribution.