Analyzing the Effect of Redundant Parity-Checks Using EXIT Charts

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Abstract: *Short and medium length codes with redundant parity-check equations are known to exhibit a superior performance with iterative decoding. We give a tentative explanation of this effect using extrinsic-information transfer (EXIT) charts and quantify the resulting performance improvement. Our simulations show that the performance improvement predicted by EXIT charts is slightly larger than the actual improvement and we explain why this is the case.*

Keywords: LDPC, convergence, EXIT charts, redundant parity-checks

1. INTRODUCTION

The performance of low-density parity-check (LDPC) codes is mainly influenced by the girth of the corresponding factor graph and the minimum distance of the code. One way of constructing LDPC codes with large girth and large minimum distance is based on partial geometries [3]. Simulations showed that the performance of these codes is superior to randomly constructed codes, especially for short and medium block lengths. Codes derived from partial geometries have a high number of redundant parity-checks. The good performance of these codes can be attributed to the redundant checks.

The aim of this paper is to explain the influence of redundant parity-checks on the convergence of the decoder by using extrinsic information transfer (EXIT) charts. We will use EXIT charts to quantify the improvement in block error probability due to redundant parity-checks.

The rest of this paper is organized as follows: in section 2, the influence of redundant parity-checks on EXIT charts is described. Section 3 compares codes with and without redundant parity-checks. In section 4, the convergence behavior of short length LDPC codes is investigated and differences to the asymptotic case (for which EXIT charts can make exact predictions) are explained.

2. REDUNDANT CHECKS & EXIT CHARTS

Assume a regular LDPC code with variable node degree d_v and check node degree d_c. If a check node with degree d_c that is redundant (i.e. it puts no additional constraint on the codeword), is added to the factor graph, the average variable node degree is increased while the check node degree remains unchanged. Adding (or removing) a redundant check does not change the code and therefore, the minimum distance remains unchanged as well.

The degrees of variable and check nodes determine the curves in the EXIT chart. For the case of the BEC, the curves of the EXIT chart are given by the following expression [1]:

$$I_E^C(I_A^C) = (I_A^C)^{d_c-1}, \tag{1}$$

$$I_E^V(I_A^V) = 1 - q \cdot (1 - I_A^V)^{d_V-1}, \tag{2}$$

where q is the erasure probability of the BEC. By adding a redundant check node, the curve for the check node decoder is not changed. Due to the higher average degree d_v of the variable nodes, the curve for the variable node decoder gets closer to the top left corner of the EXIT chart and therefore, the distance between the curves is increased. This effect allows the decoder to converge faster and the threshold where the curves intersect is decreased.

3. COMPARISON OF CODES

It is not known how to construct random codes with distinct redundant check equations of degree d_c. To analyze the influence of the redundant parity-checks, we took the inverse approach. We used LDPC codes that contain a high number of linearly dependent rows by design, and we removed the redundant parity-checks. These codes are constructed from partial geometries [3].

For the simulations we assume transmission over an AWGN channel with binary input. We compared two different representations of the same code:

- *Parity-check matrix with redundant rows*
  The matrix $\boldsymbol{H}_{red.}$ is obtained from a design based on partial geometries. The dimension of

Table 1: Matrices used for the Comparison.

	$\boldsymbol{H}_{red.}$	$\boldsymbol{H}_{full}$
n	221	221
m	221	82
rank	82	82
d_v	13	$2 \ldots 12$
d_c	13	13
girth	6	6

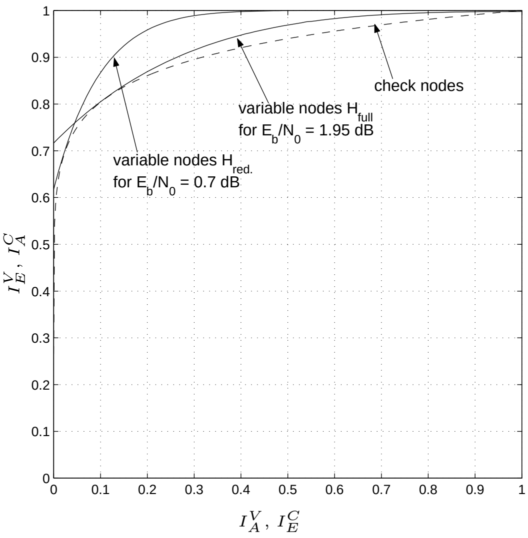

Figure 1: EXIT chart for $\boldsymbol{H}_{red.}$ and $\boldsymbol{H}_{full}$.

the matrix is 221×221 (which corresponds to a *design rate* $R_d = 0$) and the rank is 82 (which corresponds to a true rate $R = 0.629$). This code is regular with $d_v = 13$ and $d_c = 13$.

- *Parity-check matrix with full rank*
  This matrix $\boldsymbol{H}_{full}$ is obtained by removing all redundant rows of the matrix $\boldsymbol{H}_{red.}$. Therefore, the dimension of the matrix is 82×221 ($R_d = R = 0.629$) and it represents the same code. However, while the check node degree remains unchanged, the variable node degrees are irregular between 2 and 12.

By design, the girth of $\boldsymbol{H}_{red.}$ is 6 [2]. Removing rows from the parity-check matrix can not decrease the girth. Therefore, the girth of $\boldsymbol{H}_{full}$ is at least 6. The parameters and properties of the two parity-check matrices are shown in table 1.

Figure 1 shows the EXIT chart for these two code representations. The variable node curve of $\boldsymbol{H}_{full}$ intersects the check node curve if E_b/N_0 is less than 1.95dB. For the matrix $\boldsymbol{H}_{red.}$ this threshold is at 0.7dB. Adding redundant parity-checks should improve the convergence behavior of the sum-product algorithm, since it decreases the threshold by 1.25dB.

Figure 2 shows the block error rate of the two

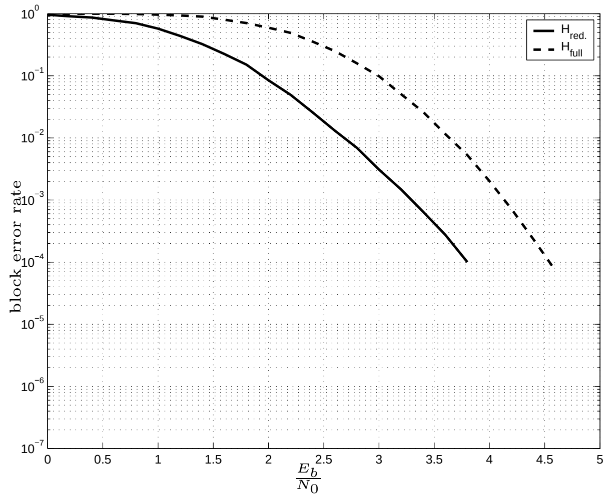

Figure 2: Block error rate for $\boldsymbol{H}_{red.}$ and $\boldsymbol{H}_{full}$.

different representations of the code. At a block error rate of 10^{-4}, the difference in required E_b/N_0 is approximately 0.75dB which is lower than the result predicted by EXIT chart analysis.

To explain this difference, we have to take a close look at the convergence behavior of LDPC codes with short block lengths.

4. CONVERGENCE FOR SHORT BLOCK LENGTHS

EXIT chart analysis is only valid for the asymptotic case of infinite block length. If we assume transmission over an AWGN channel, EXIT chart analysis shows that the decoder will converge to a codeword if E_b/N_0 is above a threshold and that the decoder will converge to a vector that is not a codeword if E_b/N_0 is below this threshold. However, this analysis does not describe the convergence for short block lengths. For the case of short block lengths we present simulation results of the convergence behavior.

For short block lengths, the decoding trajectories differ from the EXIT chart analysis [4]. At low E_b/N_0, the decoder mostly converges to a vector that is not a valid codeword. At high E_b/N_0, the decoder converges to a codeword, but the codeword can be different from the transmitted one. The probability of this type of error is small if the minimum distance of the code is sufficiently large.

Between these regions—in the waterfall region—we observed an additional type of error where the decoder does not converge but starts to oscillate. A simple way to distinguish these three types of error is to observe the evolution of the averaged magnitude of the log-likelihood values (LLR), which represent the reliability of the decision. Figure 3 shows the averaged magnitudes of the LLR values for the error types that occur at short block lengths.

The influence of these error types on the block error rate is shown in figure 4, where the total block error rate is separated into the three different types of error.

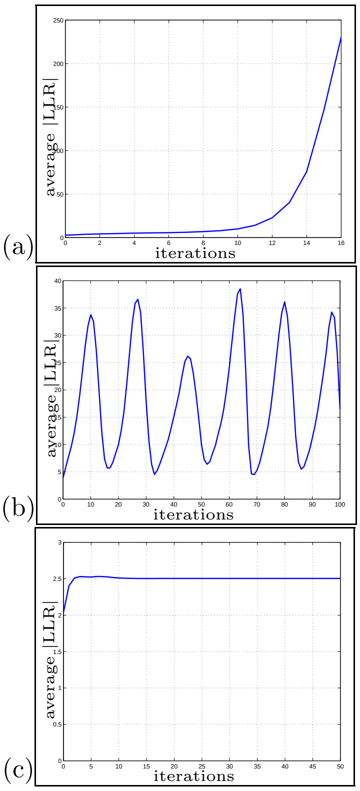

Figure 3: Averaged |LLR| for (a) convergence to a wrong codeword or successful decoding, (b) oscillations, (c) convergence to a vector that is not a codeword.

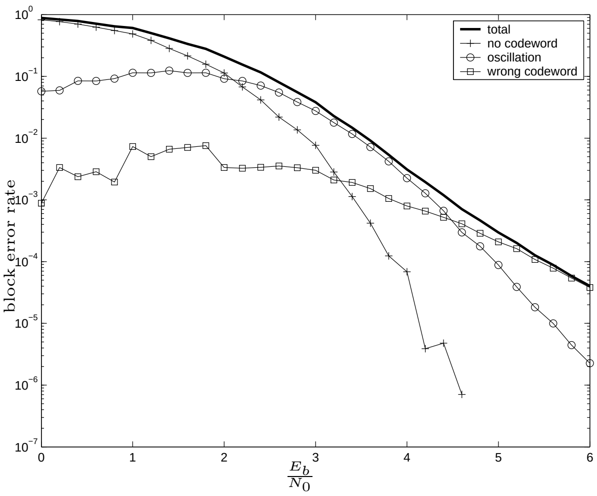

Figure 4: Block error rate for a regular LDPC code of rate 0.5 with block length 100 and $d_v = 3$.

The only type of error that occurs both in the asymptotic case and in the case of short block lengths is the convergence to a vector that is not a valid codeword. This type of error corresponds to an intersection of the variable node and check node curve in the EXIT chart.

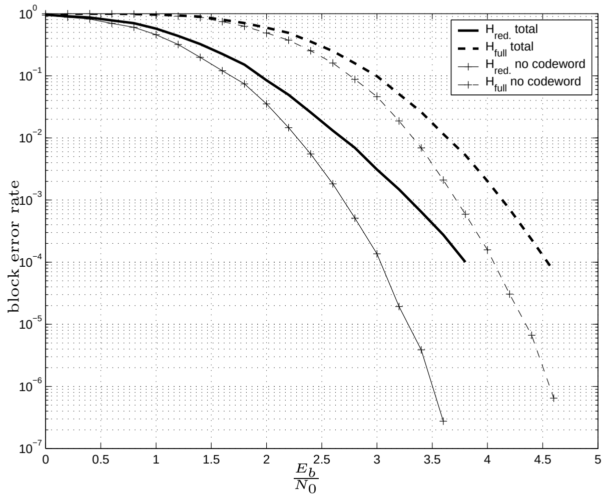

Figure 5: Block error rate for $\boldsymbol{H}_{red.}$ and $\boldsymbol{H}_{full}$.

Figure 5 shows the influence of this type of error on the block error rate for both representations of the code used in the previous section. The difference between the curves that describe the convergence to a vector that is not a codeword is approximately 1.1dB which corresponds approximately to the prediction by EXIT chart analysis (the threshold was decreased by 1.25dB).

The influence of redundant parity-checks on the other types of error is low. Therefore, for short block lengths the overall gain due to redundant parity-checks is less than predicted with EXIT charts.

5. CONCLUSION

By using EXIT charts, we analyzed the influence of redundant rows in the parity-check matrix for infinite block lengths. Adding redundant rows results in a decreased threshold. However, simulations of short LDPC codes showed that the gain due to redundant rows is less than predicted with EXIT charts.

To explain this deviation, we analyzed the convergence behavior of LDPC codes with short block lengths and compared it with the case of infinite block lengths. Three different error types were identified and we showed their influence on the block error rate. We showed that only one type of can be explained with EXIT charts and that the gained E_b/N_0 for this type of error coincides with the EXIT chart analysis.

6. ACKNOWLEDGMENT

We would like to thank Sarah Johnson for supplying the codes from partial geometries.

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